AN INTEGRAL OF CHOI KWOK PUI APPEARING IN A POISSON PROCESS APPROXIMATION

In the calculation of an immigration-death process, close to a Poisson process, it was needed to evaluate the expectation

$$\int_0^\infty e^{-t} E\left(\frac{1}{Y(t)+Z(t)+1}\right)\, dt$$

where $Y(t)$ and $Z(t)$ are independent, $Y(t)$ is distributed as Binomial(m, e^{-t}) and $Z(t)$ is Poisson distributed with mean $\lambda(1-e^{-t})$. This leads to the evaluation of the double integral

$$(1) \qquad F(\lambda, m) \quad = \quad \int_0^1 \int_0^1 (1-r+u\,r)^m e^{-\lambda\, u\, r}\, du\, dr.$$

Here $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$.

The integral is given by

$$(2)\ F(\lambda, m) \quad = \quad \frac{m!}{\lambda^{m+1}} \left[\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} \right) \times \int_0^1 \frac{1-e^{-\lambda\, r}}{r}\, dr + \sum_{k=0}^{m-1} \frac{\lambda^k}{k!} \sum_{\nu=1}^{m-k} \frac{(-\lambda)^\nu}{\nu\, \nu!} \right].$$

The integral appearing in (2) is evaluated by Mathematica as

$$(3) \qquad \int_0^1 \frac{1-e^{-\lambda\, r}}{r}\, dr \quad = \quad \gamma + \ln\lambda + \Gamma(0, \lambda)$$

where γ is the Euler constant

$$(4) \qquad \gamma \quad = \quad \lim_{n\to\infty} 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \sim 0.577215$$

and $\Gamma(0, x)$ is the incomplete gamma function defined by

$$(5) \qquad \Gamma(0, x) \quad := \quad \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

The sum appearing in the first term in (2) can be expressed as

$$(6) \qquad \sum_{k=0}^{m} \frac{\lambda^k}{k!} \quad = \quad \frac{e^\lambda}{m!}\Gamma(m+1, \lambda)$$

so that

$$F(\lambda, m) = \frac{e^\lambda}{\lambda^{m+1}}\Gamma(m+1, \lambda)\left[\gamma + \Gamma(0,\lambda) + \ln\lambda\right] + \frac{m!}{\lambda^{m+1}} \sum_{k=0}^{m-1} \frac{\lambda^k}{k!} \sum_{\nu=1}^{m-k} \frac{(-\lambda)^\nu}{\nu\, \nu!}.$$

General information about this function can be obtained in [2], Chapter 45.

To evaluate the function $F(\lambda, m)$, let $v = 1-u$ and expand the term $(1-r\,v)^m$ to obtain

$$F(\lambda, m) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \int_0^1 r^{m-k} e^{-\lambda\, r} \int_0^1 e^{\lambda\, v\, r} v^{m-k}\, dv\, dr. \tag{7}$$

Now

$$\int_0^1 e^{\lambda\, v\, r} v^{m-k}\, dv = \lambda^{-m-1+k} r^{-m-1+k} \int_0^{\lambda\, r} x^{m-k} e^x\, dx. \tag{8}$$

The indefinite integral of $x^{m-k}e^x$ can be obtained by repeated integration by parts and it can be found in [1], 2.321.2:

$$\int x^{m-k} e^x\, dx = e^x \sum_{j=0}^{m-k} \frac{(-1)^j (m-k)!}{(m-k-j)!} x^{m-k-j} \tag{9}$$

The formula (2) appears directly from here. In the sum (7) one should combine the term corresponding to $k = m$ with the inner term corresponding to $j = m-k$ to produce terms involving the integrand $(1-e^{-\lambda\, r})/r$. These are integrable near $r = 0$.

The function $F(\lambda, m)$ can be simplified even further using

$$\begin{aligned}
\sum_{k=0}^{m-1} \sum_{j=1}^{m-k} \frac{(-1)^j \lambda^{k+j}}{j j! k!} &= \sum_{j+k \le m} \frac{(-1)^j \lambda^{k+j}}{j j! k!} \\
&= \sum_{r=1}^{m} \sum_{j=1}^{r} \frac{(-1)^j \lambda^r}{j j! (r-j)!} \\
&= \sum_{r=1}^{m} \frac{\lambda^r}{r!} \sum_{j=1}^{r} \frac{(-1)^j}{j} \binom{r}{j} \\
&= \sum_{r=1}^{m} \frac{\lambda^r}{r!} \sum_{j=1}^{r} \frac{1}{j} \\
&= \sum_{r=1}^{m} \frac{\lambda^r H_r}{r!}
\end{aligned}$$

where $H_r = 1 + 1/2 + \cdots + 1/r$ is the harmonic number. Therefore

$$F(\lambda, m) = \frac{e^\lambda}{\lambda^{m+1}} \Gamma(m+1, \lambda) \left[\gamma + \Gamma(0, \lambda) + \ln \lambda\right] + \frac{m!}{\lambda^{m+1}} \sum_{r=1}^{m} \frac{\lambda^r H_r}{r!}.$$

References

[1] I.S. Gradshteyn - I.M. Rhyzik: *Tables of Integrals, Series and Products*, 6th edition, Alan Jeffrey and D. Zwillinger editors, Academic Press, 2000.

[2] J. Spanier - K. Oldham: em An atlas of functions. Springer-Verlag, 1987.