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The 2-adic Valuation of the Coefficients of a Polynomial

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Abstract. In this paper we compute the 2-adic valuations of some polynomials associated with the definite integral

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}.$$

1. Introduction.

In this paper we present a study of the coefficients of a polynomial defined in terms of the definite integral

$$N_{0,4}(a;m) \;=\; \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{1.1}$$

where m is a positive integer and $a > -1$ is a real number.

Apart from their intrinsic interest, these polynomials form the basis of a new algorithm for the definite integration of rational functions.

An elementary calculation shows that

$$P_m(a) \;:=\; \frac{2^{m+3/2}}{\pi}(a+1)^{m+1/2}N_{0,4}(a;m) \tag{1.2}$$

is a polynomial of degree m in a with rational coefficients. Let

$$P_m(a) = \sum_{l=0}^{m} d_l(m)a^l. \tag{1.3}$$

Then it can be shown that $d_l(m)$ is equal to

$$\sum_{j=0}^{l}\sum_{s=0}^{m-l}\sum_{k=s+l}^{m}(-1)^{k-l-s}2^{-3k}\binom{2k}{k}\binom{2m+1}{2(s+j)}\binom{m-s-j}{m-k}\binom{s+j}{j}\binom{k-s-j}{l-j}$$

from which it follows that $d_l(m)$ is a *rational number* with only a power of 2 in its denominator. Extensive calculations have shown that, with rare exceptions, the numerators of $d_l(m)$ contain a single large prime divisor and its remaining factors are very small. For example

$$d_6(30) \;=\; 2^{12}\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31\cdot 37\cdot 6393245948809857765 31.$$

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Similarly, $d_{10}(200)$ has 197 digits with a prime factor of length 137 and its second largest divisor is 797. This observation lead us to investigate the arithmetic properties of $d_l(m)$. In this paper we discuss the 2-adic valuation of these $d_l(m)$.

The fact that the coefficients of $P_m(a)$ are *positive* is less elementary. This follows from a hypergeometric representation of $N_{0,4}(a;m)$ that implies the expression

$$d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}. \tag{1.4}$$

We have produced a proof of (1.4) that is independent of this hypergeometric connection and is based on the Taylor expansion

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{P_{k-1}(a)}{2^{k+1}\,(a+1)^k}\, c^k \right); \tag{1.5}$$

see [**1**] for details.

The expression (1.4) can be used to efficiently compute the coefficients $d_l(m)$ when l is large relative to m. In Section 8 we derive a representation of the form

$$d_l(m) = \frac{1}{l!m!2^{m+l}} \left(\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1) \right)$$

where $\alpha_l(m)$ and $\beta_l(m)$ are polynomials in m of degrees l and $l-1$ respectively. For example

$$d_1(m) = \frac{1}{m!2^{m+1}} \left((2m+1) \prod_{k=1}^{m} (4k-1) - \prod_{k=1}^{m} (4k+1) \right). \tag{1.6}$$

This representation can now be used to efficiently examine the coefficients $d_l(m)$ when l is small compared to m. In Section 7 we prove that

$$\nu_2(d_1(m)) = 1 - 2m + \nu_2\left(\binom{m+1}{2} \right) + s_2(m)$$

where $s_2(m)$ is the sum of the binary digits of m.

2. The polynomial $\mathbf{P_m(a)}$.

Let

$$N_{0,4}(a;m) = \int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{m+1}}.$$

Then

$$P_m(a) = \frac{2^{m+3/2}}{\pi} (a+1)^{m+1/2} N_{0,4}(a;m) \tag{2.1}$$

is a polynomial in a with positive rational coefficients. The proof is elementary and is presented in [**1**]. It is based on the change of variables $x = \tan\theta$ and $u = 2\theta$ that yields

$$N_{0,4}(a;m) = 2^{-m-1} \int_0^{\pi} \frac{(1+\cos u)^{2m+1}}{((1+a)+(1-a)\cos^2 u)^{m+1}}\, du.$$

Expanding the numerator and employing the standard substitution $z = \tan u$ produces

(2.2)
$$N_{0,4}(a;m) = 2^{-2m-3/2} \sum_{\nu=0}^{m} \binom{2m+1}{2\nu} \frac{(a-1)^{m-\nu}}{(a+1)^{m-\nu+1/2}} \sum_{k=0}^{m-\nu} \binom{m-\nu}{k} \frac{2^k}{(a-1)^k} B(m-k+1/2,1/2)$$

where B is Euler's beta function, defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The expression (2.1) now produces the first formula for $d_l(m)$ given in the Introduction.

