AN INTEGRAL WITH THREE PARAMETERS

GEORGE BOROS AND VICTOR H. MOLL

Abstract. In this paper we give an exact expression for the integral

$$I(a,b;r) = \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2}$$

in terms of Euler's beta function. Several classical integrals are deduced from it. The expression for $I(a,b;r)$ provides a method for unifying a large class of integrals.

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1. Introduction

In this paper we present a closed form evaluation of the integral

$$I(a,b;r) \quad := \quad \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2}.$$

This *master formula* is used to evaluate a large number of definite integrals. Some of these are well-known, by which we mean that they can be computed by a symbolic language or can be found in a table of integrals. We have used Mathematica 3.0 and Maple V as sources for the former and Gradshteyn and Ryzhik [3] for the latter. We point out that we only tried the most naive method of symbolic evaluation, so when we claim that a certain integral cannot be evaluated by one of these languages, we mean that the integral cannot be evaluated as stated. The variety of definite integrals that can be deduced from (1.1) is immense, and a systematic classification would be desirable. In this paper we have chosen to examine a selection of these integrals that can be expressed in terms of elementary functions and the gamma function and its derivatives. For instance, in (5.4) we show that

$$\int_0^\infty \left[(x^2+1) \sqrt[3]{x(x^2+1)^2} \right]^{-1} \times \ln \left(\frac{x}{x^2+1} \right) dx \quad = \quad \left[-\frac{1}{2} \Gamma \left(\frac{1}{3} \right) \right]^3.$$

We consider this to be one of our most interesting results.

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As a consequence of our expression for (1.1) we derive a formula of Liouville [5]:

$$\int_0^\infty \left[(ax + b/x)^2 + c \right]^{-p-1} dx = \frac{\sqrt{\pi}\,\Gamma(p+1/2)}{2ac^{p+1/2}\,\Gamma(p+1)}.$$

This expression is correct only for $b < 0$, and we provide the correct result for $b > 0$ in Section 2. This formula has been misquoted in several tables of integrals (see for example [3] 3.257).

In Section 2 we evaluate (1.1). Examples involving specific choices of the parameters are presented in Section 3. In Section 4 we give additional examples obtained by differentiation with respect to the parameter r. The last two sections contain a discussion of the case $a = 1$.

2. Derivation of the master formula

Conditions on the parameters a, b and r that guarantee convergence of the integral $I(a, b; r)$ defined in (1.1) will always be assumed; these *are the only* restrictions on the parameters. In particular, $a > -1$ and $r > \frac{1}{2}$ are sufficient, but r *is not* required to be an integer.

We show first that the integral I is *independent* of b:

Lemma 2.1. *Suppose f satisfies the functional equation $f(1/x) = x^2 f(x)$. Then $\int_0^\infty f(x)/(x^b + 1)\, dx$ is independent of b.*

The proof follows easily from differentiation with respect to b and the change of variable $x \to 1/x$.

The lemma applies to (1.1), so it suffices to evaluate $I(a, b; r)$ for the special case $b = 2$:

$$\begin{aligned} J(a;r) := I(a,2;r) &= \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^r \frac{dx}{x^2} \\ &= \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^r dx. \end{aligned}$$

The last expression is obtained by the substitution $x \to 1/x$. We conclude that

$$J(a;r) = \frac{1}{2} \int_0^\infty \left(\frac{x^2}{x^4 + 2ax^2 + 1} \right)^r \times \frac{x^2+1}{x^2}\, dx.$$

We now make the substitution $x = \tan\theta$ to rewrite $J(a;r)$ in its trigonometric form:

$$J(a;r) = 2^{-r+1} \int_0^\pi \left(\frac{1 - \cos\theta}{(3+a) + (1-a)\cos\theta} \right)^r \cdot \frac{d\theta}{1 - \cos\theta},$$

where we have used the symmetry of cosine about $\theta = \pi$. The change of variable $\Phi = (1-\cos\theta)/[(3+a)+(1-a)\cos\theta]$ then yields

$$J(a;r) \quad = \quad 2^{-1/2-r}(1+a)^{1/2-r}B(r-1/2,1/2),$$

where

$$B(x,y) \quad = \quad \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

is Euler's beta function. This proves:

