DEFINITE INTEGRALS BY THE METHOD OF BRACKETS. PART 1

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Abstract. A new heuristic method for the evaluation of definite integrals is presented. This *method of brackets* has its origin in methods developed for the evaluation of Feynman diagrams. We describe the operational rules and illustrate the method with several examples. The method of brackets reduces the evaluation of a large class of definite integrals to the solution of a linear system of equations.

1. Introduction

The problem of analytic evaluations of definite integrals has been of interest to scientists since Integral Calculus was developed. The central question can be stated vaguely as follows:

given a class of functions $\mathfrak{F}$ *and an interval* $[a,b] \subset \mathbb{R}$, *express the integral of* $f \in \mathfrak{F}$

$$I = \int_a^b f(x)\, dx,$$

in terms of the special values of functions in an enlarged class $\mathfrak{G}$.

For instance, by elementary arguments it is possible to show that if $\mathfrak{F}$ is the class of rational functions, then the enlarged class $\mathfrak{G}$ can be obtained by including logarithms and inverse trigonometric functions. G. Cherry has discussed in [17], [18] and [19] extensions of this classical paradigm. The following results illustrate the idea:

$$\int \frac{x^3\, dx}{\ln(x^2-1)} = \frac{1}{2}\mathrm{li}(x^4-2x^2+1) + \frac{1}{2}\mathrm{li}(x^2-1), \tag{1.1}$$

but

$$\int \frac{x^2\, dx}{\ln(x^2-1)} \tag{1.2}$$

can not be written in terms of elementary functions and the logarithmic integral

$$\mathrm{li}(x) := \int \frac{dx}{\ln x} \tag{1.3}$$

that appears in (1.1). The reader will find in [16] the complete theory behind integration in terms of elementary functions.

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Methods for the evaluation of definite integrals were also developed since the early stages of Integral Calculus. Unfortunately, these are mostly ad-hoc procedures and a general theory needs to be developed. The method proposed in this paper represents a new addition to these procedures.

The evaluations of definite integrals have been collected in tables of integrals. The earliest volume available to the authors is [8], compiled by Bierens de Haan who also presented in [9] a survey of the methods employed in the verification of the entries. These tables form the main source for the popular volume by I. S. Gradshteyn and I. M. Ryzhik [39]. Naturally any document containing a large number of entries, such as the table [39] or the encyclopedic treatise [51], is likely to contain errors. For instance, the appealing integral

$$I = \int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}, \tag{1.4}$$

with

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)^2}, \tag{1.5}$$

that appears as entry 3.248.5 in [38], the sixth edition of the mentioned table, is incorrect. The numerical value of I is 0.666377 and the right hand side of (1.4) is about 0.641275. The table [39] is in the process of being revised. After we informed the editors of the error in 3.248.5, it was taken out. There is no entry 3.248.5 in [39]. At the present time, we are unable to evaluate the integral I.

The revision of integral tables is nothing new. C. F. Lindman [47] compiled a long list of errors from the table by Bierens de Haan [10]. The editors of [39] maintain the webpage

`http://www.mathtable.com/gr/`

where the corrections to the table are stored. The second author has began in [2, 3, 55, 56, 57, 58, 59, 60] a systematic verification of the entries in [39]. It is in this task that the method proposed in the present article becomes a valuable tool.

The *method of brackets* presented here, even though it is heuristic and still lacking a rigorous description, is quite powerful. Moreover, it is quite simple to work with: the evaluation of a definite integral is reduced to solving a linear system of equations. Many of the entries of [39] can be derived using this method. The basic idea behind it is the assignement of a *bracket* $\langle a \rangle$ to any parameter a. This is a symbol associated to the divergent integral

$$\int_0^\infty x^{a-1}\, dx. \tag{1.6}$$

The formal rules for operating with these brackets are described in Section 3 and their justification is work-in-progress, we expect to report in the near future. The rest of the paper provides a list of examples illustrating the new technique.

Given a formal sum

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} \tag{1.7}$$

we associate to the integral of f a *bracket series* written as

$$\int_0^\infty f(x)\,dx \doteq \sum_n a_n \langle \alpha n + \beta \rangle, \tag{1.8}$$

to keep in mind the formality of the method described in this paper. Convergence issues are ignored at the present time. Moreover only integrals over the half-line $[0,\infty)$ will be considered.

Note. In the evaluation of these formal sums, the index $n \in \mathbb{N}$ will be replaced by a number n^* defined by the vanishing of the bracket. Observe that it is possible that $n^* \in \mathbb{C}$. For book-keeping purposes, specially in cases with many indices, we write $\sum_n$ instead of the usual $\sum_{n=0}^{\infty}$. After the brackets are eliminated, those indices that remain recover their original nature.

The rules of operation described below assigns a *value* to the bracket series. The claim is that for a large class of integrands, including all the examples described here, this formal procedure provides the actual value of the integral. Many of the examples involve the hypergeometric function

$${}_pF_q(z) := \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n \cdots (a_p)_n}{(b_1)_n\,(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.9}$$

This series converges absolutely for all $z \in \mathbb{C}$ if $p \leq q$ and for $|z| < 1$ if $p = q+1$. The series diverges for all $z \neq 0$ if $p > q+1$ unless the series terminates. The special case $p = q+1$ is of great interest. In this special case and with $|z| = 1$, the series

$${}_{q+1}F_q(a_1, \cdots, a_{q+1}; b_1, \cdots, b_q; z) \tag{1.10}$$

converges absolutely if $\text{Re}\left(\sum b_j - \sum a_j\right) > 0$. The series converges conditionally if $z = e^{i\theta} \neq 1$ and $0 \geq \text{Re}\left(\sum b_j - \sum a_j\right) > -1$ and the series diverges if $\text{Re}\left(\sum b_j - \sum a_j\right) \leq -1$.

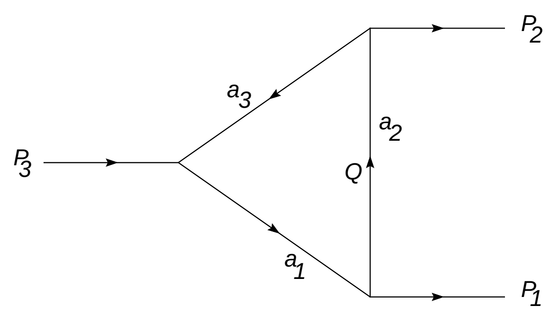

FIGURE 1. The triangle

The last section of this paper employs the method of brackets to evaluate certain definite integrals associated to a Feynman diagram. From the present point of view, a *Feynman diagram* is simply a generic graph G that contains $E+1$ *external* lines and N *internal* lines or *propagators* and L *loops*. All but one of these external lines are assumed independent. The internal and external lines represent particles that transfer momentum among the vertices of the diagram. Each of these particles carries a *mass* $m_i \geq 0$ for $i = 1, \cdots, N$. The vertices represent the interaction of these particles and conservation of momentum at each vertex assigns the momentum corresponding to the internal lines. A Feynman diagram has an associated integral given by the *parametrization* of the diagram. For example, in Figure 1 we have three external lines represented by the momentum P_1, P_2, P_3 and one loop. The parameters a_i are arbitrary real numbers. The integral associated to this diagram is given by

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}} \\
&\times \exp(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) \exp\left(- \frac{C_{11} P_1^2 + 2 C_{12} P_1 \cdot P_2 + C_{22} P_2^2}{x_1+x_2+x_3} \right) \mathbf{dx},
\end{aligned}
$$

where $\mathbf{dx} := dx_1\, dx_2\, dx_3$. The evaluation of this integral in terms of the variables $P_i \in \mathbb{R}^4$, $m_i \in \mathbb{R}$ and $a_i \in \mathbb{R}$ is the *solution of the Feynman diagram*. The functions C_{ij} are polynomials described in Section 13.

The method of brackets presented here has its origin in quantum field theory (QFT). A version of the method of brackets was developed to address one of the fundamental questions in QFT: the evaluation of loop integrals arising from Feynman diagrams. As described above, these are directed graphs depicting the interaction of particles in the model. The loop integrals depend on the dimension D and one of the (many) intrinsic difficulties is related to their divergence at $D = 4$, the dimension of the physical world. A correction to this problem is obtained by taking $D = 4 - 2\epsilon$ and considering a Laurent expansion in powers of ϵ. This is called the *dimensional regularization* [11] and the parameter ϵ is the *dimensional regulator*.

