A SPECIAL RATIONAL FUNCTION WITH VANISHING INTEGRAL

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Abstract. The integral of a rational function proposed as a question in Mathematics Stack Exchange is evaluated. The integrand has a polynomial of degree 4 as denominator. A natural extension to degree 8 is shown to vanish.

1. Introduction

Question 258746 in Mathematics Stack Exchange asks for the evaluation of

$$I_1(\alpha,\beta) = \int_{-\infty}^{\infty} \frac{dw}{(\alpha^2-w^2)^2+\beta^2w^2}. \tag{1.1}$$

It is convenient to expand the integrand and introduce the scaling to obtain

$$\begin{aligned} I_1(\alpha,\beta) &= 2\int_0^{\infty} \frac{dw}{w^4+(\beta^2-2\alpha^2)w^2+\alpha^4} \\ &= \frac{2}{\alpha^3}\int_0^{\infty} \frac{dt}{t^4+2at^2+1}, \end{aligned} \tag{1.2}$$

where $a = \beta^2/2\alpha^2 - 1$. The value of $I_1(\alpha,\beta)$ is now obtained from the next result.

Theorem 1.1. *For $a > -1$ and $m \in \mathbb{N}\cup\{0\}$, define*

$$N_{0,4}(a;m) := \int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.3}$$

Then

$$N_{0,4}(a;m) = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+1/2}}, \tag{1.4}$$

where $P_m(a)$ is the polynomial

$$P_m(a) = \sum_{\ell=0}^{m}\left[2^{-2m}\sum_{k=\ell}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\ell}\right]a^{\ell}. \tag{1.5}$$

The special case $m=0$ gives

$$N_{0,4}(a;0) = \int_0^{\infty}\frac{dx}{x^4+2ax^2+1} = \frac{\pi}{2\sqrt{2}\sqrt{a+1}} \tag{1.6}$$

and this produces

$$I_1(\alpha,\beta) = \frac{\pi}{\alpha^2\beta} \tag{1.7}$$

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directly.

An elementary proof of Theorem 1.1, using only the value of the Wallis' integral

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \tag{1.8}$$

appears in [4]. The reader will find in [1] a variety of proofs, including the original one in [2].

2. An email request

In a recent email, the author was asked for the evaluation of

$$I_2(p,q) = \int_{-\infty}^\infty \frac{x^4-p+qx^2}{(x^4-p)^2+(qx^2)^2}\,dx, \text{ for } p,\, q>0. \tag{2.1}$$

This is a natural extension of the original question about the integral I_1 in (1.1).

A brute force computation of $I_2(p,q)$ using `Mathematica` gives

$$I_2(1,1) = 0 \text{ and } I_2(2,3) = 0. \tag{2.2}$$

On the other hand, if one asks for the value of $I_2(p,q)$ with p and q kept as parameters, produces a result with a variety of restrictions such as

$$\text{Re}\left[\left(p - \tfrac{1}{2}q^2 - \tfrac{1}{2}\sqrt{-4pq^2+q^4}\right)^{1/4}\right] > 0. \tag{2.3}$$

This is not a natural restriction, since (2.1) converges for any value $p,\, q>0$.

Symbolic examples suggest that $I_2(p,q)=0$. But more seems to be true. Let

$$f(x;p,q) = \frac{x^4-p+qx^2}{(x^4-p)^2+(qx^2)^2} \tag{2.4}$$

then the examples above satisfy

$$\int_0^1 f(x;p,q)\,dx = -\int_1^\infty f(x;p,q)\,dx, \tag{2.5}$$

and this gives $I_2(p,q)=0$. An elementary approach to (2.5), following techniques developed in the classical book [5], is presented next.

Lemma 2.1. *Assume $g(x)$ satisfies $g(1/x) = -x^2 g(x)$. Then* $\displaystyle\int_0^\infty g(x)\,dx = 0.$

Proof. Split the integral into $[0,\,1]$ and $[1,\,\infty)$ and make the change of variables $x \mapsto 1/x$ in the second interval. □

It is unfortunate that $f(x;p,q)$ does not satisfy the hypothesis of Lemma 2.1. A different approach is required. This is presented next.