3. The triple sum for $\mathbf{d_l(m)}$.

The expression for the coefficients $d_l(m)$ given in the Introduction can be written as

$$\sum_{j=0}^{l} \sum_{s=0}^{m-l} \sum_{k=s+l}^{m} (-1)^{k-l-s} 2^{-3k} \binom{2k}{k} \binom{2m+1}{2(s+j)} \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{l-j}.$$

(3.1)

This expression follows directly from expanding (2.3) and the value

$$B(j+1/2,1/2) = \frac{\pi}{2^{2j}} \binom{2j}{j}.$$

It follows that $d_l(m)$ is a rational number whose denominator is a power of 2, therefore

Lemma 3.1. *Let p be an odd prime. Then*

$$\nu_p(d_l(m)) \geq 0.$$

The positivity of $d_l(m)$ remains to be seen.

4. The single sum expression for $\mathbf{d_l(m)}$.

An alternative form of the coefficients $d_l(m)$ is obtained by recognizing $N_{0,4}(a;m)$ as a hypergeometric integral. A standard argument shows that

$$N_{0,4}(a;m) = \frac{\pi\binom{2m}{m}}{2^{m+3/2}(a+1)^{m+1/2}} {}_2F_1\left[-m, m+1; 1/2-m; (1+a)/2\right]$$

where ${}_2F_1$ is a hypergeometric function, defined by

$${}_2F_1[a,b,c;z] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $(r)_k$ is the rising factorial

$$(r)_k = r(r+1)(r+2)\cdots(r+k-1).$$

It follows that $P_m(a)$ is the *Jacobi polynomial* of degree m with *parameters* $m+1/2$ and $-(m+1/2)$. Therefore the coefficients are given by

$$d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l} \tag{4.1}$$

from which their positivity is obvious. We have obtained a proof of (4.1) that is independent of hypergeometric considerations and is based on the presence of $P_m(a)$ in the Taylor expansion (1.5). See [**1**] for details.

The formula (4.1) is very efficient for the calculation of the coefficients $d_l(m)$ when l approximately equal to m. For instance, we have

$$\begin{aligned} d_m(m) &= 2^{-m}\binom{2m}{m}; \\ d_{m-1}(m) &= 2^{-(m+1)}\binom{2m}{m}. \end{aligned}$$

The expression (4.1), rewritten in the form

$$d_l(m) = 2^{-(2m-l)} \sum_{k=l}^{m} 2^{k-l} \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l},$$

shows that

$$\nu_2(d_l(m)) \geq l - 2m. \tag{4.2}$$

5. Basics on valuations.

Here we describe what is required on valuations.
Given a prime p and a rational number r, there exist unique integers a, b, m with $p \not| a, b$ such that

$$r = \frac{a}{b} p^m. \tag{5.1}$$

The integer m is the $p-$adic valuation of r and we denote it by $\nu_p(r)$.
Now recall a basic result of number theory which states that

$$\nu_p(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{p^k} \right\rfloor. \tag{5.2}$$

Naturally the sum is finite and we can end it at $k = \lfloor \log_p m \rfloor$.
There is a famous result of Legendre [**2, 4**] for the $p-$adic valuation of $m!$. It states that

$$\nu_p(m!) = \frac{m - s_p(m)}{p-1} \tag{5.3}$$

where $s_p(m)$ is the sum of the base$-p$ digits of m. In particular

$$\nu_2(m!) = m - s_2(m). \tag{5.4}$$

6. The constant term.

The calculation of the 2-adic valuation of the coefficients can be made very explicit for the first few. We begin with the case of the constant term.
We first compute

$$N_{0,4}(0;m) = \int_0^\infty \frac{dx}{(x^4+1)^{m+1}}$$

via the change of variable $u = x^4$, yielding

$$\begin{aligned} N_{0,4}(0;m) &= \frac{1}{4}B(1/4, m+3/4) \\ &= \frac{\pi}{m!2^{2m+3/2}} \prod_{k=1}^{m}(4k-1). \end{aligned}$$

Therefore

(6.1) $$d_0(m) = \frac{1}{m!2^m} \prod_{k=1}^{m}(4k-1).$$

Theorem 6.1. *The 2-adic valuation of the constant term $d_0(m)$ is given by*

$$\begin{aligned} \nu_2(d_0(m)) &= -\left(m + \nu_2(m!)\right) \\ &= s_2(m) - 2m. \end{aligned}$$

