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Some intriguing definite integrals

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Dedicated to Professor Dr. Herbert Gajewski on the occasion of his sixtieth birthday

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Abstract

We calculate some definite integrals which (up to now) computer algebra systems like Maple or Mathematica are unable to evaluate. The first one is a simply looking integral involving cos and log, the others are some integrals containing polylogarithmic functions. It is shown that they can be evaluated by rational combinations of ζ–functions and products of ζ–functions at positive integers.

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1. Introduction

To justify our doing, we quote J.J. SYLVESTER (1814–1897):

> *"It seems to be expected of every pilgrim up the slopes of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock."* ([S])

The first integral we consider is

$$I(a)=\int_0^\pi (1+\cos x)\log(a+\cos x)dx, \quad a\ge 1.$$

We show that

$$I(a)=\pi\left\{a-\sqrt{a^2-1}+\log\left(\frac{a+\sqrt{a^2-1}}{2}\right)\right\}. \tag{1.1}$$

The other integrals contain polylogarithmic functions. The polylogarithmic function $\mathcal{L}_p$ is defined for any complex p and any complex $|z|<1$ by the power series

$$\mathcal{L}_p(z)=\sum_{n=1}^\infty \frac{z^n}{n^p}. \tag{1.2}$$

We show that

$$I_m=\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds, \quad J_m=\int_{-1}^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds$$

can be expressed for $m=1,2,...$ in terms of ζ–functions at positive integers by

$$I_m=(-1)^{m-1}\left\{(m+1)\zeta(2m+1)-2\sum_{k=1}^{[m/2]}\zeta(2k)\zeta(2m+1-2k)\right\} \tag{1.3}$$

and

$$\begin{aligned} J_m= \quad & (-1)^{m-1}\Big\{(2-2^{-2m})\zeta(2m+1) \\ & -2^{2-2m}\sum_{k=1}^{[m/2]}[2^{2p-1}+2^{2m-2p}-1]\zeta(2k)\zeta(2m+1-2k)\Big\}. \end{aligned} \tag{1.4}$$

2. The proof of the identity (1.1)

Already Euler knew that

$$\int_0^\pi \log\sin x\,dx=2\int_0^{\pi/2}\log\sin x\,dx=-\pi\log 2.$$

Consider

$$A=\int\limits_0^{\pi/2}\cos^2 x\,\log\sin x\,dx,\quad B=\int\limits_0^{\pi/2}\sin^2 x\,\log\sin x\,dx.$$

The preceding line shows that

$$A+B=\int\limits_0^{\pi/2}\log\sin x\,dx=-\frac{\pi}{2}\log 2,$$

and obviously holds

$$A-B=\int\limits_0^{\pi/2}\cos 2x\,\log\sin x\,dx.$$

Integration by parts

$$\int\limits_0^{\pi/2}f'(x)g(x)\,dx=f(x)g(x)\Big|_0^{\pi/2}-\int\limits_0^{\pi/2}f(x)g'(x)\,dx$$

with

$$f'=\cos 2x,\quad g=\log\sin x,\quad f=\frac{1}{2}\sin 2x,\quad g'=\frac{\cos x}{\sin x}$$

gives

$$\begin{aligned}\int\limits_0^{\pi/2}\cos 2x\,\log\sin x\,dx &= \frac{1}{2}\sin 2x\,\log\sin x\Big|_0^{\pi/2}-\frac{1}{2}\int\limits_0^{\pi/2}\sin 2x\,\frac{\cos x}{\sin x}\,dx\\ &= -\int\limits_0^{\pi/2}\cos^2 x\,dx=-\frac{\pi}{4},\end{aligned}$$

and, consequently,

$$A=-\frac{\pi}{8}(1+\log 4),\quad B=\frac{\pi}{8}(1-\log 4).$$

Next consider

$$C=\int_0^{\pi}(1+\cos x)\,\log(1+\cos x)\,dx.$$

Using $1+\cos x=2\cos^2\frac{x}{2}$ and some obvious substitutions we get

$$\begin{aligned}C &= 2\int_0^{\pi}\cos^2\frac{x}{2}\log\left(2\cos^2\frac{x}{2}\right)\,dx=4\int\limits_0^{\pi/2}\cos^2 z\,\log\left(2\cos^2 z\right)dz\\ &= 4\int\limits_0^{\pi/2}\sin^2 y\,\log\left(2\sin^2 y\right)dy=4\int\limits_0^{\pi/2}\sin^2 y\,[\log 2+2\log\sin y]dy\\ &= 4\log 2\int\limits_0^{\pi/2}\sin^2 y\,dy+8\int\limits_0^{\pi/2}\sin^2 y\,\log\sin y\,dy=\pi\log 2+8B,\end{aligned}$$