Expanding the denominator in (2.1) and using the symmetry of the integrand gives

$$I_2(p,q) = 2\int_0^\infty \frac{x^4+qx^2-p}{x^8+(q^2-2p)x^4+p^2}\,dx. \tag{2.6}$$

In order to compute $I_2(p,q)$ introduce the notation

$$T_k(a) = \int_0^\infty \frac{t^k\,dt}{t^8+2at^4+1}. \tag{2.7}$$

Lemma 2.2. *The integral* $I_2(p,q)$ *is given by*

$$I_2(p,q) = 2p^{-3/4}T_4(a) + 2qp^{-5/4}T_2(a) - 2p^{-3/4}T_0(a) \tag{2.8}$$

with $a = q^2/2p - 1$.

Proof. Make the change of variables $x = p^{1/4}t$ so that

$$x^8 + (q^2-2p)x^4 + p^2 = p^2\left(t^8 + 2at^4 + 1\right). \tag{2.9}$$

The rest is elementary. □

The integrals $T_k(a)$ are evaluated in the next section.

3. The integrals T_k

This section presents the evaluation of the integrals T_k. The first result was established in [3]. The conditions $a_1 > \max\left\{-a_2-1, -\text{sign}(a_2+4)\times\left(a_2^2/8+1\right)\right\}$ guarantee the convergence of the integral below. In particular, if $a_2 = 0$ this becomes $a_1 > -1$.

Theorem 3.1. *Define*

$$M_8(a_1,a_2;r) := \int_0^\infty \left(\frac{x^4}{x^8+a_2x^6+2a_1x^4+a_2x^2+1}\right)^r dx, \tag{3.1}$$

where $r \in \mathbb{N}$. *Then*

$$M_8(a_1,a_2;r) = c^{1/4-r}N_{0,4}\left(\frac{a_2+4}{2\sqrt{c}}; r-1\right), \tag{3.2}$$

where $c = 2(a_1+a_2+1)$.

Proof. The change of variable $x \mapsto 1/x$ yields a new form of the integral M_8:

$$M_8(a_1,a_2;r) = \int_0^\infty \left(\frac{x^4}{x^8+a_2x^6+2a_1x^4+a_2x^2+1}\right)^r \frac{dx}{x^2}. \tag{3.3}$$

Computing the average of these two forms and letting $x = \tan\theta$ and then $\psi = 2\theta$ produces

$$M_8(a_1,a_2;r) = 2^{-r+1}\int_0^\pi \frac{(1-C)^{2r-1}\,d\psi}{\left[(a_1-a_2+1)C^2+7(2-a_1-a_2)C+(17+3a_2+a_1)\right]^r},$$

where $C = \cos\psi$. The substitution $z = \cot\,\psi$ then gives

$$M_8(a_1,a_2;r) = 2^{-r+1}\int_0^\infty \frac{dz}{\left(8z^4+2(a_2+4)z^2+(a_1+a_2+1)\right)^r}. \tag{3.4}$$

The change of variable $z \mapsto (8/(a_1+a_2+1))^{1/4}t$ and scaling (1.6) yield (3.2). □

The special case $a_2 = 0$ and $r = 1$ gives the value of $T_4(a)$.

Corollary 3.2. The integral $T_4(a)$ is given by

$$T_4(a) = \frac{\pi}{2^{9/4}\sqrt{a+1}\sqrt{\sqrt{2}+\sqrt{a+1}}} = \frac{\pi}{2^{9/4}}\frac{\left[\sqrt{2}-\sqrt{1+a}\right]^{1/2}}{\sqrt{1-a^2}}. \tag{3.5}$$

Proof. Theorem 3.1 gives

$$T_4(a) = c^{1/4}N_{0,4}\left(\frac{2}{\sqrt{c}}, 0\right), \tag{3.6}$$

and the result follows from (1.6). □

Corollary 3.3. For $a > -1$, the identity $T_2(a) = T_4(a)$ holds.

Proof. The change of variables $x \mapsto 1/x$ gives the result. □

It does not seem possible to obtain an expression for the remaining integral

$$T_0(a) = \int_0^\infty \frac{dx}{x^8 + 2ax^4 + 1} \tag{3.7}$$

by the previous methods. For a different approach, let $t = x^4$ to obtain

$$T_0(a) = \frac{1}{4} \int_0^\infty \frac{t^{-3/4}\, dt}{t^2 + 2at + 1}. \tag{3.8}$$