The method of brackets discussed in this paper is based on previous results by I. G. Halliday, R. M. Ricotta and G. V. Dunne [26], [27] and [40]. The work involves an analytic extension of D to negative values, so the method was labelled NDIM (negative dimensional integration method). The validity of this continuation is based on the observation that the objects associated to a Feynman diagram (loop integrals as well as the functions linked to propagators) are analytic in the dimension D. A. Suzuki and A. Schmidt employed this technique to the evaluation of diagrams with two loops [68], [69]; three loops [71]; tensorial integrals [70] and massive integrals with one loop [66], [72], [67]. An extensive use of this method as well as an analysis of the solutions was provided by C. Anastasiou and E. Glover in [5] and [6]. The conclusion of these studies is that the NDIM method is inadequate to the evaluation of Feynman diagrams with an arbitrary number of loops. The proposed solutions involve hypergeometric functions with a large number of parameters. By establishing new procedural rules I. Gonzalez and I.Schmidt [36] and [37] have concluded that the modification of the previous procedures permits now the evaluation of more complex Feynman diagrams. One of the results of [36], [37] is the justification of the method of brackets in terms of arguments derived from fractional calculus. The authors have given NDIM the alternative name IBFE (Integration by Fractional Expansion).

From the mathematical point of view, the NDIM method has been used to provide evaluation of a very limited type of integrals [64], [65]. The examples presented in this paper show great flexibility of the method of brackets. A systematic study of integrals arising from Feynman diagrams is in preparation.

2. A detour on definite integrals

The literature contains a large variety of techniques for the evaluation of definite integrals. Elementary techniques are surveyed in classical texts such as [28] and [32]. The text [7] contains an excellent collection of problems solved by the method of contour integration. The reader will find in [13] a discussion of several elementary analytic methods involved in the evaluation of integrals.

It is hard to predict the type of techniques required for the evaluation of a specific definite integral. For instance, [73] contains a detailed account of the proof of

$$\int_{\pi/4}^{\pi/2} \ln \ln \tan x \, dx = \frac{\pi}{2} \ln \left(\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right), \tag{2.1}$$

that appears as formula 4.229.7 in [39]. This particular example involves the use of L-functions

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{2.2}$$

where χ is a character. This is a generalization of the Riemann zeta function $\zeta(s)$ (corresponding to $\chi \equiv 1$). Vardi's technique has been extended in [52] to provide a systematic study of integrals of the form

$$I_Q = \int_0^1 Q(x) \ln \ln 1/x \, dx, \tag{2.3}$$

that gives evaluations such as

$$\int_0^1 \frac{x \ln \ln 1/x \, dx}{(x+1)^2} = \frac{1}{2}(-\ln^2 2 + \gamma - \ln \pi + \ln 2), \tag{2.4}$$

$$\int_0^\infty \ln x \ln \tanh x \, dx = \frac{\gamma \pi^2}{8} - \frac{3}{4}\zeta'(2) + \frac{\pi^2 \ln 2}{12}. \tag{2.5}$$

Here $\gamma = -\Gamma'(1)$ is Euler's constant.

A second class of examples appeared during the evaluation of definite integrals related to the Hurwitz zeta function

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}. \tag{2.6}$$

In [29] the authors found an evaluation that generalizes the classical integral

$$L_1 := \int_0^1 \ln \Gamma(q) \, dq = \ln \sqrt{2\pi}, \tag{2.7}$$

namely

$$L_2 := \int_0^1 \ln^2 \Gamma(q) \, dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma L_1}{3} + \frac{4}{3} L_1^2 - \frac{A \zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2}, \tag{2.8}$$

where L_1 is in (2.7) and $A = \gamma + \ln 2\pi$. The natural next step, namely the evaluation of

$$L_3 := \int_0^1 \ln^3 \Gamma(q) \, dq, \tag{2.9}$$

remains to be completed. In [30] and [31] the reader will find a relation between L_3 and the Tornheim sums $T(m,k,n)$, for $m,k,n \in \mathbb{N}$. These sums are defined by

$$T(a,b,c) := \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{j_1^a \, j_2^b \, (j_1+j_2)^c}. \tag{2.10}$$

The special case

$$T(n,0,m) = \sum_{j_2>j_1} \frac{1}{j_1^m j_2^n}, \tag{2.11}$$

corresponds to the multiple zeta value (MZV) $\zeta(n,m)$ of depth 2. The MZV is given by

$$\zeta(n_1,n_2,\cdots,n_r) := \sum_{j_1>j_2>\cdots>j_r} \frac{1}{j_1^{n_1} \, j_2^{n_2} \cdots j_r^{n_r}}, \tag{2.12}$$

where the parameter r is called the depth of the sum. These series were initially considered by Euler and have recently appeared in many different places. The reader will find in [44] a description of how these sums are connected to knots and Feynman diagrams. These diagrams are a very rich source of interesting integrals. The last section of this paper is dedicated to the evaluation of some of these integrals by the method of brackets.

The computation of hyperbolic volumes of 3-manifolds provides a different source of interesting integrals. Mostow's rigidity theorem states that a finite volume 3-manifold has a unique hyperbolic structure. In particular its volume is a topological invariant. An interesting class of such 3-manifolds is provided by hyperbolic knots or link complement in S^3. The reader will find information about this topic in the articles by C. Adams and J. Weeks in [53]. It turns out that their hyperbolic structure can be given in terms of hyperbolic tetrahedra [1]. Milnor [54] describes how the volume of these tetrahedra can be expressed in terms of the Clausen function

$$\mathrm{Cl}_2(\theta) := -\int_0^{\theta} \log|2\sin\frac{u}{2}|\,du. \tag{2.13}$$

The reader will find in [49] a discussion on arithmetic properties of 3-manifolds. In particular, Chapter 11 has up to date information on their volumes.

Zagier [74] provided an arithmetic version of these computations in his study of the Dedekind zeta function

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}, \tag{2.14}$$

for a number field K that is not totally real. Here $N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$ and the sum runs over all the nonzero integral ideals of K. In the case of totally real number fields a classical result of Siegel shows that $\zeta_K(2m)$ is a rational multiple of $\pi^{2nm}/\sqrt{D}$, where n and D denote the degree and the discriminant of K, respectively. Little is known in the non-totally real situation. Zagier [74] proves that $\zeta_K(2)$ is given by a finite sum of values of

$$A(x) := \int_0^x \frac{1}{1+t^2} \log\frac{4}{1+t^2}\,dt. \tag{2.15}$$

The function A can be written as

$$A(x) = \mathrm{Cl}_2(2\cot^{-1} x). \tag{2.16}$$

Morever, he conjectured that $\zeta_k(2m)$ can be given in terms of

$$A_m(x) := \frac{2^{2m-1}}{(2m-1)!}\int_0^\infty \frac{t^{2m-1}\,dt}{x\sinh^2 t + x^{-1}\cosh^2 t}. \tag{2.17}$$

The conjecture is established in the special case where K is an abelian extension of $\mathbb{Q}$. The example $K = \mathbb{Q}(\sqrt{-7})$ yields

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{\pi^2}{3\sqrt{7}}\left(A\left(\cot\tfrac{\pi}{7}\right) + A\left(\cot\tfrac{2\pi}{7}\right) + A\left(\cot\tfrac{4\pi}{7}\right)\right) \tag{2.18}$$

and also

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{2\pi^2}{7\sqrt{7}}\left(2A(\sqrt{7}) + A(\sqrt{7}+2\sqrt{3}) + A(\sqrt{7}-2\sqrt{3})\right), \tag{2.19}$$

leading to the new Clausen identity

$$A\left(\cot\tfrac{\pi}{7}\right) + A\left(\cot\tfrac{2\pi}{7}\right) + A\left(\cot\tfrac{4\pi}{7}\right) = \tfrac{6}{7}\left(2A(\sqrt{7}) + A(\sqrt{7}+2\sqrt{3}) + A(\sqrt{7}-2\sqrt{3})\right).$$

Zagier stated in 1986 that there was no direct proof of this identity. To this day this has elluded considerable effort. The famous text of Lewin [46] has such parametric identities but it misses this one. R. Crandall [23] has worked out a theory in which certain Clausen identities are seen to be equivalent to the vanishing of log-rational integrals.