Proof: This follows directly from (6.1). The second expression comes from (5.4). □

Using the single sum formula for $d_0(m)$ we obtain

Corollary 6.2.

$$\begin{aligned} \nu_2\left(\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \right) &= m - \nu_2(m!) \\ &= s_2(m). \end{aligned}$$

Corollary 6.3. *The 2-adic valuation of the constant term $d_0(m)$ satisfies*

$$\nu_2(d_0(m)) \geq 1 - 2m$$

with equality if and only if m is a power of 2.

We now present a different proof of Corollary 3 that is based on the expression

(6.2) $$d_0(m) = \frac{1}{m!2^m} \prod_{k=1}^{m}(4k-1)$$

and the single sum formula

$$
\begin{aligned}
2^{2m}d_0(m) &= \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} \\
&= \binom{2m}{m} + 2\sum_{k=1}^{m} 2^{k-1}\binom{2m-2k}{m-k}\binom{m+k}{m}. \qquad (6.3)
\end{aligned}
$$

Proof: From (6.3) it follows that

$$\nu_2(d_0(m)) \;\geq\; 1-2m$$

because the central binomial coefficient is an even number. Now from (6.2) we obtain

$$\nu_2(d_0(m)) \;=\; -(m+\nu_2(m!)). \qquad (6.4)$$

From (5.2) we have

$$\nu_2(m!) \;=\; \sum_{k=1}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor .$$

Thus, from (6.4),

$$\nu_2(d_0(m)) \;=\; -\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor .$$

We know $\nu_2(d_0(m)) \geq 1-2m$, so it suffices to determine when equality occurs. Indeed, the equation

$$\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor \;=\; 2m-1 \qquad (6.5)$$

can be solved explicitly. Write $m = 2^e r$ with r odd, and say $2^N < r < 2^{N+1}$. Then

$$\sum_{k=0}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor \;=\; 2^e\cdot r + 2^{e-1}\cdot r + \cdots + r + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{r}{2^2} \right\rfloor + \cdots \left\lfloor \frac{r}{2^N} \right\rfloor$$

and (6.5) leads to

$$r-1 \;=\; \sum_{k=1}^{N} \left\lfloor \frac{r}{2^j} \right\rfloor < \sum_{k=1}^{N} \frac{r}{2^j} \leq \sum_{k=1}^{\infty} \frac{r}{2^j} = \frac{r}{2}$$

and we conclude that $r=1$. The proof is finished.

7. The linear term.

From the triple sum we obtain

$$d_1(m) \;=\; \sum_{s=0}^{m-1}\sum_{k=s+1}^{m} (-1)^{k-s-1}2^{-3k}(m-s)\binom{2k}{k}\binom{2m+2}{2s+1}\binom{m-s-1}{m-k}.$$

Differentiating (2.1) and $d_1(m) = P_m'(0)$ we produce

$$d_1(m) = \frac{1}{m!2^{m+1}} \left((2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1) \right).$$

Therefore the linear coefficient is given in terms of

$$A_1(m) := (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1) \tag{7.1}$$

so that

$$d_1(m) = \frac{A_1(m)}{m!2^{m+1}}. \tag{7.2}$$

We prove

Theorem 7.1. *The 2-adic valuation of the linear coefficient $d_1(m)$ is given by*

$$\nu_2(d_1(m)) = 1 - 2m + \nu_2\left(\binom{m+1}{2}\right) + s_2(m).$$

Recall that the inequality $\nu_2(d_1(m)) \geq 1-2m$ follows directly from the single sum expression. The theorem determines the exact value of the correction term.

Proof: We prove

$$\begin{aligned} \nu_2\left(A_1(m)\right) &= \nu_2(2m(m+1)) \\ &= 2 + \nu_2\left(\binom{m+1}{2}\right). \end{aligned}$$

The result then follows from (5.4) and (7.2).