hence

$$C = \pi(1 - \log 2).$$

This result is known to Maple and Mathematica. To evaluate

$$I(a) = \int_0^\pi (1 + \cos x) \log(a + \cos x) dx, \quad a \geq 1,$$

differentiate with respect to a and obtain

$$I'(a) = \int_0^\pi \frac{1 + \cos x}{a + \cos x} dx = \pi - (a - 1) \int_0^\pi \frac{dx}{a + \cos x}.$$

The classical substitution $t = \tan(x/2)$ (or computer algebra, or a classical table of integrals like [RG]) shows that

$$\int_0^\pi \frac{dx}{a + \cos x} = \frac{\pi}{\sqrt{a^2 - 1}}.$$

So we have

$$I'(a) = \pi - \pi \sqrt{\frac{a - 1}{a + 1}}.$$

Integration yields

$$I(a) - I(1) = \int_1^a I'(s)\, ds = \pi(a - 1) - \pi \int_1^a \sqrt{\frac{s - 1}{s + 1}}\, ds.$$

The remark that $I(1) = C$ and the elementary integral

$$\int_1^a \sqrt{\frac{s - 1}{s + 1}}\, ds = \sqrt{a^2 - 1} - \log(a + \sqrt{a^2 - 1})$$

(simply checked by differentiating both sides) proves (1.1).

3. The proof of the identities (1.3), (1.4)

A standard reference for the properties of polylogarithmic functions is the book of L. Lewin ([L]). According to A.B. Goncharov ([G]) the history of these functions can be traced back to Leibniz and J.Bernoulli. In the last time there seems to be an growing interest in these functions ([G],[M],[Z]). For the index $m = 1$ obviously holds

$$\mathcal{L}_1(z) = -\log(1 - z).$$

The computer algebra systems Maple and Mathematica know that

$$I_1 = \int_0^1 \frac{(\mathcal{L}_1(s))^2}{s} ds = \int_0^1 \frac{(\log(1 - s))^2}{s} ds = 2\zeta(3)$$

and

$$J_1 = \int_{-1}^{1} \frac{(\mathcal{L}_1(s))^2}{s}\, ds = \int_{-1}^{1} \frac{(\log(1-s))^2}{s}\, ds = \frac{7}{4}\zeta(3).$$

The corresponding integrals for polylogarithmic functions of higher indices are unknown to these computer algebra systems. We evaluate these integrals using formulas proved in [BBG] (some of them go back to Euler [E]).

For $Re(p) > 1$ the Riemann zeta function is defined by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Since the series (1.2) for $Re(p) > 1$ converges still on $|z| = 1$ we have

$$\mathcal{L}_p(1) = \zeta(p), \quad \mathcal{L}_p(-1) = (2^{1-p} - 1)\zeta(p) \quad \text{for} \quad Re(p) > 1.$$

Moreover, $\mathcal{L}_1(-1) = -\log 2$ and differentiation of (1.2) with respect to z yields the well–known relation

$$\mathcal{L}_p'(z) = \frac{\mathcal{L}_{p-1}(z)}{z}.$$

For positive integers $l \geq 2$ define

$$S_l(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{z^k}{k} \right), \quad A_l = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{1}{k} \right), \quad B_l = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{(-1)^k}{k} \right).$$