J. Borwein and D. Broadhurst [15] identified a large number of finite volume hyperbolic 3-manifolds whose volumes are expressed in the form

$$\frac{a}{b}\mathrm{vol}(\mathfrak{M}) = \frac{(-D)^{3/2}}{(2\pi)^{2n-4}}\frac{\zeta_K(2)}{2\zeta(2)}. \tag{2.20}$$

Here K is a field associated to the manifold $\mathfrak{M}$ (the so-called invariant trace field) and n and D are the degree and discriminant of K, respectively. The authors offer a systematic numerical study of the rational numbers $\dfrac{a}{b}$. The identity of Zagier described above yields the remarkable identity

$$\int_{\pi/3}^{\pi/2} \ln\left|\frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}}\right| dt = A(\sqrt{7}) + \frac{1}{2}A(\sqrt{7}+2\sqrt{3}) + \frac{1}{2}A(\sqrt{7}-2\sqrt{3}).$$

This example corresponds to the link 6_1^3 with discriminant $D = -7$. Zagier's result gives $\dfrac{a}{b} = 2$ in (2.20).

Coffey [20], [21] has studied the integral above, that also appears in the reduction of a multidimensional Feynman integral [48]. One of his goals is to produce a more direct proof of Zagier remarkable identity as well as the many others that have been numerically verified in [15].

The subject of evaluation of definite integrals has a rich history. We expect that the method of brackets developed in this paper will expand the class of integrals that can be expressed in analytic form.

3. The method of brackets

The method of brackets discussed in this paper is based on the assignment of a *bracket* $\langle a \rangle$ the parameter a. In the examples presented here $a \in \mathbb{R}$, but the extension to $a \in \mathbb{C}$ is direct. The formal rules for operating with these brackets are described next.

Definition 3.1. Let f be a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}. \tag{3.1}$$

The symbol

$$\int_0^\infty f(x)\, dx \doteq \sum_n a_n \langle \alpha n + \beta \rangle \tag{3.2}$$

represents a *bracket series* assignement to the integral on the left. Rule 3.2 describes how to evaluate this series.

Definition 3.2. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)} \tag{3.3}$$

will be called the *indicator of* n.

The symbol ϕ_n gives a simpler form for the bracket series associated to an integral. For example,

$$\int_0^\infty x^{a-1} e^{-x}\, dx \doteq \sum_n \phi_n \langle n + a \rangle. \tag{3.4}$$

The integral is the gamma function $\Gamma(a)$ and the right-hand side its bracket expansion.

Rule 3.1. For $\alpha \in \mathbb{C}$, the expression

$$(a_1 + a_2 + \cdots + a_r)^\alpha \tag{3.5}$$

is assigned the bracket series

$$\sum_{m_1, \cdots, m_r} \phi_{1,2,\cdots,r}\, a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + m_1 + \cdots + m_r \rangle}{\Gamma(-\alpha)}, \tag{3.6}$$

where $\phi_{1,2,\cdots,r}$ is a short-hand notation for the product $\phi_{m_1}\phi_{m_2}\cdots\phi_{m_r}$.

Rule 3.2. The series of brackets

$$\sum_n \phi_n f(n) \langle an + b \rangle \tag{3.7}$$

is given the *value*

$$\frac{1}{a} f(n^*) \Gamma(-n^*) \tag{3.8}$$

where n^* solves the equation $an + b = 0$.

Rule 3.3. A two-dimensional series of brackets

$$\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1 + a_{12}n_2 + c_1\rangle\langle a_{21}n_1 + a_{22}n_2 + c_2\rangle \tag{3.9}$$

is assigned the *value*

$$\frac{1}{|a_{11}a_{22} - a_{12}a_{21}|} f(n_1^*, n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*) \tag{3.10}$$

where (n_1^*, n_2^*) is the unique solution to the linear system

$$\begin{aligned} a_{11}n_1 + a_{12}n_2 + c_1 &= 0, \\ a_{21}n_1 + a_{22}n_2 + c_2 &= 0, \end{aligned} \tag{3.11}$$

obtained by the vanishing of the expressions in the brackets. A similar rule applies to higher dimensional series, that is,

$$\sum_{n_1} \cdots \sum_{n_r} \phi_{1,\cdots,r} f(n_1,\cdots,n_r)\langle a_{11}n_1 + \cdots a_{1r}n_r + c_1\rangle \cdots \langle a_{r1}n_1 + \cdots a_{rr}n_r + c_r\rangle$$

is assigned the value

$$\frac{1}{|\det(A)|} f(n_1^*,\cdots,n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*), \tag{3.12}$$

where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets. The value is not defined if the matrix A is not invertible.

Rule 3.4. In the case where the assignment leaves free parameters, any divergent series in these parameters is discarded. In case several choices of free parameters are available, the series that converge in a common region are added to contribute to the integral.

A typical place to apply Rule 3.4 is where the hypergeometric functions ${}_pF_q$, with $p = q + 1$, appear. In this case the convergence of the series imposes restrictions on the internal parameters of the problem. Example 13.2, dealing with a Feynman diagram with a *bubble*, illustrates the latter part of this rule.

Note. To motivate Rule 3.1 start with the identity

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-Ax}\, dx, \tag{3.13}$$

and apply it to $A = a_1 + \cdots + a_r$ to produce

$$\begin{aligned} (a_1 + \cdots + a_r)^\alpha &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty x^{-\alpha-1} \exp\left[-(a_1 + \cdots + a_r)x\right]\, dx \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty x^{-\alpha-1} e^{-a_1x} \cdots e^{-a_rx}\, dx. \end{aligned}$$

Expanding the exponentials we obtain

$$(a_1 + \cdots + a_r)^\alpha \doteq \frac{1}{\Gamma(-\alpha)} \sum_{m_1} \cdots \sum_{m_r} \phi_{1,\cdots,r} a_1^{m_1} \cdots a_r^{m_r} \int_0^\infty x^{-\alpha+m_1+\cdots+m_r-1}\, dx$$

and thus

$$(a_1 + \cdots + a_r)^\alpha \doteq \sum_{m_1} \cdots \sum_{m_r} \phi_{1,\cdots,r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + m_1 + \cdots + m_r\rangle}{\Gamma(-\alpha)}. \tag{3.14}$$

This is Rule 3.1.

4. Wallis' formula

The evaluation

$$J_{2,m} := \int_0^\infty \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m} \tag{4.1}$$

is historically one of the earliest closed-form expressions for a definite integral. The change of variables $x = \tan\theta$ converts it into its trigonometric form

$$J_{2,m} := \int_0^{\pi/2} \cos^{2m}\theta\, d\theta = \frac{\pi}{2^{2m+1}}\binom{2m}{m}. \tag{4.2}$$

An elementary argument shows that $J_{2,m}$ satisfies the recurrence

$$J_{2,m} = \frac{2m-1}{2m} J_{2,m-1} \tag{4.3}$$

and then one simply checks that the right hand side of (4.2) satisfies the same recurrence with matching initial conditions. A second elementary proof of (4.1) is presented in [14]: using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ one obtains the recurrence

$$J_{2,m} = 2^{-m}\sum_{i=0}^{\lfloor m/2\rfloor}\binom{m}{2i} J_{2,i}, \tag{4.4}$$

and the inductive proof follows from the identity

$$\sum_{i=0}^{\lfloor m/2\rfloor} 2^{-2i}\binom{m}{2i}\binom{2i}{i} = 2^{-m}\binom{2m}{m}. \tag{4.5}$$

This can be established using automatic methods developed by H. Wilf and D. Zeilberger in [61].

The proof of Wallis' formula by the method of brackets starts with the expansion of the integrand as

$$(1+x^2)^{-m-1} \doteq \sum_{n_1}\sum_{n_2} \phi_{1,2}\frac{\langle m+1+n_1+n_2\rangle}{\Gamma(m+1)} x^{2n_2}. \tag{4.6}$$

The corresponding integral $J_{2,m}$ is assigned the bracket series

$$J_{2,m} \doteq \sum_{n_1}\sum_{n_2}\phi_{1,2}\frac{1}{\Gamma(m+1)}\langle m+1+n_1+n_2\rangle\langle 2n_2+1\rangle. \tag{4.7}$$

Rule 3.2 then shows that

$$J_{2,m} = \frac{1}{2}\frac{\Gamma(-n_1^*)\Gamma(-n_2^*)}{\Gamma(m+1)}, \tag{4.8}$$

where (n_1^*, n_2^*) is the solution to the linear system of equations

$$\begin{aligned} m+1+n_1+n_2 &= 0, \\ 2n_2+1 &= 0. \end{aligned} \tag{4.9}$$

Therefore $n_1^* = -(m+\frac{1}{2})$ and $n_2^* = -\frac{1}{2}$. We conclude that

$$J_{2,m} = \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(m)}. \tag{4.10}$$

This is exactly the right-hand side of (4.1).