This integral is a special case of entry 3.252.11 in [6]

$$\int_0^\infty \frac{z^{\nu-1}\, dz}{(z^2 + 2az + 1)^{\mu+1/2}} = \frac{2^\mu \Gamma(1+\mu) B(-\nu + 2\mu + 1, \nu) P_{\mu-\nu}^{-\mu}(a)}{(a^2-1)^{\mu/2}} \tag{3.9}$$

where $P_\nu^\mu(z)$ is the associated Legendre function. This is a special function with hypergeometric representation

$$P_\nu^\mu(a) = \frac{1}{\Gamma(1-\mu)} \left(\frac{a+1}{a-1} \right)^{\mu/2} {}_2F_1 \left(\left. \begin{matrix} -\nu, \nu+1 \\ 1-\mu \end{matrix} \right| \frac{1-a}{2} \right) \tag{3.10}$$

given in entry 8.702 of [6]. This yields

$$\int_0^\infty \frac{z^{\nu-1}\, dz}{(z^2 + 2az + 1)^{\mu+1/2}} = \frac{2^\mu B(2\mu + 1 - \nu, \nu)}{(a+1)^\mu} \, {}_2F_1 \left(\left. \begin{matrix} \nu - \mu, 1 + \mu - \nu \\ 1 + \mu \end{matrix} \right| \frac{1-a}{2} \right). \tag{3.11}$$

Using the parameters $\nu = \frac{1}{4}$ and $\mu = \frac{1}{2}$, the expression (3.8) becomes

$$T_0(a) = \frac{3\pi}{8\sqrt{a+1}} {}_2F_1 \left(\left. \begin{matrix} -\frac{1}{4}, \frac{5}{4} \\ \frac{3}{2} \end{matrix} \right| \frac{1-a}{2} \right). \tag{3.12}$$

The functional equation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ and $\Gamma(x+1) = x\Gamma(x)$ have been used in the simplification.

The final step uses entry 9.121.30 of [6]

$${}_2F_1 \left(\left. \begin{matrix} 1 + \frac{n}{2}, 1 - \frac{n}{2} \\ \frac{3}{2} \end{matrix} \right| z^2 \right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}} \tag{3.13}$$

with $n = 5/2$ and $z = \sqrt{(1-a)/2}$ to obtain

$$T_0(a) = \frac{\pi\sqrt{2}}{4\sqrt{1-a^2}} \sin \left(\frac{3}{2} \arcsin \sqrt{\frac{1-a}{2}} \right). \tag{3.14}$$

Using the identity $\sin(3u) = 3\sin u - 4\sin^3 u$ gives the final expression for $T_0(a)$.

Proposition 3.4. The integral $T_0(a)$ is given by

$$\begin{aligned} T_0(a) &= \frac{\pi}{2^{9/4}\sqrt{1-a^2}} \left[\sqrt{2} - \sqrt{1+a} \right]^{1/2} \left[1 + \sqrt{2}\sqrt{1+a} \right] \\ &= T_4(a) \left[1 + \sqrt{2}\sqrt{1+a} \right]. \end{aligned} \tag{3.15}$$

The values of the integrals $T_k(a)$ produce the value of $I_2(p,q)$.

Theorem 3.5. *Let* p, $q > 0$. *Then the integral*

$$I_2(p,q) = \int_{-\infty}^{\infty} \frac{x^4 - p + qx^2}{(x^4-p)^2 + (qx^2)^2}\, dx \tag{3.16}$$

vanishes.

Proof. Lemma 2.2 is now used to evaluate $I_2(p,q)$ with $a = q^2/2p^2 - 1$. The factor

$$1 + \sqrt{2}\sqrt{1+a} = 1 + q/\sqrt{p} \tag{3.17}$$

gives

$$\begin{aligned} I_2(p,q) &= 2p^{-3/4}T_4(a) + 2qp^{-5/4}T_2(a) - 2p^{-3/4}T_0(a) \\ &= 2T_4(a)\left[p^{-3/4} + qp^{-5/4} - p^{-3/4}\left(1 + q/\sqrt{p}\right)\right] \\ &= 0, \end{aligned} \tag{3.18}$$

as claimed. □

The values of $T_k(a)$ gives a generalization of the vanishing of $I_2(p,q)$.

Theorem 3.6. *Assume* $(A\sqrt{p} + B)p + (\sqrt{p} + q)C = 0$. *Then*

$$\int_{-\infty}^{\infty} \frac{Ax^4 + Bx^2 + C}{(x^4-p)^2 + (qx^2)^2}\, dx = 0. \tag{3.19}$$

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