Define

$$B_m = \prod_{k=1}^{m}(4k+1) - 1$$

and

$$C_m = (2m+1)\prod_{k=1}^{m}(4k-1) - 1.$$

Then evidently $A_1(m) = B_m - C_m$.
We show

a) $\nu_2(B_m) = 2 + \nu_2\left(\binom{m+1}{2}\right)$

b) $\nu_2(C_m) \geq 3 + \nu_2\left(\binom{m+1}{2}\right)$

from which the result follows immediately.

a) We have

$$\begin{aligned} B_m &= \prod_{k=1}^{m}(4k+1)-1 \\ &= \left(\sum_{j=1}^{m+1} 4^{m+1-j}\begin{bmatrix} m+1 \\ j \end{bmatrix}\right)-1 \\ &= \sum_{j=1}^{m} 2^{2(n+1-j)}\begin{bmatrix} m+1 \\ j \end{bmatrix} \\ &= 2^2\begin{bmatrix} m+1 \\ m \end{bmatrix} + \sum_{k=2}^{m} 2^{2k}\begin{bmatrix} m+1 \\ m+1-k \end{bmatrix} \\ &= 2^2\binom{m+1}{2} + \sum_{k=2}^{m} 2^{2k}\begin{bmatrix} m+1 \\ m+1-k \end{bmatrix} \end{aligned}$$

where $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]$ is an (unsigned) Stirling numbers of the first kind, i.e.,

$$x(x+1)\cdots(x+m-1) = \sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix} x^k.$$

To prove a), it suffices to show that

$$\nu_2\left(2^2\binom{m+1}{2}\right) < \nu_2\left(2^{2k}\begin{bmatrix} m+1 \\ m+1-k \end{bmatrix}\right)$$

for $2 \le k \le m$.
To do this we observe that there exist integers $C_{k,i}$ $(k \ge 1,\ i \ge 0)$ such that

$$\begin{bmatrix} m \\ m-k \end{bmatrix} = \sum_{i=0}^{k-1}\binom{m}{2k-i}C_{k,i}$$

see [**3**, p. 152]. For example

$$\begin{aligned} \begin{bmatrix} m \\ m-1 \end{bmatrix} &= \binom{m}{2} \\ \begin{bmatrix} m \\ m-2 \end{bmatrix} &= 3\binom{m}{4}+2\binom{m}{3} \\ \begin{bmatrix} m \\ m-3 \end{bmatrix} &= 15\binom{m}{6}+20\binom{m}{5}+6\binom{m}{4} \\ \begin{bmatrix} m \\ m-4 \end{bmatrix} &= 105\binom{m}{8}+210\binom{m}{7}+130\binom{m}{6}+24\binom{m}{5}. \end{aligned}$$

Hence the rational number

$$u := \frac{m(m-1)\cdots(m-k)}{(2k)!}$$

divides $\left[\begin{smallmatrix} m \\ m-k \end{smallmatrix}\right]$ in the sense that the quotient

$$\frac{\left[\begin{smallmatrix} m \\ m-k \end{smallmatrix}\right]}{u}$$

is an integer.

It follows that

$$\begin{aligned}\nu_2\left(\begin{bmatrix} m \\ m-k \end{bmatrix}\right) &\geq \nu_2(m(m-1)\cdots(m-k)) - \nu_2((2k)!) \\ &= \nu_2(m(m-1)\cdots(m-k)) - 2k + s_2(k)\end{aligned}$$

where we have used (5.3).
Hence, provided $k \geq 3$,

$$\nu_2\left(\begin{bmatrix} m+1 \\ m+1-k \end{bmatrix}\right) \geq \nu_2((m+1)m(m-1)\cdots(m+1-k)) - 2k + s_2(k)$$

so that

$$\begin{aligned}\nu_2\left(2^{2k}\begin{bmatrix} m+1 \\ m+1-k \end{bmatrix}\right) &\geq \nu_2((m+1)m) + \nu_2((m-1)(m-2)) + s_2(k) \\ &\geq \nu_2((m+1)m) + 1 + 1 \\ &> \nu_2\left(2^2\binom{m+1}{2}\right)\end{aligned}$$

provided $m \geq 3$. (For $m = 1,\ 2$ it is easy to check $\nu_2(B_m) = 2$.)