Theorem 2.2. *Let*

$$\begin{aligned}
I_1 &= \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1}\right)^r \cdot \frac{x^2+1}{x^b+1}\cdot\frac{dx}{x^2} \\
I_2 &= \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1}\right)^r \cdot \frac{dx}{x^2} \\
I_3 &= \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1}\right)^r dx \\
I_4 &= \frac{1}{2}\int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1}\right)^r \cdot \frac{x^2+1}{x^2}dx.
\end{aligned}$$

Then $I_1 = I_2 = I_3 = I_4$ *and each has the common value*

$$I(a,b;r) \quad = \quad 2^{-1/2-r}(1+a)^{1/2-r}B(r-1/2,1/2).$$

A simple scaling produces:

Corollary 2.3.

$$\int_0^\infty \left(\frac{x^2}{bx^4+2ax^2+c}\right)^r dx \quad = \quad \frac{B(r-1/2,1/2)}{2^{r+1/2}\sqrt{b}\left(a+\sqrt{bc}\right)^{r-1/2}},$$

where $b>0,\ c\geq 0,\ a>-\sqrt{bc}$ *and* $r>1/2$.

Note 1. Corollary 2.3 can be used to clarify [3] 3.257:

$$F := \int_0^\infty \left[(ax+b/x)^2+c\right]^{-p-1} dx \quad = \quad \frac{\sqrt{\pi}\,\Gamma(p+1/2)}{2ac^{p+1/2}\,\Gamma(p+1)}. \tag{2.1}$$

The evaluation of this integral can be traced back to Liouville [5]. We may assume $a>0$. Then Corollary 2.3 produces

$$\begin{aligned}
F &= \int_0^\infty \left(\frac{x^2}{a^2x^4+(2ab+c)x^2+b^2}\right)^{p+1} dx \\
&= \frac{B(p+1/2,1/2)}{2a\left[2a(b+|b|)+c\right]^{p+1/2}},
\end{aligned}$$

which is valid for $p>-1/2$ and $2a(b+|b|)+c>0$. For $b>0$ this yields

$$F \quad = \quad \frac{B(p+1/2,1/2)}{2a\left[4ab+c\right]^{p+1/2}},$$

where $c > -4ab$, and for $b < 0$ it yields

$$F = \frac{B(p+1/2, 1/2)}{2ac^{p+1/2}},$$

where $c > 0$. Thus (2.1) is correct only for $b < 0$.

Liouville considered only the case $b < 0$, but his method can also be applied to $b > 0$. The integral I_3 is written as

$$\int_0^\infty \left[(x-1/x)^2 + 2(a+1)\right]^{-r} \, dx \tag{2.2}$$

$$= \frac{1}{2}\int_0^\infty \left[(x-1/x)^2 + 2(a+1)\right]^{-r} \times \left(1+1/x^2\right) \, dx, \tag{2.3}$$

where (2.3) is produced by averaging (2.2) with the integral obtained by the substitution $x \to 1/x$. The expression in Theorem 1 follows directly from here.

3. Special cases

We now compute examples of $I(a,b;r)$ for specific choices of the parameters, some of which are well-known. We tried to compute each example using both Maple V and Mathematica 3.0 on a SUN Ultra 1; in those cases where an answer was thereby obtained, we indicate the amount of CPU time taken. In several cases symbolic integration fail to give an answer.

Example 3.1: $a = 1/2,\ r = 3$ in I_2:

$$\int_0^\infty \frac{x^4\,dx}{(x^4+x^2+1)^3} = \frac{\pi}{48\sqrt{3}}.$$

Mathematica 3.0 computes this in 29.91 seconds.

