A BY-PRODUCT OF AN INTEGRAL EVALUATION

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Abstract. Schur polynomials are used to effectively prove certain evaluations of definite integrals.

The authors have presented in [2] a collection of proofs of the identity

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2} \cdot \frac{P_m(a)}{[2(a+1)]^{m+1/2}}, \tag{1}$$

where the hypergeometric form of the polynomial $P_m(a)$ is given by

$$P_m(a) = 2^{-2m} \binom{2m}{m} \, {}_2F_1 \left(\begin{matrix} -m \quad m+1 \\ -m+\frac{1}{2} \end{matrix} ; \frac{a+1}{2} \right). \tag{2}$$

This can be expressed as

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l \tag{3}$$

with

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}. \tag{4}$$

The coefficiens $d_{l,m}$ have remarkable properties, described in [7]. After [2] several other proofs have appeared: the authors, in joint work with C. Vignat [3], produced one using a method of Schwinger developed to deal with integrals arising in Feynman diagrams, C. Koutschan [6] gave an automatic proof and M. Apagodou [4] proved it in the form

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{1}{4m!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} 2^j \left(\tfrac{j}{2} - \tfrac{3}{4} \right)! \, \left(m + \tfrac{j}{2} - \tfrac{1}{4} \right)! \, a^j \tag{5}$$

proved via the Almkvist-Zeilberger algorithm [5]. Apagodu also presents the generalization

$$\int_0^\infty \frac{dx}{(x^{2k}+2ax^k+1)^{m+1}} = \frac{1}{2k} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j}{2} - \frac{1}{2k}\right) \, \Gamma\left(\frac{j}{2} + m + 1 - \frac{1}{2k}\right)}{j! \, m!} (-2a)^j. \tag{6}$$

The goal of this short note is to provide a new proof of this extension based on the results of [1]. The notation is reviewed here for the convenience of the reader. A vector $\mu = (\mu_1, \mu_2, \ldots)$ means a finite sequence of real numbers. μ is further called a partition if $\mu_1 \geq \mu_2 \geq \ldots$ and all the parts μ_j are positive integers. Write $\mathbf{1}^n$ for the partition with n ones, and with $\lambda(n)$ denote the partition

$$\lambda(n) = (n-1, n-2, \ldots, 1).$$

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Vectors and partitions may be added componentwise. In case they are of different length, the shorter one is padded with zeroes. For instance, one has $\lambda(n+1) = \lambda(n) + \mathbf{1}^n$. Likewise, vectors and partitions may be multiplied by scalars. In particular, $a \cdot \mathbf{1}^n$ is the partition with n a's.

Fix n and consider $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. Let $\mu = (\mu_1, \mu_2, \ldots)$ be a vector of length at most n. The corresponding alternant a_μ is defined as the determinant

$$a_\mu(\boldsymbol{q}) = \left| q_i^{\mu_j} \right|_{1 \le i,j \le n}.$$

Again, μ is padded with zeroes if necessary. Note that the alternant $a_{\lambda(n)}$ is the classical Vandermonde determinant

$$a_{\lambda(n)}(\boldsymbol{q}) = \left| q_i^{n-j} \right|_{1 \le i,j \le n} = \prod_{1 \le i < j \le n} (q_i - q_j).$$

The Schur function s_μ associated with the vector μ can now be defined as

$$s_\mu(\boldsymbol{q}) = \frac{a_{\mu+\lambda(n)}(\boldsymbol{q})}{a_{\lambda(n)}(\boldsymbol{q})}.$$

If μ is a partition with integer entries this is a symmetric polynomial. Indeed, as μ ranges over the partitions of m, of length at most n, the Schur functions $s_\mu(\boldsymbol{q})$ form a basis of the homogeneous symmetric polynomials in $\boldsymbol{q}$ of degree m.

The evaluation of integrals with rational integrands is provided by the next result established in [1]:

Theorem. Let $\mathbf{q} = (q_1, \cdots, q_n)$ with $\operatorname{Re} q_k > 0$. Further, let $\alpha > 0$ and $0 < \beta < \alpha n$ such that β is not an integral multiple of α. Then

$$\int_0^\infty \frac{x^{\beta-1}\, dx}{\prod_{j=1}^n (x^\alpha + q_j^\alpha)} = \frac{\pi/\alpha}{\sin(\pi\beta/\alpha)} \frac{s_\lambda(\mathbf{q})}{s_\mu(\mathbf{q})},$$

where

$$\lambda = (\alpha - 1)\lambda(n) - \beta \cdot \mathbf{1}^{n-1}, \text{ and } \mu = (\alpha-1)\lambda(n+1) - (\beta - 1) \cdot \mathbf{1}^n. \tag{7}$$

To prove (6) choose w such that $2a = w^k + w^{-k}$, so that the integral is rewritten in the form

$$\int_0^\infty \frac{dx}{(x^{2k} + 2ax^k + 1)^{m+1}} = \int_0^\infty \frac{dx}{(x^k + w^k)^{m+1}\,(x^k + w^{-k})^{m+1}}. \tag{8}$$

Now apply the Theorem with $q_1 = \cdots = q_{m+1} = w$, $q_{m+2} = \cdots = q_{2m+2} = w^{-1}$ and

$$\lambda = (k-1)\lambda(2m+2) - \mathbf{1}^{2m+1}, \;\; \mu = (k-1)\lambda(2m+2). \tag{9}$$

This produces the result.

The above two theorems have different types of outputs; that is Apagodu's result is an infinite series while the Theorem stated here gives a rational function of its parameters. Combining the two expressions gives the following interesting identity:

Corollary. Preserve the notation as above. Then

$$\frac{1}{m!} \sum_{j=0}^\infty \frac{(-1)^j}{j!} \Gamma\left(\tfrac{j}{2} + \tfrac{1}{2k}\right) \Gamma\left(\tfrac{j}{2} + m + 1 - \tfrac{1}{2k}\right) (w^k + w^{-k})^j = \frac{2\pi}{\sin(\pi/k)} \frac{s_\lambda(\mathbf{q})}{s_\mu(\mathbf{q})}.$$

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