On the other hand, if $k = 2$, then

$$\begin{aligned}\begin{bmatrix} m \\ m-2 \end{bmatrix} &= 3\binom{m}{4} + 2\binom{m}{3} \\ &= \frac{1}{24}m(m-1)(m-2)(3m-1),\end{aligned}$$

so if m is even, $m \geq 4$, we have

$$\begin{aligned}\nu_2\left(\begin{bmatrix} m \\ m-2 \end{bmatrix}\right) &= \nu_2\left(\frac{m(m-1)}{2}\right) + \nu_2(m-2) - \nu_2(12) \\ &\geq \nu_2\left(\frac{m(m-1)}{2}\right) + 1 - 2 \\ &= \nu_2\left(\frac{m(m-1)}{2}\right) - 1\end{aligned}$$

while if m is odd, $m \geq 3$, we have

$$\begin{aligned}\nu_2\left(\begin{bmatrix} m \\ m-2 \end{bmatrix}\right) &= \nu_2\left(\frac{m(m-1)}{2}\right) + \nu_2(3m-1) - \nu_2(12) \\ &\geq \nu_2\left(\frac{m(m-1)}{2}\right) + 1 - 2 \\ &= \nu_2\left(\frac{m(m-1)}{2}\right) - 1\end{aligned}$$

so in either event

$$\nu_2\left(\begin{bmatrix} m+1 \\ m-1 \end{bmatrix}\right) \geq \nu_2\left(\binom{m+1}{2}\right) - 1.$$

Hence

$$\begin{aligned} \nu_2\left(2^4 \begin{bmatrix} m+1 \\ m-1 \end{bmatrix}\right) &\geq \nu_2\left(\binom{m+1}{2}\right) + 3 \\ &> \nu_2\left(2^m \binom{m+1}{2}\right) \end{aligned}$$

as desired.

We now prove b):

$$C_m = (2m+1)\prod_{k=1}^{m}(4k-1) - 1.$$

We have

$$\begin{aligned} \prod_{k=1}^{m}(4k-1) &= 4^m \prod_{k=1}^{m}(k - 1/4) \\ &= -4^{m+1}\sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix} (-1/4)^k \\ &= (-1)^m \sum_{k=1}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix} (-4)^{m+1-k} \\ &= (-1)^m \sum_{k=1}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix} (-4)^{m+1-k} \end{aligned}$$

thus

$$C_m = \left((-1)^m(2m+1)\sum_{k=1}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix} (-4)^{m+1-k}\right) - 1.$$

When m is even, we have

$$\begin{aligned} C_m &= (2m+1) - (2m+1)\cdot 4\begin{bmatrix} m+1 \\ m \end{bmatrix} - 1 + (2m+1)\sum_{k=2}^{m} \begin{bmatrix} m+1 \\ m+1-k \end{bmatrix}(-4)^k \\ &= -2m^2(2m+3) + (2m+1)\sum_{k=2}^{m} \begin{bmatrix} m+1 \\ m+1-k \end{bmatrix}(-4)^k \end{aligned}$$

so, as in the proof of a), we have

$$\begin{aligned}\nu_2(C_m) &\geq \min\left(\nu_2(2m^2), \nu_2\left(4^2\begin{bmatrix} m+1 \\ m-1\end{bmatrix}\right),\right. \\ &\qquad \left.\nu_2\left(4^4\begin{bmatrix} m+1 \\ m-1\end{bmatrix}\right), \cdots, \nu_2\left(4^m\begin{bmatrix} m+1 \\ 1\end{bmatrix}\right)\right) \\ &\geq \min\left(1+2\nu_2(m),\ 3+\nu_2\left(\tbinom{m+1}{2}\right)\right) \\ &\geq 3+\nu_2\left(\tbinom{m+1}{2}\right)\end{aligned}$$

since m is even.

On the other hand, when m is odd we observe that

$$C_m + 1 = (2m+1)\prod_{k=1}^{m}(4k-1)$$

and

$$C_{m+1} + 1 = (2m+3)(4m+3)\prod_{k=1}^{m}(4k-1)$$

so

$$\frac{C_{m+1}+1}{(2m+3)(4m+3)} = \frac{C_m+1}{2m+1}$$

and hence

$$\begin{aligned} C_m &= \frac{(C_{m+1}+1)(2m+1)}{(2m+3)(2m+3)} - 1 \\ &= \frac{(2m+1)C_{m+1} - 8(m+1)^2}{(2m+3)(4m+3)} \qquad (7.3)\end{aligned}$$

so

$$\begin{aligned}\nu_2(C_m) &\geq \min\left(\nu_2(C_{m+1}), 2\nu_2(m+1)+3\right) \\ &\geq 3+\nu_2\left(\binom{m+1}{2}\right)\end{aligned}$$

since m is odd.

This completes the proof.

The corresponding question of the 3-adic valuation of $d_1(m)$ seems to be more difficult. We propose.