Example 3.2: $a = 7/2,\ r = 5/2$ in I_2:

$$\int_0^\infty \frac{x^3\,dx}{(x^4+7x^2+1)^{5/2}} = \frac{2}{243}.$$

In this case Mathematica gives the correct answer in 2.09 seconds, but Maple V gives $-17/1215$, clearly incorrect; if we introduce the change of variable $x^2 = u$, Maple V does provide the correct answer.

Example 3.3: $a = 7,\ r = 5/4$ in I_2:

$$\int_0^\infty \frac{\sqrt{x}\,dx}{(x^4+14x^2+1)^{5/4}} = \frac{\Gamma^2(3/4)}{4\sqrt{2\pi}}.$$

This could not be evaluated symbolically.

Example 3.4: $a = 1/2,\ r = 3/4$ in I_2:

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^4+x^2+1)^{3/4}} = \frac{\pi^{3/2}}{\Gamma^2(3/4)\ \sqrt[4]{12}}.$$

This could not be done symbolically either.

Example 3.5: $b = 102,\ a = (1+2\sqrt{2})/2,\ r = 1$ in I_1:

$$\int_0^\infty \frac{dx}{\left[x^4+(1+2\sqrt{2})x^2+1\right]\left[x^{100}-x^{98}+\cdots+1\right]} = \frac{\pi}{2(1+\sqrt{2})}.$$

Example 3.6: $a = 1,\ r = 1$ in I_1 produces the evaluation of a well-known integral:

$$\int_0^\infty \frac{dx}{(x^2+1)(x^b+1)} = \frac{\pi}{4}$$

(see [2], page 262). This can be transformed via $x = \tan\theta$ to the familiar form

$$\int_0^{\pi/2} \frac{d\theta}{1+(\tan\theta)^b} = \frac{\pi}{4}.$$

Example 3.7: $a = 1$ and arbitrary r in I_3:

$$\int_0^{\pi/2} \sin^{2r-2}\theta\cos^{2r-2}\theta\, d\theta = 2^{1-2r}B(r-1/2,1/2)$$

after the change of variable $x = \tan\theta$. The substitution $2\theta = \nu$ yields Wallis' integral:

$$\int_0^{\pi/2} \sin^\delta \nu d\ \nu = \frac{1}{2}B\left(\frac{\delta+1}{2},\frac{1}{2}\right),$$

where $\delta := 2r-2 > -1$ (see [2], page 54).

4. Differentiation results

The function

$$\begin{aligned} I(a,b;r) &= \int_0^\infty \left(\frac{x^2}{x^4+2ax^2+1}\right)^r \cdot \frac{x^2+1}{x^b+1}\cdot\frac{dx}{x^2} \\ &= 2^{-1/2-r}(1+a)^{1/2-r}\times B(r-1/2,1/2) \qquad (4.1)\end{aligned}$$

may be differentiated with respect to each of the independent variables a, b and r. Consistent with Lemma 1, $\partial I/\partial b$ is zero, and $\partial I/\partial a$ produces an equivalent form of (4.1). One can, however, obtain additional results by differentiation with respect to r. For example, with $a = -1/2$ in I_2, we have

$$\int_0^\infty \left(\frac{x^2}{x^4 - x^2 + 1} \right)^r \cdot \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2} \Gamma(r - 1/2)/\Gamma(r). \tag{4.2}$$

Differentiation of (4.2) produces

$$\int_0^\infty \left(\frac{x^2}{x^4 - x^2 + 1} \right)^r \times \ln \left(\frac{x^2}{x^4 - x^2 + 1} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2} \times \frac{\Gamma(r)\Gamma'(r-1/2) - \Gamma(r-1/2)\Gamma'(r)}{\Gamma^2(r)}. \tag{4.3}$$

Several interesting integrals can now be evaluated by specifying a value of the parameter r. Such calculations produce *nice results* in terms of well-known constants provided we know the values of the function Γ and its derivatives at the arguments r and $r - 1/2$.