5. The integral representation of the gamma function

The exponential in the integral

$$I = \int_0^\infty x^{a-1} e^{-x}\, dx \tag{5.1}$$

is expanded in power series to obtain

$$x^{a-1}e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+a-1}}{n!} = \sum_{n=0}^{\infty} \phi_n x^{n+a-1}. \tag{5.2}$$

Therefore, the integral (5.1) gets assigned the bracket series

$$I \doteq \sum_n \phi_n \langle a+n \rangle. \tag{5.3}$$

Rule 3.2 assigns the value $\Gamma(a)$ to (5.3). This is precisely the value of the integral:

$$\int_0^\infty x^{a-1}e^{-x}\, dx = \Gamma(a). \tag{5.4}$$

Rule 3.2 was developed from this example.

6. A Fresnel integral

In this section we verify the evaluation of Fresnel integral

$$\int_0^\infty \sin(ax^2)\, dx = \frac{\pi}{2\sqrt{2a}}. \tag{6.1}$$

The reader will find in [7] the standard evaluation using contour integrals and other elementary proofs in [33] and [45].

In order to apply the method of brackets, use the hypergeometric representation

$$\frac{\sin z}{z} = {}_0F_1\left[-; \tfrac{3}{2}; -\tfrac{z^2}{4} \right],$$

that can be written as

$$\sin z = \sum_{n=0}^{\infty} \phi_n \frac{z^{2n+1}}{\left(\frac{3}{2}\right)_n 4^n}. \tag{6.2}$$

Therefore

$$\int_0^\infty \sin(ax^2)\, dx \doteq \sum_n \phi_n \frac{a^{2n+1}}{\left(\frac{3}{2}\right)_n 4^n} \langle 4n+3 \rangle. \tag{6.3}$$

According to Rule 3.2, the assignment of the right-hand side is obtained by evaluating the function

$$g(n) := \frac{a^{2n+1}}{\left(\frac{3}{2}\right)_n 4^n} \tag{6.4}$$

at the solution of $4n^*+3=0$. Therefore the integral (6.1) has the value

$$\tfrac{1}{4}g(-\tfrac{3}{4}) = \frac{a^{-1/2}\Gamma(\frac{3}{4})}{\left(\frac{3}{2}\right)_{-3/4}4^{1/4}},$$

where the factor $\frac{1}{4}$ comes from the term $4n+3$ in the bracket. Using $(a)_m = \Gamma(a+m)/\Gamma(a)$, we obtain

$$\left(\frac{3}{2}\right)_{-3/4} = \frac{2\Gamma(\frac{3}{4})}{\sqrt{\pi}}. \tag{6.5}$$

We conclude that the assigned value is $\pi/2\sqrt{2a}$. As expected, this is consistent with (6.1).

The method also give the evaluation of

$$I = \int_0^\infty x^{b-1}\sin(ax^c)\,dx. \tag{6.6}$$

The change of variables $t=x^c$ transforms (6.6) into

$$I = \frac{1}{c}\int_0^\infty t^{b/c-1}\sin(at)\,dt, \tag{6.7}$$

and this is formula 3.761.4 in [39] with value

$$\int_0^\infty x^{b-1}\sin(ax^c)\,dx = \frac{\Gamma(b/c)}{ca^{b/c}}\sin\left(\frac{\pi b}{2c}\right). \tag{6.8}$$

To verify this result by the method of brackets, start with the expansion

$$x^{b-1}\sin(ax^c) = \sum_{n=0}^\infty \phi_n \frac{a^{2n+1}}{\left(\frac{3}{2}\right)_n 2^{2n}}x^{2nc+c+b-1} \tag{6.9}$$

and associate to it the bracket series

$$\int_0^\infty x^{b-1}\sin(ax^c)\,dx \doteq \sum_n \phi_n \frac{a^{2n+1}}{\left(\frac{3}{2}\right)_n 2^{2n}}\langle 2nc+c+b\rangle. \tag{6.10}$$

Apply Rule 3.2 to obtain

$$I = \frac{1}{2c}\frac{a^{2n_*+1}}{\left(\frac{3}{2}\right)_{n^*}2^{2n^*}}\Gamma(-n^*), \tag{6.11}$$

where n^* solve $2nc+b+c=0$; that is, $n^*=-1/2-b/2c$. Then (6.11) yields

$$I = \frac{\Gamma(\frac{3}{2})2^{b/c}}{ca^{b/c}\Gamma(1-\frac{b}{2c})}\Gamma\left(\tfrac{1}{2}+\tfrac{b}{2c}\right). \tag{6.12}$$

with $x=b/2c$ to transform (6.12) into (6.8). To transform (6.12) into (6.8), simplify (6.12) using the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}, \tag{6.13}$$

and the duplication formula

$$\Gamma(x+\tfrac{1}{2}) = \frac{\Gamma(2x)\sqrt{\pi}}{\Gamma(x)2^{2x-1}}, \tag{6.14}$$

with $x=b/2c$.

Note. The method developed by Flanders [33] is based on showing that

$$F(t) := \int_0^\infty e^{-tx^2} \cos x^2 \, dx \quad \text{and} \quad G(t) := \int_0^\infty e^{-tx^2} \sin x^2 \, dx \tag{6.15}$$

satisfy the functional equation

$$F^2(t) - G^2(t) = 2F(t)G(t) = \frac{\pi}{4(1+t^2)}. \tag{6.16}$$

The latter can be solved to obtain the values

$$F(t) = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{1+t^2}+t}{1+t^2}} \text{ and } G(t) = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{1+t^2}-t}{1+t^2}}. \tag{6.17}$$

A second elementary proof was obtained by Leonard [45]. Converting (6.15) into the Laplace transform of $\cos x/2\sqrt{x}$ and $\sin x/2\sqrt{x}$ respectively, he shows that

$$F(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{u^2+t}{1+(u^2+t)^2} \, du \quad \text{and} \quad G(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{du}{1+(u^2+t)^2}. \tag{6.18}$$

The evaluation of these integrals described in [45], is elementary but long. A shorter argument follows from the formula

$$f(a) := \int_0^\infty \frac{dx}{x^4+2ax^2+1} = \frac{\pi}{2\sqrt{2(1+a)}}. \tag{6.19}$$

Indeed, the values

$$G(t) = (1+t^2)^{-3/4} f\left(\frac{t}{\sqrt{t^2+1}} \right) \text{ and } F(t) = (1+t^2)^{-1/4} G(t/\sqrt{1+t^2}) + tG(t),$$

follow from (6.19) by a change of variable $v = (1+t^2)^{1/4}u$. The evaluation of the quartic integral (6.19) by the method of brackets is discussed in detail in Section 12.