Problem 7.2. *Prove the existence of a sequence of positive integers m_j such that $\nu_3(d_1(m_j)) = 0$. Extensive calculations show that*

$$m_{j+1} - m_j \in \{2, 7, 20, 61, 182, \cdots\} \qquad (7.4)$$

where the sequence $\{q_j\}$ in (7.4) is defined by $q_1 = 2$ and $q_{j+1} = 3q_j + (-1)^{j+1}$. It would be of interest to know whether $\nu_3(d_1(m))$ is unbounded: the maximum value for $2 \leq m \leq 20000$ is 12, so perhaps $\nu_3(d_1(m)) = O(\log m)$ as $m \to \infty$.

8. The general situation.

In this section we prove the existence of polynomials $\alpha_l(x)$ and $\beta_l(x)$ with positive integer coefficients such that

$$d_l(m) = \frac{1}{l!m!2^{m+l}} \left(\alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1) \right).$$

These polynomials are efficient for the calculation of $d_l(m)$ if l is small relative to m, so they complement the results of Section 4.
For example

$$\begin{aligned} \alpha_0(m) &= 1 \\ \alpha_1(m) &= 2m+1 \\ \alpha_2(m) &= 2(2m^2+2m+1) \\ \alpha_3(m) &= 4(2m+1)(m^2+m+3) \\ \alpha_4(m) &= 8(2m^4+4m^3+26m^2+24m+9). \end{aligned}$$

and

$$\begin{aligned} \beta_0(m) &= 0 \\ \beta_1(m) &= 1 \\ \beta_2(m) &= 2(2m+1) \\ \beta_3(m) &= 12(m^2+m+1) \\ \beta_4(m) &= 8(2m+1)(2m^2+2m+9). \end{aligned}$$

The proof consists in computing the expansion of $P_m(a)$ via the Leibnitz rule:

$$P_m(a) = \frac{2^{m+3/2}}{\pi} \sum_{j=0}^{l} \binom{l}{j} \left(\frac{d}{da}\right)^{l-j} (a+1)^{m+1/2}\Big|_{a=0} \left(\frac{d}{da}\right)^{j} N_{0,4}(a;m)\Big|_{a=0}.$$

We have

$$\left(\frac{d}{da}\right)^{r} (a+1)^{m+1/2}\Big|_{a=0} = 2^{-2r}\frac{(2m+2)!}{(m+1)!}\frac{(m-r+1)!}{(2m-2r+2)!} \tag{8.1}$$

and

$$\left(\frac{d}{da}\right)^{r} N_{0,4}(a;m)\Big|_{a=0} = (-1)^r \frac{(m+r)!}{m!} 2^r \int_0^\infty \frac{x^{2r}}{(x^4+1)^{m+r+1}}\, dx. \tag{8.2}$$

The integral is evaluated via the change of variable $t = x^4$ as

$$\int_0^\infty \frac{x^{2r}\, dx}{(x^4+1)^{m+r+1}} = \frac{1}{4} B\left(\frac{r}{2}+\frac{1}{4}, m+\frac{r}{2}+\frac{3}{4}\right).$$

This yields

$$\left(\frac{d}{da}\right)^{r} N_{0,4}(a;m)\Big|_{a=0} = \frac{(-1)^r (2r)!}{2^{2r+2m+3/2}} \frac{\pi}{m!r!} \prod_{l=1}^{m} (4l-1+2r). \tag{8.3}$$

Therefore

$$P_m^{(l)}(0) = \frac{l!(2m+2)!}{2^{m+2l}m!(m+1)!}\sum_{j=0}^{l}\frac{(-1)^j(m-l+j+1)!(2j)!}{j!^2(l-j)!(2m-2l+2j+2)!}\prod_{\nu=1}^{m}(4\nu-1+2j).$$

We now split the sum according to the parity of j. In the case j is odd $(=2t-1)$ we use

$$\prod_{\nu=1}^{m}(4\nu-1+2j) = \prod_{\nu=1}^{m}(4\nu+1)\left(\prod_{\nu=m+1}^{m+t-1}(4\nu+1)\Big/\prod_{\nu=1}^{t-1}(4\nu+1)\right)$$

and if j is even $(=2t)$ we employ

$$\prod_{\nu=1}^{m}(4\nu-1+2j) = \prod_{\nu=1}^{m}(4\nu-1)\left(\prod_{\nu=m+1}^{m+t}(4\nu-1)\Big/\prod_{\nu=1}^{t}(4\nu-1)\right).$$