Example 4.1. $r = 1$ in (4.3):

$$\int_0^\infty \left(\frac{1}{x^4 - x^2 + 1} \right) \times \ln \left(\frac{x^2}{x^4 - x^2 + 1} \right) dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(1)\Gamma'(1/2) - \Gamma(1/2)\Gamma'(1)}{\Gamma^2(1)} = -\pi \ln 2.$$

In this calculation we have used the values $\Gamma'(1) = -\gamma$ and $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2\ln 2)$. Here γ is Euler's constant. This cannot be evaluated symbolically.

Example 4.2. $r = 3/2$ n (4.3):

$$\int_0^\infty \frac{x}{(x^4 - x^2 + 1)^{3/2}} \times \ln \left(\frac{x^2}{x^4 - x^2 + 1} \right) dx = 2(\ln 2 - 1).$$

A direct Mathematica evaluation yields *divergent integral*. The same calculation adapted to the range $0 \leq x \leq 1$ produces the correct answer in 2.75 seconds.

Example 4.3. $r = 3/4$ in (4.3):

$$\int_0^\infty \left(\frac{x^2}{x^4 - x^2 + 1} \right)^{3/4} \times \ln \left(\frac{x^2}{x^4 - x^2 + 1} \right) dx = -\frac{\sqrt{\pi}}{2\sqrt{2}} \Gamma^2(1/4).$$

We could not evaluate this example symbolically even after transforming the problem to $0 \leq x \leq 1$.

Note 2. Further differentiation of (4.2) produces more examples of integrals that can be evaluated in closed form. These calculations are now given in terms of the values of Γ, Γ' and Γ'' at the arguments r and $r - 1/2$. For example, $r = 1$ yields

$$\int_0^\infty \frac{1}{(x^4 - x^2 + 1)} \times \ln^2 \left(\frac{x^2}{x^4 - x^2 + 1} \right) dx = \frac{\pi}{2} \left(\frac{\pi^2}{3} + 4 \ln^2 2 \right).$$

5. The case $a = 1$

In this section we discuss in detail the case $a = 1$. Theorem 1 with I_2 yields

$$\int_0^\infty \left(\frac{x}{x^2+1} \right)^{2r} \frac{dx}{x^2} = 2^{-2r} B(r-1/2, 1/2).$$

Let

$$G(r) := \frac{\Gamma(r-1/2)}{\Gamma(r) 2^{2r-1}} = \frac{2^{1-2r}}{\sqrt{\pi}} B(r-1/2, 1/2).$$

Then

$$\frac{\sqrt{\pi}}{2} G(r) = \int_0^\infty \left(\frac{x}{x^2+1} \right)^{2r} \frac{dx}{x^2}.$$

Differentiation produces

$$\int_0^\infty \left(\frac{x}{x^2+1} \right)^{2r} \ln \left(\frac{x}{x^2+1} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4} G'(r),$$

$$= \frac{\sqrt{\pi}}{4} G(r) \left[\psi(r-1/2) - \psi(r) - 2 \ln 2 \right]. \tag{5.1}$$

Here $\psi(r) = \Gamma'(r)/\Gamma(r)$ is the logarithmic derivative of $\Gamma(r)$.

In order to obtain *explicit* results, in addition to knowing the values of Γ and Γ' at the arguments r and $r - 1/2$, we also need to know the values of ψ and ψ'.

Example 5.1. Let $r = 1$. Then (5.1) yields

$$\int_0^\infty \frac{1}{(x^2+1)^2} \times \ln \left(\frac{x}{x^2+1} \right) dx = -\frac{\pi \ln 2}{2},$$

where we have used the appropriate values of ψ to compute $G'(1)$. This *cannot* be evaluated using Mathematica. We now use [3] 4.234.6

$$\int_0^\infty \frac{\ln x \, dx}{(A^2 + B^2 x^2)(1+x^2)} = \frac{\pi B}{2A(B^2 - A^2)} \ln \left(\frac{A}{B} \right)$$

in the limiting case $A \to 1, \; B \to 1$ to obtain

$$\int_0^\infty \frac{\ln x \, dx}{(x^2+1)^2} = -\frac{\pi}{4}.$$

Combining this with the previous result gives

$$\int_0^\infty \frac{\ln(x^2+1) \, dx}{(x^2+1)^2} = \frac{\pi}{4}(2 \ln 2 - 1).$$

This can be evaluated in Mathematica in 1.37 seconds.