7. An integral of beta type

In this section we present the evaluation of

$$I = \int_0^\infty \frac{x^a \, dx}{(E+Fx^b)^c}. \tag{7.1}$$

The change of variables $x = C^{1/b}t^{1/b}$, with $C = E/F$, yields

$$I = \frac{C^u}{bE^c} \int_0^\infty \frac{t^{u-1} \, dt}{(1+t)^c} \tag{7.2}$$

where $u = (a+1)/b$. The new integral evaluates as $B(c-u, u)$ where $B(x,y)$ is the classical beta function; see [39], formula 8.380.3. We conclude that

$$I = \frac{C^u}{bE^c} B(c-u, u). \tag{7.3}$$

To evaluate this integral by the method of brackets, the integrand $(E+Fx^b)^{-c}$ is expanded as

$$\sum_{n_1} \sum_{n_2} \phi_{1,2} E^{n_1} F^{n_2} x^{bn_2} \frac{\langle c+n_1+n_2 \rangle}{\Gamma(c)}. \tag{7.4}$$

Replacing in (7.1) we obtain

$$\begin{aligned} I &\doteq \sum_{n_1}\sum_{n_2} \phi_{1,2} E^{n_1} F^{n_2} \frac{\langle c+n_1+n_2\rangle}{\Gamma(c)} \int_0^\infty x^{a+bn_2+1-1}\,dx \\ &\doteq \sum_{n_1}\sum_{n_2} \phi_{1,2} E^{n_1} F^{n_2} \frac{1}{\Gamma(c)} \langle c+n_1+n_2\rangle\langle a+bn_2+1\rangle. \end{aligned}$$

To obtain the value assigned to the two dimensional sum, solve

$$\begin{aligned} c+n_1+n_2 &= 0 \\ a+bn_2+1 &= 0, \end{aligned}$$

to produce the solution $n_1^* = \frac{a+1}{b} - c$ and $n_2^* = -\frac{a+1}{b}$. Therefore

$$I = \frac{1}{b\Gamma(c)} E^{n_1^*} F^{n_2^*} \Gamma(-n_1^*)\Gamma(-n_2^*), \tag{7.5}$$

and this reduces to the value in (7.3).

8. A combination of powers and exponentials

In this section we employ the method of brackets and evaluate the integral

$$I := \int_0^\infty \frac{x^{\alpha-1}\,dx}{(A+B\exp(Cx^\beta))^\gamma}, \tag{8.1}$$

with α, β, γ, A, B, $C \in \mathbb{R}$. To evaluate this integral we consider the bracket series

$$\left(A+B\exp(Cx^\beta)\right)^{-\gamma} \doteq \sum_{n_1,n_2} A^{n_1}B^{n_2}\exp(Cn_2x^\beta)\frac{\langle \gamma+n_1+n_2\rangle}{\Gamma(\gamma)}. \tag{8.2}$$

The exponential function is expanded as

$$\begin{aligned} \exp(Cn_2x^\beta) &= \sum_{n_3=0}^\infty \frac{C^{n_3}n_2^{n_3}}{\Gamma(n_3+1)} x^{\beta n_3} \\ &= \sum_{n_3=0}^\infty C^{n_3}(-n_2)^{n_3}\phi_{n_3}x^{n_3}. \end{aligned}$$

Therefore, the integral (8.1) is assigned the bracket series

$$I \doteq \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{A^{n_1}B^{n_2}C^{n_3}(-n_2)^{n_3}\langle\alpha+\beta n_3\rangle\langle\gamma+n_1+n_2\rangle}{\Gamma(\gamma)}.$$

The vanishing of the two brackets leads to the system

$$\begin{aligned} \alpha+\beta n_3 &= 0 \\ \gamma+n_1+n_2 &= 0, \end{aligned}$$

and we have to choose a free parameter between n_1 and n_2. Observe that $n_3 = -\alpha/\beta$ is determined by the method.

Choice 1: take n_2 to be free. Then $n_1^* = -\gamma - n_2$ and $n_3^* = -\alpha/\beta$. This leads to

$$I = \sum_{n_2=0}^\infty \frac{B^{n_2}\Gamma(\alpha/\beta)\Gamma(\gamma+n_2)}{A^{\gamma+n_2}C^{\alpha/\beta}\beta\Gamma(\gamma)(-n_2)^{\alpha/\beta}}. \tag{8.3}$$

This is impossible due to the presence of the term $n_2^{\alpha/\beta}$ leading to a divergent series. These divergent series are discarded.

Choice 2: take n_1 as the free variable. Then $n_3^* = -\alpha/\beta$ and $n_2^* = -\gamma - n_1$. This time we obtain

$$I = \frac{\Gamma(\alpha/\beta)}{\Gamma(\gamma)} \frac{1}{B^\gamma C^{\alpha/\beta}\,\beta} \sum_{n_1=0}^{\infty} (-1)^{n_1} \frac{\Gamma(\gamma+n_1)}{\Gamma(1+n_1)} \frac{(A/B)^{n_1}}{(\gamma+n_1)^{\alpha/\beta}}. \tag{8.4}$$

This formula cannot be expressed in term of more elementary special functions.

In the special case $\gamma = 1$ we obtain

$$I = -\frac{\Gamma(\nu)}{a\beta c^\nu} \text{PolyLog}(\nu, -a/b). \tag{8.5}$$

with $\nu = \alpha/\beta$. The polylogarithm function appearing here is defined by

$$\text{PolyLog}(z,k) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}. \tag{8.6}$$

Specializing to $A = B = C = \alpha = \gamma = 1$ and $\beta = 2$ we obtain

$$\int_0^\infty \frac{dx}{1+e^{x^2}} = -\frac{\sqrt{\pi}}{2}(\sqrt{2}-1)\zeta\left(\tfrac{1}{2}\right). \tag{8.7}$$

Of course, this integral can be evaluated by simply expanding the integrand as a geometric series.

9. The Mellin transform of a quadratic exponential

The Mellin transform of a function $f(x)$ is defined by

$$\mathcal{M}(f)(s) := \int_0^\infty x^{s-1} f(x)\, dx. \tag{9.1}$$

Many of the integrals appearing in [39] are of this type. For example, 3.462.1 states that

$$\mathcal{M}\left(e^{-\beta x^2 - \gamma x}\right)(s) = \int_0^\infty x^{s-1} e^{-\beta x^2 - \gamma x}\, dx = (2\beta)^{-s/2}\Gamma(s) e^{\gamma^2/(8\beta)} D_{-s}\left(\frac{\gamma}{\sqrt{2\beta}}\right). \tag{9.2}$$

Here $D_p(z)$ is the parabolic cylinder function defined by (formula 9.240 in [39])

$$D_p(z) = 2^{p/2} e^{-z^2/4} \left(\frac{\sqrt{\pi}}{\Gamma((1-p)/2)} {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma(-p/2)} {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\right).$$

A direct application of the method of brackets gives

$$\int_0^\infty x^{s-1} e^{-\beta x^2 - \gamma x}\, dx \doteq \sum_{n_1} \sum_{n_2} \phi_{1,2} \beta^{n_1} \gamma^{n_2} \langle s + 2n_1 + n_2 \rangle. \tag{9.3}$$

The equation $s + 2n_1 + n_2 = 0$ gives two choices for a free index. Taking $n_2^* = -2n_1 - s$ leads to the series

$$\begin{aligned} \sum_{n_1=0}^{\infty} \frac{1}{\Gamma(n_1+1)} \left(-\frac{\beta}{\gamma^2}\right)^{n_1} (s)_{2n_1} &= \sum_{n_1=0}^{\infty} \frac{1}{\Gamma(n_1+1)} \left(-\frac{4\beta}{\gamma^2}\right)^{n_1} \left(\frac{s}{2}\right)_{n_1} \left(\frac{s+1}{2}\right)_{n_1} \\ &= {}_2F_0\left(\frac{s}{2}, \frac{s+1}{2} \Big| -\frac{4\beta}{\gamma^2}\right). \end{aligned}$$

This choice of a free index is excluded because the resulting series diverges. The second choice is $n_1^* = -n_2/2 - s/2$ and this yields the series

$$\frac{1}{2\beta^{s/2}} \sum_{n_2=0}^{\infty} \frac{\rho^{n_2}}{\Gamma(n_2+1)} \Gamma\left(\frac{n_2}{2} + \frac{s}{2}\right), \tag{9.4}$$

where $\rho = -\gamma/\sqrt{\beta}$. To write (9.4) in hypergeometric form we separate it into two sums according to the parity of n_2 and obtain

$$\frac{1}{2\beta^{s/2}} \left(\Gamma\left(\frac{s}{2}\right) \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(1)_{2n}} \left(\frac{s}{2}\right)_n + \Gamma\left(\frac{1+s}{2}\right) \sum_{n=0}^{\infty} \frac{\rho^{2n+1}}{(2)_{2n}} \left(\frac{1+s}{2}\right)_n \right).$$

The identity

$$(a)_{2n} = 4^n \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n \tag{9.5}$$

gives the final representation of the sum as

$$\frac{1}{2\beta^{s/2}} \left[\Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{s}{2}, \frac{1}{2}; \frac{1}{2}\rho^2\right) + \rho\Gamma\left(\frac{1+s}{2}\right) {}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; \frac{1}{2}\rho^2\right) \right]. \tag{9.6}$$

This is (9.2).