Obviously $A_l = S_l(1)$, $B_l = S_l(-1)$, and differentiation gives for $|z| < 1$

$$S_l'(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} z^{k-1} \right) \qquad \text{and with} \qquad \sum_{k=1}^{n} z^{k-1} = \frac{1-z^n}{1-z}$$

follows

$$S_l'(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\frac{1-z^n}{1-z} \right) = \frac{1}{(1-z)} \sum_{n=1}^{\infty} \frac{1}{n^l} - \frac{1}{(1-z)} \sum_{n=1}^{\infty} \frac{z^n}{n^l}$$

or

$$S_l'(z) = \frac{\zeta(l)}{(1-z)} - \frac{\mathcal{L}_l(z)}{(1-z)}.$$

By integration over $[0, x]$ for real $x = z$, $|x| < 1$ we obtain with $S_l(0) = 0$

$$S_l(x) = -\zeta(l) \log(1-x) - \int_0^x \frac{\mathcal{L}_l(s)}{(1-s)}\, ds.$$

Correspondingly, integration over $[-1, x]$ gives with $S_l(-1) = B_l$

$$S_l(x) = B_l - \zeta(l)\{\log(1-x) - \log 2\} - \int_{-1}^{x} \frac{\mathcal{L}_l(s)}{(1-s)}\, ds.$$

Integration by parts

$$\int_0^x f'(s)g(s)ds = f(s)g(s)|_0^x - \int_0^x f(s)g'(s)ds$$

on the right hand side with

$$f'(s) = \frac{1}{1-s}\,, \quad g(s) = \mathcal{L}_l(s), \quad f(s) = -\log(1-s), \quad g'(s) = \mathcal{L}_l'(s) = \frac{\mathcal{L}_{l-1}(s)}{s}$$

shows that

$$S_l(x) = \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) - \int_0^x \frac{\mathcal{L}_{l-1}(s)\log(1-s)}{s}\,ds$$

or

$$S_l(x) = \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) + \int_0^x \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds.$$

The same manipulation on the interval $[-1, x]$ gives

$$S_l(x) = B_l + \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) + \log\, 2\{\zeta(l) - \mathcal{L}_l(-1)\} + \int_{-1}^x \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds.$$

For $l \geq 2$, we have

$$\lim_{x\to 1-} \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) = 0.$$

This can be seen using l'Hopital's rule

$$\lim_{x\to 1-} \frac{[\mathcal{L}_l(x) - \zeta(l)]}{1/\log(1-x)} = \lim_{x\to 1-} (1-x)\,(\log(1-x))^2 \mathcal{L}_l'(x) = 0.$$

So we obtain

$$A_l = \int_0^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds \tag{3.1}$$

and

$$A_l = B_l + \log\, 2\{\zeta(l) - \mathcal{L}_l(-1)\} + \int_{-1}^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds \tag{3.2}$$

Repeating integration by parts for $l \geq 3$

$$\int_0^1 f'(s)g(s)ds = f(s)g(s)|_0^1 - \int_0^1 f(s)g'(s)ds$$

with

$$f' = \frac{\mathcal{L}_1(s)}{s}, \quad g = \mathcal{L}_{l-1}(s), \quad f = \mathcal{L}_2(s) \quad g' = \mathcal{L}_{l-1}'(s) = \frac{\mathcal{L}_{l-2}(s)}{s}$$

gives

$$\begin{aligned} \int_0^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds &= \mathcal{L}_2(s)\mathcal{L}_{l-1}(s)|_0^1 - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds \\ &= \zeta(2)\,\zeta(l-1) - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds. \end{aligned}$$

In the same way we obtain by partial integration on the interval $[-1, 1]$

$$\int_{-1}^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds = \zeta(2)\,\zeta(l-1) - \mathcal{L}_2(-1)\mathcal{L}_{l-2}(-1) - \int_{-1}^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds$$

We use this in (3.1) and find

$$A_l = \zeta(2)\,\zeta(l-1) - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds,$$

With (3.2) we get, respectively,

$$\begin{aligned} A_l = B_l \quad &+ \quad \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} \\ &+ \quad \zeta(2)\,\zeta(l-1) - \mathcal{L}_{l-1}(-1)\mathcal{L}_2(-1) - \int_{-1}^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds. \end{aligned}$$

Going on this way one shows by induction that for $j = 1, ..., [l/2]$

$$A_l = \sum_{k=1}^{j-1} (-1)^{k-1}\zeta(k+1)\zeta(l-k) + (-1)^{j-1}\int_0^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \tag{3.3}$$

and, especially for $l = 2m,\ j = m$:

$$A_{2m} = \sum_{k=1}^{m-1} (-1)^{k-1}\zeta(k+1)\zeta(2m-k) + (-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds. \tag{3.4}$$

