A QUESTION OF EDGAR REICH

Problem. Prove that the integral

$$h(b) := \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{1-x^2+\sqrt{x^4+2bx^2+1}}{\sqrt{x^4+2bx^2+1}}\,dx,$$

for $|b| \leq 1$ is identically equal to 1.

A direct Mathematica calculation gives incorrect results. For example,

$$h(1/2) \quad = \quad \frac{2i}{\pi}\mathrm{ArcTanh}[2/\sqrt{3}]$$

and using *ComplexExpand* and *FullSimplify* simplifies to

$$h(1/2) \quad = \quad 1 + \frac{\ln(7+4\sqrt{3})}{\pi}i.$$

A more *dangerous* error occurs at $b = 1/3$, where Mathematica gives $h(1/3) = 1/2$. This error is hard to detect, due to the fact that the answer *seems reasonable*, at least it is real.

Victor Adamchik informed me that the reasons behind these errors is that the Mathematica evaluation involves dealing with branch cuts of the elliptic integrals appearing in h. He has provided the following elementary proof of the value of h.

Let $c = \sqrt{1-b^2}$, so that $b^2 + c^2 = 1$. The change of variables $x \to \sqrt{cy-b}$ transforms the integral into

$$h(b) \quad = \quad \int_{b/c}^\infty \frac{\sqrt{1+b+c(\sqrt{1+y^2}-y)}}{2\sqrt{cy-b}\sqrt{1+y^2}}\,dy$$

and the second change of variable $y = (z^2-1)/(2z)$ yields

$$h(b) \quad = \quad \frac{1}{\pi}\int_{(b+1)/c}^\infty \frac{\sqrt{c+(b+1)z}}{z\sqrt{cz^2-2bz-c}}\,dz.$$

Factoring the quadratic polynomial as

$$\begin{aligned} cz^2-2bz-c \quad &= \quad c\left(z-\tfrac{b-1}{c}\right)\left(z-\tfrac{b+1}{c}\right) \\ &= \quad c(z+1/d)(z-d) \end{aligned}$$

where $d = (b+1)/c$ so that $(b-1)/c = -1/d$. This yields

$$\frac{\sqrt{c+(b+1)z}}{z\sqrt{cz^2-2bz-c}} \quad = \quad \frac{\sqrt{d}}{z\sqrt{z-d}}$$

and finally the integral is

$$h(b) \quad = \quad \frac{\sqrt{d}}{\pi}\int_d^\infty \frac{dz}{z\sqrt{z-d}}.$$

The above integral is elementary

$$\int_d^\infty \frac{dz}{z\sqrt{z-d}} \quad = \quad \int_0^\infty \frac{2\,dy}{y^2+d} = \frac{\pi}{\sqrt{d}}.$$

The evaluation is complete.