The special case $s = 1$ gives

$$\int_0^\infty e^{-\beta x^2 - \gamma x}\, dx = \tfrac{1}{2\sqrt{\beta}} \left[\Gamma(\tfrac{1}{2})\ {}_1F_1\left(\tfrac{1}{2}; \tfrac{1}{2}; \tfrac{1}{2}\rho^2\right) + \rho\, {}_1F_1\left(1; \tfrac{3}{2}; \tfrac{1}{2}\rho^2\right) \right]. \tag{9.7}$$

The first hypergeometric sum evaluates to $e^{\gamma^2/4\beta}$ and using the representation of the *error function*

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{9.8}$$

as

$$\text{erf}(x) = \frac{2x}{\sqrt{\pi}} e^{-x^2} {}_1F_1\left(1; \tfrac{3}{2}; x^2\right), \tag{9.9}$$

(given as 8.253.1 in [39]) we find the value of the second hypergeometric sum. The conclusion is that

$$\int_0^\infty e^{-\beta x^2 - \gamma x}\, dx = \frac{\sqrt{\pi}}{2\beta} \exp\left(\frac{\gamma^2}{4\beta}\right) \left(1 - \text{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right)\right). \tag{9.10}$$

This can be checked directly by completing the square in the integrand.

10. A multidimensional integral from Gradshteyn and Ryzhik

The method of brackets can also be used to evaluate some multidimensional integrals. Consider the following integral

$$I_n := \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}\, dx_1 dx_2 \cdots dx_n}{\left(1 + (r_1 x_1)^{q_1} + \cdots + (r_n x_n)^{q_n}\right)^s}, \tag{10.1}$$

which appears as 4.638.3 in [39] with an incorrect evaluation.

The first step in the evaluation of I_n is to expand the denominator of the integrand using Rule 3.1 as

$$\frac{1}{(1+(r_1x_1)^{q_1}+\cdots+(r_nx_n)^{q_n})^s} \doteq \sum_{k_0,k_1,\cdots,k_n} \phi_{0,\cdots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s+k_0+\cdots+k_n\rangle}{\Gamma(s)}.$$

Next the integral is assigned the value

$$I_n \doteq \sum_{k_0,k_1,\cdots,k_n} \phi_{0,\cdots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s+k_0+\cdots+k_n\rangle}{\Gamma(s)} \prod_{j=1}^{n} \langle p_j+q_jk_j\rangle. \tag{10.2}$$

The evaluation of this bracket sum involves the values

$$k_0 = -s+\sum_{j=1}^{n} \frac{p_j}{q_j} \text{ and } k_j = -\frac{p_j}{q_j} \text{ for } 1 \leq j \leq n. \tag{10.3}$$

We conclude that

$$I_n = \frac{1}{\Gamma(s)} \Gamma\left(s-\sum_{j=1}^{n} \frac{p_j}{q_j}\right) \prod_{j=1}^{n} \frac{\Gamma\left(\frac{p_j}{q_j}\right)}{q_jr_j^{p_j}}. \tag{10.4}$$

The table [39] has the exponents of r_j written as p_jq_j instead of p_j. This has now been corrected.

11. An example involving Bessel functions

The Bessel function $J_\nu(x)$ is defined by the series

$$J_\nu(x) = \frac{1}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+\nu}}{2^{2k} k! \Gamma(\nu+k+1)}, \tag{11.1}$$

and it admits the hypergeometric representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu\, \Gamma(1+\nu)} \, {}_0F_1\left(\begin{matrix} - \\ 1+\nu \end{matrix} \middle| \frac{-x^2}{4} \right). \tag{11.2}$$

The method of brackets will now be employed to evaluate the integral

$$I := \int_0^\infty x^{-\lambda} J_\nu(\alpha x) J_\mu(\beta x)\, dx. \tag{11.3}$$

Three integrals of this type form Section 6.574 of [39].

Replacing the hypergeometric form in the integral, we have

$$\begin{aligned} I \ &\doteq \ \frac{\left(\frac{\alpha}{2}\right)^\nu \left(\frac{\beta}{2}\right)^\mu}{\Gamma(\nu+1)\Gamma(\mu+1)} \\ &\times \int_0^\infty \sum_{n_1,n_2} \phi_{1,2} \frac{\alpha^{2n_1}\, \beta^{2n_2}}{4^{n_1+n_2}\, (\nu+1)_{n_1}\, (\mu+1)_{n_2}} x^{2n_1+2n_2-\lambda+\nu+\mu}\, dx. \end{aligned}$$

Therefore, the bracket series associated to the integral (11.3) becomes

$$\begin{aligned} I \ &\doteq \ \frac{2^{-\nu-\mu}\alpha^\nu\beta^\mu}{\Gamma(\nu+1)\Gamma(\mu+1)} \\ &\times \sum_{n_1}\sum_{n_2} \frac{\phi_{1,2}}{4^{n_1+n_2}} \frac{\alpha^{2n_1}\, \beta^{2n_2}}{(\nu+1)_{n_1}\, (\mu+1)_{n_2}} \langle 2n_1+2n_2-\lambda+\nu+\mu+1\rangle. \end{aligned}$$

The vanishing of the brackets yields the value $n_1^* = \frac{1}{2}(\lambda - \nu - \mu - 1) - n_2$ and it follows that

$$I = \frac{2^{-\nu-\mu}}{\Gamma(\nu+1)\Gamma(\mu+1)} \sum_{n_2=0}^{\infty} \frac{\phi_2}{4^{n_1^*+n_2}} \frac{\alpha^{2n_1^*}\beta^{2n_2}}{(\nu+1)_{n_1^*}(\mu+1)_{n_2}} \frac{\Gamma(-n_1^*)}{2}.$$

Writing the Pochhammer symbol $(\nu+1)_{n_1^*}$ in terms of the gamma function we obtain

$$\begin{aligned} I &= \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda \Gamma(\mu++1)} \\ &\times \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{(\beta^2/\alpha^2)^{n_2}}{\Gamma(\nu+1+\frac{1}{2}(\lambda-\nu-\mu-1)-n_2)} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1)+n_2)}{(\mu+1)_{n_2}}. \end{aligned}$$

In order to write this in hypergeometric terms, we start with

$$\begin{aligned} I &= \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda \Gamma(\mu++1)} \\ &\times \sum_{n_2=0}^{\infty} (-1)^{n_2} \frac{(\frac{1}{2}(\nu+\mu-\lambda+1))_{n_2} (\beta^2/\alpha_2)^{n_2}}{(\frac{1}{2}(\lambda+\nu-\mu+1))_{-n_2} (\mu+1)_{n_2} \Gamma(n_2+1)}, \end{aligned}$$

and use the identity

$$(c)_{-n} = \frac{(-1)^n}{(1-c)_n}, \tag{11.4}$$

to obtain

$$\begin{aligned} I &= \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1)}{\Gamma(\mu+1)\Gamma(\frac{1}{2}(\lambda+\nu-\mu+1))} \\ &\times \sum_{n_2=0}^{\infty} (\tfrac{1}{2}(1-\lambda-\nu+\mu))_{n_2} (\tfrac{1}{2}(\nu+\mu-\lambda+1))_{n_2} \frac{1}{(\mu+1)_{n_2}\Gamma(n_2+1)} \left(\frac{\beta^2}{\alpha^2}\right)^{n_2}, \end{aligned}$$

that can be written as

$$\begin{aligned} I &= \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1))}{\Gamma(\mu+1)\Gamma(\frac{1}{2}(\lambda+\nu-\mu+1))} \\ &\times {}_2F_1\left(\begin{matrix} \frac{1}{2}(1-\lambda-\nu+\mu) & \frac{1}{2}(\nu+\mu-\lambda+1) \\ & \mu+1 \end{matrix} \Bigg| \frac{\beta^2}{\alpha^2} \right). \end{aligned}$$

This solution is valid for $|\beta^2/\alpha^2| < 1$ and it corresponds to formula 6.574.3 in [39]. The table contains an error in this formula, the power of β is written as ν instead of μ. To obtain a formula valid for $|\beta^2/\alpha^2| > 1$ we could proceed as before and obtain 6.574.1 in [39]. Alternatively exchange (ν, α) by (μ, β) and use the formula developed above.

