2001h:33029 33E20

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A rational Landen transformation. The case of degree six. (English. English summary)

Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), 83–91, *Contemp. Math.*, 251, *Amer. Math. Soc., Providence, RI*, 2000.

Let $r(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$ for $a, b > 0$, $\theta \in (0, \pi/2)$, and let $I(a,b) = \int_0^{\pi/2} d\theta/r(\theta)$. If $a_0 = a$, $b_0 = b$, $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$, $n = 0, 1, 2, 3, \cdots$, then both sequences converge quadratically to the common limit, the arithmetic-geometric mean $\mathrm{AGM}(a,b)$ of a and b. In the 1770's J. Landen proved that $I(a,b) = I(a_1,b_1) = I(a_n,b_n)$ for all $n = 2, 3, \cdots$. Then clearly $I(a,b) = \pi/(2\mathrm{AGM}(a,b))$. Thus the computation of the integral $I(a,b)$ is reduced to the much simpler task of the computation of the $\mathrm{AGM}(a,b)$. This fact had far-reaching applications and generalizations in the works of J. Lagrange, K. F. Gauss, A. M. Legendre, C. G. J. Jacobi: it turned out that a wide class of special functions, among them elliptic integrals and some other transcendental functions, can be computed in this way. For a monograph on this topic, see [J. Wimp, *Computation with recurrence relations*, Pitman, Boston, MA, 1984; MR 85f:65001]. In particular, Carlson and Gustafson [e.g. B. C. Carlson, *Special functions of applied mathematics*, Academic Press, New York, 1977; MR **58** #28707; B. C. Carlson and J. L. Gustafson, SIAM J. Math. Anal. **25** (1994), no. 2, 288–303; MR 95b:33056] applied the Landen transformation for the numerical computation of elliptic integrals and also introduced new standard forms for these integrals well suited for numerical implementation (discussed in [W. Press and S. Teukolsky, "Elliptic integrals", Comput. Phys., **1990**, Jan/Feb, 92–96; per revr.; see also T. Morita, Numer. Math. **82** , no. 4, 677–688; MR 2000i:33030]). These new standard forms involve certain symmetric algebraic integrals with limits 0 and ∞.

In the paper under review the authors prove that integrals $\int_0^\infty (p(x)/q(x))dx$, where p and q are even polynomials of orders 4 and 6, respectively, are invariant under certain Landen-type recursive substitutions involving the coefficients of p and q. The coefficients of p and q are assumed to be positive. These sequences are shown to converge (in numerical calculations the convergence appears to be quadratic). The authors also announce that they have similar algorithms for higher order polynomials. It seems likely that this discovery will have many applications. The proofs rely on classical analysis.

{Reviewer's remark: Some historical remarks on elliptic integrals and the arithmetic geometric mean can be found in the monograph [G. D. Anderson, M. K. Vamanamurthy and M. K. Vuorinen, *Conformal invariants, inequalities, and quasiconformal maps*, Wiley, New York, 1997; MR 98h:30033 (pp. 68–69, 79)].}

{For the entire collection see 2001b:00015}

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