Analogously we obtain for the interval $[-1, 1]$

$$\begin{aligned} A_l = \quad & B_l + \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} \\ & + \sum_{k=1}^{j-1} (-1)^{k-1}\zeta(k+1)\zeta(l-k) \\ & - \sum_{k=1}^{j-1} (-1)^{k-1}\mathcal{L}_{k+1}(-1)\mathcal{L}_{l-k}(-1) + (-1)^{j-1}\int_{-1}^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds \end{aligned}$$

and, especially for $l = 2m,\ j = m$:

$$\begin{aligned} A_{2m} = \quad & B_{2m} + \log 2\{\zeta(2m) - \mathcal{L}_{2m}(-1)\} \\ & + \sum_{k=1}^{m-1} (-1)^{k-1}\zeta(k+1)\zeta(2m-k) \\ & - \sum_{k=1}^{m-1} (-1)^{k-1}\mathcal{L}_{k+1}(-1)\mathcal{L}_{2m-k}(-1) + (-1)^{m-1}\int_{-1}^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds \end{aligned}$$

The quantities $A_l,\ B_l$ are related to the Euler sums

$$\sigma_h(s,t) = \sum_{n=1}^{\infty} \frac{1}{n^t}\left(\sum_{k=1}^{n-1} \frac{1}{k^s}\right), \quad s = 1, 2, ..., \quad t = 2, 3, ...,$$

and

$$\sigma_a(s,t) = \sum_{n=1}^{\infty} \frac{1}{n^t}\left(\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k^s}\right), \quad s = 1, 2, ..., \quad t = 2, 3, ...,$$

considered in [BBG]. Indeed, we have

$$A_l = \sigma_h(1,l) + \zeta(l+1), \quad B_l = -\sigma_a(1,l) + \mathcal{L}_{l+1}(-1). \tag{3.5}$$

We quote from [BBG] (p.278) (also proved in [N])

$$2\sigma_h(1,l) = l\zeta(l+1) - \sum_{k=1}^{l-2} \zeta(k+1)\,\zeta(l-k)$$

and (p.290)

$$2\sigma_a(1,l) = 2\eta(1)\,\zeta(l) - l\,\zeta(l+1) + \sum_{k=1}^{l} \eta(k)\,\eta(l+1-k),$$

where

$$\eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1-2^{1-p})\,\zeta(p) \quad \text{for } Re(p) > 1, \quad \eta(1) = \log 2.$$

With (3.5) we find

$$A_l = \left(\frac{l}{2}+1\right)\zeta(l+1) - \frac{1}{2}\sum_{k=1}^{l-2} \zeta(k+1)\,\zeta(l-k), \tag{3.6}$$

and, using $\mathcal{L}_l(-1) = -\eta(l)$,

$$B_l = -\zeta(l)\log 2 + \frac{l}{2}\zeta(l+1) - \frac{1}{2}\sum_{k=1}^{l} \eta(k)\,\eta(l+1-k) - \eta(l+1).$$

Especially, after some transformation,

$$A_{2m} = (m+1)\zeta(2m+1) - \sum_{k=1}^{m-1} \zeta(k+1)\,\zeta(2m-k),$$

$$B_{2m} = -\{\zeta(2m)+\eta(2m)\}\log 2 + m\zeta(2m+1) - \sum_{k=1}^{m-1} \eta(k+1)\,\eta(2m-k) - \eta(2m+1).$$

We use the expression for A_{2m} in (3.4) and find

$$(-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds = (m+1)\zeta(2m+1) - \sum_{k=1}^{m-1}[1+(-1)^{k-1}]\zeta(k+1)\,\zeta(2m-k).$$

In the sum the terms with k even cancel and we obtain, changing the sense of the summation index k ,

$$(-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds = (m+1)\zeta(2m+1) - 2\sum_{k=1}^{[m/2]} \zeta(2k)\,\zeta(2m+1-2k).$$

So the identity (1.3) for I_m is proved.