12. A new evaluation of a quartic integral

The integral

$$N_{0,4}(a; m) := \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} \tag{12.1}$$

is given by

$$N_{0,4}(a,m) = \frac{\pi}{2}\frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}}, \tag{12.2}$$

where P_m is the polynomial

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l, \tag{12.3}$$

with coefficients

$$d_{l,m} = 2^{-2m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}. \tag{12.4}$$

The sequence $\{d_{l,m} : 0 \leq l \leq m\}$ have remarkable arithmetical and combinatorial properties [50].

The reader will find in [4] a survey of the many different proofs of (12.2) available in the literature. One of these proofs follows from the hypergeometric representation (12.5)

$$N_{0,4}(a,m) = 2^{m-\frac{1}{2}}(a+1)^{-m-\frac{1}{2}}B\left(2m+\tfrac{3}{2}\right) {}_2F_1\left(\begin{matrix}-m & & m+1\\ & m+\frac{3}{2} & \end{matrix}\Big| \frac{1-a}{2}\right).$$

New proofs of this evaluation keep on appearing. For instance, the survey [4] does not include the recent automatic proof by C. Koutschan and V. Levandovskyy [43]. The goal of this section is to provide yet another proof of the identity (12.2) using the method of brackets.

The bracket series for $I \equiv N_{0,4}(a,m)$ is formed by the usual procedure. The result is

$$I \doteq \frac{1}{\Gamma(m+1)}\sum_{n_1,n_2,n_3}\phi_{1,2,3}(2a)^{n_2}\langle 4n_1+2n_2+1\rangle\langle m+1+n_1+n_2+n_3\rangle. \tag{12.6}$$

The expression (12.6) contains two brackets and three indices. Therefore the final result will be a single series on the free index. We employ the following notation: I is the original bracket series, the symbol I_j denotes the series I after eliminating the index n_j. Similarly $I_{i,j}$ denotes the series I after first eliminating n_i (to produce I_i) and then eliminating n_j.

Case 1: n_3 is the free index. Eliminate first n_1 from the bracket $\langle 4n_1+2n_2+1\rangle$ to obtain $n_1^* = -\frac{1}{2}n_2 - \frac{1}{4}$. The resulting bracket series is

$$I_1 \doteq \sum_{n_2,n_3}\phi_{2,3}\frac{(2a)^{n_2}\Gamma(\frac{1}{2}n_2+\frac{1}{4})}{4\Gamma(m+1)}\langle m+\tfrac{3}{4}+\tfrac{1}{2}n_2+n_3\rangle. \tag{12.7}$$

The next step is to eliminate n_2 to get $n_2^* = -2m-\frac{3}{2}-2n_3$ and obtain

$$I_{1,2} = \frac{1}{2\Gamma(m+1)(2a)^{2m+3/2}}\sum_{n_3=0}^{\infty}\frac{\phi_3}{(2a)^{n_3}}\Gamma(-m-\tfrac{1}{2}-n_3)\Gamma(2m+\tfrac{3}{2}+2n_3). \tag{12.8}$$

In order to simplify these expressions, we employ

$$\Gamma(x+m) = (x)_m\Gamma(x),\ \Gamma(x-m) = (-1)^m\Gamma(x)/(1-x)_m \tag{12.9}$$

and

$$(x)_{2m} = 2^{2m}\left(\tfrac{1}{2}x\right)_m \left(\tfrac{1}{2}(x+1)\right)_m, \tag{12.10}$$

for $x \in \mathbb{R}$ and $m \in \mathbb{N}$. We obtain

$$\Gamma(-m-\tfrac{1}{2}-n_3) = \frac{(-1)^{n_3}\Gamma(-\tfrac{1}{2}-m)}{(\tfrac{3}{2}+m)_{n_3}}$$

and

$$\Gamma(2m+\tfrac{3}{2}+2n_3) = \Gamma(2m+\tfrac{3}{2})(m+\tfrac{3}{4})_{n_3}(m+\tfrac{5}{4})_{n_3}2^{2n_3}.$$

These yield

$$I_{1,2} = \frac{\Gamma(-\tfrac{1}{2}-m)\Gamma(2m+\tfrac{3}{2})}{2\Gamma(m+1)(2a)^{2m+3/2}} \sum_{n_3=0}^{\infty} \frac{(m+3/4)_{n_3}\,(m+5/4)_{n_3}}{(m+3/2)_{n_3} n_3!} a^{-2n_3}, \tag{12.11}$$

or

$$I_{1,2} = \frac{\Gamma(-\tfrac{1}{2}-m)\Gamma(2m+\tfrac{3}{2})}{2\Gamma(m+1)(2a)^{2m+3/2}}\; {}_2F_1\left(\begin{matrix} m+\tfrac{3}{4} \qquad m+\tfrac{5}{4} \\ m+\tfrac{3}{2} \end{matrix} \Bigg| \frac{1}{a^2}\right). \tag{12.12}$$

Note. The reader can check that $I_{1,2} = I_{2,1}$, so the value of the sum for the quartic integral does not depend on the order in which the indices n_1 and n_2 are eliminated. The reader can also verify that this occurs in the next two cases described below; that is, $I_{1,3} = I_{3,1}$ and $I_{2,3} = I_{3,2}$.

Case 2: n_1 is the free index. A similar argument yields

$$I_{2,3} = \frac{\Gamma(m+\tfrac{1}{2})\Gamma(\tfrac{1}{2})}{2\Gamma(m+1)(2a)^{1/2}}\; {}_2F_1\left(\begin{matrix} \tfrac{1}{4} \qquad \tfrac{3}{4} \\ \tfrac{1}{2}-m \end{matrix} \Bigg| \frac{1}{a^2}\right). \tag{12.13}$$

Case 3: n_2 is the free index. Eliminate n_1 from the bracket series (12.6) to produce

$$I_1 \stackrel{\bullet}{=} \sum_{n_2,n_3} \phi_{2,3} \frac{(2a)^{n_2}\Gamma(\tfrac{1}{2}n_2+\tfrac{1}{4})}{4\Gamma(m+1)} \langle m+\tfrac{3}{4}+\tfrac{1}{2}n_2+n_3\rangle, \tag{12.14}$$

and now eliminate n_3 to obtain $n_3^* = -m-\frac{3}{4}-\frac{1}{2}n_2$. This yields

$$I_{1,3} = \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} (-1)^{n_2}\frac{(2a)^{n_2}}{n_2!}\Gamma(\tfrac{1}{2}n_2+\tfrac{1}{4})\Gamma(m+\tfrac{3}{4}+\tfrac{1}{2}n_2). \tag{12.15}$$

In order to obtain a hypergeometric representations of these expressions, we separate the last series according to the parity of n_2:

$$\begin{aligned} I_{1,3} &= \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} \frac{(2a)^{2n_2}}{(2n_2)!}\,\Gamma(n_2+\tfrac{1}{4})\Gamma(n_2+m+\tfrac{3}{4}) \\ &- \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} \frac{(2a)^{2n_2+1}}{(2n_2+1)!}\,\Gamma(n_2+\tfrac{3}{4})\Gamma(n_2+m+\tfrac{5}{4}). \end{aligned}$$

Using the standard formulas (12.9) and (12.10), we can write this in the form

$$\begin{aligned} I_{1,3} &= \frac{\Gamma(\tfrac{1}{4})\Gamma(m+\tfrac{3}{4})}{4\Gamma(m+1)}\; {}_2F_1\left(\begin{matrix} \tfrac{1}{4} \qquad m+\tfrac{3}{4} \\ \tfrac{1}{2} \end{matrix} \Bigg| a^2\right) - \\ & \frac{a\Gamma(\tfrac{3}{4})\Gamma(m+\tfrac{5}{4})}{2\Gamma(m+1)}\; {}_2F_1\left(\begin{matrix} \tfrac{3}{4} \qquad m+\tfrac{5}{4} \\ \tfrac{3}{2} \end{matrix} \Bigg| a^2\right). \end{aligned}$$