We conclude that

$$d_l(m) = X(m,l)\prod_{\nu=1}^{m}(4\nu-1) - Y(m,l)\prod_{\nu=1}^{m}(4\nu+1)$$

with

$$X(m,l) = \frac{(2m+2)!}{2^{m+2l}m!(m+1)!}\sum_{t=0}^{\lfloor l/2\rfloor}\frac{(m-l+2t+1)!(4t)!}{(2t)!^2(l-2t)!(2m-2l+4t+2)!}\frac{\prod_{\nu=m+1}^{m+t}(4\nu-1)}{\prod_{\nu=1}^{t}(4\nu-1)}$$

and

$$Y(m,l) =$$

$$\frac{(2m+2)!}{2^{m+2l}m!(m+1)!}\sum_{t=1}^{\lfloor (l+1)/2\rfloor}\frac{(m-l+2t)!(4t-2)!}{(2t-1)!^2(l-2t+1)!(2m-2l+4t)!}\frac{\prod_{\nu=m+1}^{m+t-1}(4\nu+1)}{\prod_{\nu=1}^{t-1}(4\nu+1)}.$$

The quotients of factorials appearing above can be simplified via

$$\frac{(m+1)!}{(m-l+2t+1)!} = \prod_{j=1}^{l-2t}(j+m-l+2t+1)$$

and

$$\frac{(2m+2)!}{(2m-2l+4t+2)} = 2^{l-2t}\left(\prod_{i=1}^{l-2t}(i+m-l+2t+1)\right)\left(\prod_{i=1}^{l-2t}(2i+2m-2l+4t+1)\right).$$

We conclude that

$$d_l(m) = \frac{1}{l!m!2^{m+l}}\left(\alpha_l(m)\prod_{\nu=1}^{m}(4\nu-1) - \beta_l(m)\prod_{\nu=1}^{m}(4\nu+1)\right)$$

with

$$\alpha_l(m) = l!\sum_{t=0}^{\lfloor l/2\rfloor}\frac{\binom{4t}{2t}}{2^{2t}(l-2t)!}\frac{\prod_{\nu=m+1}^{m+t}}{\prod_{\nu=1}^{t}(4\nu-1)}\left(\prod_{\nu=1}^{t}(4\nu-1)\right)\left(\prod_{\nu=m-(l-2t-1)}^{m}(2\nu+1)\right)$$

and

$$\beta_l(m) = l! \sum_{t=1}^{\lfloor (l+1)/2 \rfloor} \frac{\binom{4t-2}{2t-1}}{2^{2t-1}(l-2t+1)!} \left(\frac{\prod_{\nu=m+1}^{m+t-1}(4\nu+1)}{\prod_{\nu=1}^{t-1}(4\nu+1)} \right) \left(\prod_{\nu=m-(l-2t)}^{m} (2\nu+1) \right).$$

The identity

$$\prod_{\nu=1}^{t}(4\nu-1) = \frac{(4t)!}{2^{2t}(2t)!} \left(\prod_{\nu=1}^{t-1}(4\nu+1) \right)^{-1}$$

is now employed to produce

$$\alpha_l(m) = \sum_{t=0}^{\lfloor l/2 \rfloor} \binom{l}{2t} \prod_{\nu=m+1}^{m+t}(4\nu-1) \prod_{\nu=m-(l-2t-1)}^{m}(2\nu+1) \prod_{\nu=1}^{t-1}(4\nu+1)$$

and

$$\beta_l(m) = \sum_{t=1}^{\lfloor (l+1)/2 \rfloor} \binom{l}{2t-1} \prod_{\nu=m+1}^{m+t-1}(4\nu+1) \prod_{\nu=m-(l-2t)}^{m}(2\nu+1) \prod_{\nu=1}^{t-1}(4\nu-1).$$

We have proven:

Theorem 8.1. *There exist polynomials $\alpha_l(x)$ and $\beta_l(x)$ with integer coefficients such that*

$$d_l(m) = \frac{1}{l!m!2^{m+l}} \left(\alpha_l(m) \prod_{k=1}^{m}(4k-1) - \beta_l(m) \prod_{k=1}^{m}(4k+1) \right).$$

Based on extensive numerical calculations we propose

Conjecture 8.2. *All the roots of the polynomials $\alpha_l(m)$ and $\beta_l(m)$ lie on the line $Re(m) = -1/2$.*

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