THE PARITY OF FIBONACCI NUMBERS

The Fibonacci numbers F_n are defined by the recurrence

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. The list of these numbers begins with

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55. \tag{2}$$

This data shows that the parity of these numbers begins with

$$\text{even, odd, odd, even, odd, odd,} \tag{3}$$

and this patterns seems to repeat. `How does one prove this?`

Since parity deals with the remainder after division by 2, it looks reasonable to transform

$$\text{even} \mapsto 0 \quad \text{and odd } \mapsto 1. \tag{4}$$

The set of remainders of an integer after division by 2, namely $\{0, 1\}$ is denoted by $\mathbb{Z}_2$ and is called the `set of integers modulo` 2. The remainder of x after division by 2 is denoted by $x \bmod 2$. This operation respects the operations in $\mathbb{Z}$; that is,

$$(a + b) \bmod 2 = a \bmod 2 + b \bmod 2. \tag{5}$$

Applying this to the recurrence defining Fibonacci numbers, we see that

$$F_n \bmod 2 = F_{n-1} \bmod 2 + F_{n-2} \bmod 2. \tag{6}$$

This shows that two terms in the sequence of parities of F_n determines the third one completely. Thus, if a pattern of of 0 and 1 repeats, then everything after that pair will also repeat and the sequence becomes periodic. In this case, the pattern $\{0, 1\}$, appearing in positions 0, 1, reappears in positions 3, 4, showing that the sequence $F_n \bmod 2$ is periodic with period length at most 3. The first three terms of this sequence are $\{0, 1, 1\}$ and the argument before shows that this sequence determines the values of $F_n \bmod 2$ for $n \geq 3$. This completes the proof.

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