Note 3. If we repeat the same procedure used to obtain (5.1) but use Theorem 1 with I_1 in lieu of I_2, we obtain

$$\int_0^\infty \left(\frac{x}{x^2+1} \right)^{2r} \times \ln \left(\frac{x}{x^2+1} \right) \times \frac{x^2+1}{x^2(x^b+1)} dx = \frac{\sqrt{\pi}}{4} G'(r). \tag{5.2}$$

Example 5.2. $r = 1$ and $b = 0$ in (5.2):

$$\int_0^\infty \left(\frac{1}{x^2+1} \right) \times \ln \left(\frac{x}{x^2+1} \right) dx = \frac{\sqrt{\pi}}{2} G'(1) = -\pi \ln 2.$$

Since

$$\int_0^\infty \frac{\ln x}{x^2+1} dx = 0,$$

we thus get

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1} dx \quad = \quad \pi \ln 2.$$

This cannot be evaluated directly by Mathematica.

Note 4. Differentiation of (5.1) yields

$$\int_0^\infty \left(\frac{x}{x^2+1} \right)^{2r} \times \ln^2 \left(\frac{x}{x^2+1} \right) \frac{dx}{x^2} \quad = \quad \frac{\sqrt{\pi}}{8} G''(r) \tag{5.3}$$

with

$$G''(r) \quad = \quad G(r) \left[\psi'(r-1/2) - \psi'(r) + (\psi(r-1/2) - \psi(r) - 2\ln 2)^2 \right].$$

As usual, certain values of r produce exact evaluations.

Example 5.3. $r = 1$ in (5.3):

$$\int_0^\infty \frac{1}{(x^2+1)^2} \times \ln^2 \left(\frac{x}{x^2+1} \right) dx \quad = \quad \frac{\pi}{48} \left(\pi^2 + 48 \ln^2 2 \right).$$

This cannot be evaluated in Mathematica.

We conclude this section with two examples that we find aesthetically pleasing.

Example 5.4. $r = 3/4$ in (5.1):

$$\int_0^\infty \left[(x^2+1)\sqrt{x(x^2+1)} \right]^{-1} \times \ln \left(\frac{x}{x^2+1} \right) dx \quad = \quad -\frac{1}{8\sqrt{\pi}} (\pi + 2\ln 2) \times \Gamma^2 \left(\frac{1}{4} \right).$$

Example 5.5. $r = 5/6$ in (5.1):

$$\int_0^\infty \left[(x^2+1)\sqrt[3]{x(x^2+1)^2} \right]^{-1} \times \ln \left[\frac{x}{x^2+1} \right] dx = \left(-\frac{1}{2} \Gamma \left(\frac{1}{3} \right) \right)^3. \tag{5.4}$$

This example is reminiscent of [3] 4.244.1:

$$\int_0^1 \left[\sqrt[3]{x(1-x^2)^2}\right]^{-1} \times \ln x \, dx \quad = \quad \left(-\frac{1}{2}\Gamma\left(\frac{1}{3}\right)\right)^3. \tag{5.5}$$

6. A final series of examples

Starting with the expression

$$G(r) = \frac{\Gamma(r-1/2)}{\Gamma(r)2^{2r-1}} = \frac{2}{\sqrt{\pi}} \int_0^\infty \left(\frac{x}{x^2+1}\right)^{2r} \cdot \frac{dx}{x^2}$$

and differentiating n times we obtain

(6.1)

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^{2r} \times \left[\ln\left(\frac{x}{x^2+1}\right)\right]^n \cdot \frac{dx}{x^2} \quad = \quad \frac{\sqrt{\pi}}{2^{n+1}} \frac{d^n}{dr^n} \left[\frac{\Gamma(r-1/2)}{\Gamma(r)2^{2r-1}}\right].$$