To prove (1.4), we use the expressions for A_{2m}, B_{2m} in the identity containing J_m. Several terms cancel, and we obtain after some transformations the intermediate result

$$
\begin{aligned}
(-1)^{m-1}\int_{-1}^{1}\frac{(\mathcal{L}_m(s))^2}{s}\,ds = \quad & (2-2^{-2m})\,\zeta(2m+1)\\
& -\sum_{k=1}^{m-1}[1+(-1)^{k-1}]\zeta(k+1)\,\zeta(2m-k)\\
& +\sum_{k=1}^{m-1}[1+(-1)^{k-1}]\eta(k+1)\,\eta(2m-k).
\end{aligned}
$$

Again the terms with k even cancel and with $\eta(j) = (1-2^{1-j})\zeta(j)$ this identity simplifies to

$$
\begin{aligned}
(-1)^{m-1}\int_{-1}^{1}\frac{(\mathcal{L}_m(s))^2}{s}\,ds = \quad & (2-2^{-2m})\,\zeta(2m+1)\\
& -2^{2-2m}\sum_{k=1}^{[m/2]}\{2^{2p-1}+2^{2m-2p}-1\}\zeta(2k)\,\zeta(2m+1-2k).
\end{aligned}
$$

This is identity (1.4) for J_m.

In the same way comparison of (3.6) and (3.3) yields the "mixed" integrals

$$
I_{j,l} = \int_0^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \quad l \geq 2, \quad j = 1, ...[l/2]
$$

as

$$
\begin{aligned}
I_{j,l} = \quad & (-1)^{j+1}\Bigg\{\left(\frac{l}{2}+1\right)\zeta(l+1)\\
& -\sum_{k=1}^{j-1}\left((-1)^{k-1}+\frac{1}{2}\right)\zeta(k+1)\,\zeta(l-k) - \frac{1}{2}\sum_{k=j}^{l-2}\zeta(k+1)\,\zeta(l-k)\Bigg\}.
\end{aligned}
$$

We could also give formulas for the corresponding integrals

$$
J_{j,l} = \int_{-1}^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \quad l \geq 2, \quad j = 1, ...[l/2]
$$

over the interval $[-1, 1]$. One gets the somewhat clumsy expression

$$
\begin{aligned}
J_{j,l} = \quad & (-1)^{j+1}\Bigg\{(2-2^{-l})\zeta(l+1)\\
& -\sum_{k=1}^{j-1}\left((-1)^{k-1}+\frac{1}{2}\right)\{2^{1-l+k}+2^{-k}-2^{1-l}\}\,\zeta(k+1)\,\zeta(l-k)\\
& -\frac{1}{2}\sum_{k=j}^{l-2}\{2^{1-l+k}+2^{-k}-2^{1-l}\}\zeta(k+1)\,\zeta(l-k)\Bigg\}.
\end{aligned}
$$

It is obvious that the identities for $m = 1$ mentioned in the introduction are included.

References

[BBG] Borwein, D., Borwein, J.M., Girgensohn R., Explicit Evaluation of Euler Sums, Proceedings Edinburgh Math. Soc. 38 (1995), 277–294.

[E] L. Euler, *Opera Omnia*, Ser. 1, Vol. XV, pp. 217–267, Teubner, Berlin (1917).

[G] Goncharov, A.B., The classical polylogarithms, algebraic K–theory and $\zeta_F(n)$. In: L. Corwin, I. Gelfand, J. Lepowsky (eds.), *The Gelfand Mathematical Seminars, 1990–1992*, pp.113–135, Birkhãuser Boston (1993).

[L] L. Lewin, *Dilogarithms and Associated Functions*, Macdonald, London (1958) or North Holland (1981).

[M] Milnor, J., On Polylogarithms, Hurwitz zeta functions and the Kubert Identities. L'Enseignement mathématique 29(1983), pp. 282–322.

[N] N. Nielsen, *Die Gammafunktion*, pp. 47–49, Chelsea New York (1965).

[RG] I.S. Gradshteyn, I.M. Ryshik, *Table of Integrals, Series, and Products*, 5th ed., Academic Press Boston (1994). Chelsea New York (1965).

[S] J.J. Sylvester, *The Collected Mathematical Papers*, Vol. II, p. 214 (footnote), Chelsea New York (1973).

[Z] Zagier, D., Polylogarithms, Dedekind zeta functions and the algebraic K–theory of fields. In: G. v.d. Geer, F. Oort, J. Steenbrink (eds.), Progress in Mathematics, 89 (1991), pp. 391–430, Birkhãuser Boston.