In particular, for $r = 3/2$ and $n = 3$ we have

$$\int_0^\infty \left[\left(\frac{x}{x^2+1}\right) \times \ln\left(\frac{x}{x^2+1}\right)\right]^3 \cdot \frac{dx}{x^2} \quad = \quad \frac{3}{8}\left[\zeta(3)+\zeta(2)-4\right].$$

The substitution $t = x/(x^2+1)$ then gives

$$\zeta(3) \quad = \quad 4 - \zeta(2) + \frac{8}{3}\int_0^{1/2} \frac{t}{\sqrt{1-4t^2}} \left[\ln\, t\right]^3 \, dt.$$

Now, with $r = 3/2$ and $n = 4$ we get

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^3 \times \left[\ln\left(\frac{x}{x^2+1}\right)\right]^4 \cdot \frac{dx}{x^2} \quad = \quad \frac{1}{160}\left[-3\pi^4 - 80\pi^2 + 1920 - 480\zeta(3)\right],$$

and in the answer we see a rational combination of the values of the Riemann zeta function $\zeta(j)$ at $j = 2,\ 3,$ and 4. Higher values of n produce similar algebraic combinations of the values of $\zeta(j)$. The appearance of $\zeta(j)$ is due to the expression for the Polygamma function

$$\text{PolyGamma}[n,z] = \frac{d^{n+1}}{dz^{n+1}} \log\Gamma(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}$$

and the special values at $z = 1$ and $z = 3/2$:

$$\begin{aligned} \text{PolyGamma}[j-1,1] &= (-1)^j n! \zeta(j) \\ \text{PolyGamma}[j-1,3/2] &= (-1)^j (j-1)! \left[(2^j-1)\zeta(j) - 2^j\right]. \end{aligned}$$

One last example. Formula (6.1) can be used to produce many more exact evaluations of definite integrals. For example, one can show that

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^{2j+1} \times \left[\ln\left(\frac{x}{x^2+1}\right)\right] \cdot \frac{dx}{x^2} \quad = \quad \frac{H_j - H_{2j} - 1/2j}{2j\binom{2j}{j}}.$$

for any positive integer j and a similar formula for even exponents. Here

$$H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$$

is the j-th partial sum of the harmonic series.

7. Conclusions

We have presented an elementary master formula for the closed form evaluation of an integral that depends upon three parameters. Many classical examples can be evaluated within this framework by specifying the parameters or by differentiation with respect to them. In addition, we have been able to evaluate a large number of other integrals which cannot be found in standard tables and cannot be evaluated by standard symbolic software packages.

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$$\int_0^1 \frac{t^{\mu+1/2}(1-t)^{\mu-1/2}}{(a+bt-ct^2)^{\mu+1}}.$$

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[6] J. Liouville. Sur l'integrale

$$\int_0^1 \frac{t^{\mu+1/2}(1-t)^{\mu-1/2}}{(a+bt-ct^2)^{\mu+1}}.$$

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8. Update. January 2009

This is a review of the statements from this paper.

1) The result in Theorem 2.2 has appeared as entry 3.242.11 of [1].

2) Naturally the times have been reduced. Example 3.1 takes 5.89 and Example 3.2 takes 0.194 seconds. The integrals in Examples 3.3 and 3.4 still cannot be evaluated symbolically.

3) Mathematica 6.0 is now able to compute the integral in Example 4.1. The same is true for Example 4.2.

4) Mathematica is unable to compute the integral in Example 4.3, nor the one after Note 2.

5) Mathematica is now able to evaluate the integral in Example 5.1. The same is true for the one in Example 5.2 and also the one in Example 5.3.

6) The integrals in Example 5.5 can be evaluated using Mathematica.

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Department of Mathematics, University of New Orleans, New Orleans, LA 70148

E-mail address: gboros@math.uno.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118

E-mail address: vhm@math.tulane.edu