In summary: we have obtained three series related to the integral $N_{0,4}(a,m)$. The series $I_{1,2}$ and $I_{2,3}$ are given in terms of the hypergeometric function ${}_2F_1$ with last argument $1/a^2$. These series converge when $a^2 > 1$. The remaining case $I_{1,3}$ gives ${}_2F_1$ with argument a^2, that is convergent when $a^2 < 1$. Rule 3.4 states that we must add the series $I_{1,2}$ and $I_{2,3}$ to get a valid representation for $a^2 > 1$. In conclusion, the method of brackets shows that

$$\begin{aligned} N_{0,4}(a,m) &= \frac{\Gamma(\frac{1}{4})\Gamma(m+\frac{3}{4})}{4\Gamma(m+1)}\,{}_2F_1\left(\begin{matrix}\frac{1}{4} & m+\frac{3}{4}\\ \frac{1}{2}\end{matrix}\Big| a^2\right)+\\ &- \frac{a\Gamma(\frac{3}{4})\Gamma(m+\frac{5}{4})}{2\Gamma(m+1)}\,{}_2F_1\left(\begin{matrix}\frac{3}{4} & m+\frac{5}{4}\\ \frac{3}{2}\end{matrix}\Big| a^2\right) \quad \text{for } a^2<1,\\ &= \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{2\sqrt{2a}\Gamma(m+1)}\,{}_2F_1\left(\begin{matrix}\frac{1}{4} & \frac{3}{4}\\ \frac{1}{2}-m\end{matrix}\Big| \frac{1}{a^2}\right)\\ &+ \frac{\Gamma(-\frac{1}{2})\Gamma(2m+\frac{3}{2})}{2(2a)^{2m+3/2}\Gamma(m+1)}\,{}_2F_1\left(\begin{matrix}m+\frac{3}{4} & m+\frac{5}{4}\\ m+\frac{3}{2}\end{matrix}\Big| \frac{1}{a^2}\right) \quad \text{for } a^2>1. \end{aligned}$$

The continuity of these expressions at $a=1$ requires the evaluation of ${}_2F_1(a,b;c;1)$. Recall that this is finite only when $c > a+b$. In our case, we have four hypergeometric terms and in each one of them, the corresponding expression $c-(a+b)$ equals $-\frac{1}{2}-m$. Therefore each hypergeometric term blows up as $a \to 1$. This divergence is made evident by employing the relation

$$ {}_2F_1(a,b,c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z). \tag{12.16}$$

The expression for $N_{0,4}(a,m)$ given above is transformed into

$$\begin{aligned} N_{0,4}(a,m) &= \frac{\Gamma(\frac{1}{4})\Gamma(m+\frac{3}{4})}{4\Gamma(m+1)(1-a^2)^{m+1/2}}\,{}_2F_1\left(\begin{matrix}\frac{1}{4} & -m-\frac{1}{4}\\ \frac{1}{2}\end{matrix}\Big| a^2\right)+\\ &- \frac{a\Gamma(\frac{3}{4})\Gamma(m+\frac{5}{4})}{2\Gamma(m+1)(1-a^2)^{m+1/2}}\,{}_2F_1\left(\begin{matrix}\frac{3}{4} & -m+\frac{1}{4}\\ \frac{3}{2}\end{matrix}\Big| a^2\right) \quad \text{for } a^2<1,\\ &= \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{2\sqrt{2a}\Gamma(m+1)(1-a^{-2})^{m+1/2}}\,{}_2F_1\left(\begin{matrix}\frac{1}{4}-m & -\frac{1}{4}-m\\ \frac{1}{2}-m\end{matrix}\Big| \frac{1}{a^2}\right)\\ &+ \frac{\Gamma(-\frac{1}{2})\Gamma(2m+\frac{3}{2})}{2(2a)^{2m+3/2}\Gamma(m+1)(1-a^{-2})^{m+1/2}}\,{}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{1}{4}\\ m+\frac{3}{2}\end{matrix}\Big| \frac{1}{a^2}\right) \quad \text{for } a^2>1. \end{aligned}$$

Introduce the functions

$$\begin{aligned} G_1(a,m) &= \left(\frac{3}{4}\right)_m {}_2F_1\left(\begin{matrix}\frac{1}{4} & -\frac{1}{4}-m\\ \frac{1}{2}\end{matrix}\Big| a^2\right)\\ &- 2a\left(\frac{1}{4}\right)_{m+1} {}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{1}{4}-m\\ \frac{3}{2}\end{matrix}\Big| a^2\right), \end{aligned}$$

and

$$\begin{aligned} G_2(a,m) &= \left(\frac{1}{2}\right)_m (2a)^{2m+1}\,{}_2F_1\left(\begin{matrix}\frac{1}{4}-m & -\frac{1}{4}-m\\ \frac{1}{2}-m\end{matrix}\Big| \frac{1}{a^2}\right)\\ &- (-1)^m m! 2^{-2m}\binom{4m+1}{2m}\,{}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{1}{4}\\ m+\frac{3}{2}\end{matrix}\Big| \frac{1}{a^2}\right). \end{aligned}$$

Then

$$N_{0,4}(a,m) = \frac{\pi\sqrt{2}}{4m!} \frac{G_1(a,m)}{(1-a^2)^{m+1/2}} \tag{12.17}$$

for $a^2 < 1$ and

$$N_{0,4}(a,m) = \frac{\pi}{2^{2m+5/2}\sqrt{a}m!} \frac{G_2(a,m)}{(a^2-1)^{m+1/2}} \tag{12.18}$$

for $a^2 > 1$. The functions $G_1(a,m)$ and $G_2(a,m)$ match at $a=1$ to sufficiently high order to verify the continuity at $a=1$. Morever, their blow up at $a=-1$ is a reflection of the fact that the convergence of the integral $N_{0,4}(a,m)$ requires $a > -1$.

It is possible to show that both expressions (12.17) and (12.18) reduce to (12.2). The details will appear elsewhere.

13. Integrals from Feynman diagrams

The flexibility of the method of brackets is now illustrated by evaluating examples of definite integrals appearing in the resolution of Feynman diagrams. The reader will find in [41], [63], [42] and [75] information about these diagrams. The mathematical theory behind Quantum Field Theory and in particular to the role of Feynman diagrams can be obtained from [34] and [22].

The graph G contains N *propagators* or *internal lines*, L *loops* associated to independent internal momenta $\mathbf{Q} := \{Q_1, \cdots, Q_L\}$, E independent external momenta $\mathbf{P} := \{P_1, \cdots, P_E\}$ (therefore the diagram has $E+1$ *external lines*). The momentum P_j, Q_j belong to $\mathbb{R}^4$ and the space $\mathbb{R}^4$ is equipped with the Minkowski metric. Therefore, for A, $B \in \mathbb{R}^4$, we have

$$A^2 := A_0^2 - A_1^2 - A_2^2 - A_3^2, \tag{13.1}$$

and

$$A \cdot B := A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3. \tag{13.2}$$

Finally, each propagator has a *mass* $m_j \geq 0$ associated to it, collected in the vector $\mathbf{m} = (m_1, \cdots, m_N)$.

The method of dimensional regularization (see [62] for details) gives an integral expression in the momentum space that represents the diagram in $D = 4 - 2\epsilon$ dimensions. In Minkowski space the integral is given by

$$G = G(\mathbf{P}, \mathbf{m}) := \int \frac{d^DQ_1}{i\pi^{D/2}} \cdots \int \frac{d^DQ_L}{i\pi^{D/2}} \frac{1}{(B_1^2 - m_1^2)^{a_1}} \cdots \frac{1}{(B_N^2 - m_N^2)^{a_N}}. \tag{13.3}$$

The symbol B_j represents the momentum of the j-th propagator and it is a linear combination of the internal and external momenta $\mathbf{P}$ and $\mathbf{Q}$, respectively. The vector $\mathbf{a} := (a_1, \cdots, a_N)$ captures the *powers* of the propagators and they may assume arbitrary values.

In order to simplify (13.3), we use the identity

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-Ax}\, dx \tag{13.4}$$

with $A = B_j^2 - m_j^2$ and convert it into

(13.5)
$$G = \frac{1}{\prod_{j=1}^{N} \Gamma(a_j)} \int_0^\infty \exp\left(\sum_{j=1}^{N} x_j m_j^2 \right) \int \frac{\prod_{j=1}^{L} d^D Q_j}{(i\pi^{D/2})^L} \exp\left(- \sum_{j=1}^{N} x_j B_j^2 \right) \mathbf{dx},$$

where $\mathbf{dx} = \prod_{j=1}^{N} x_j^{a_j - 1} dx_j$.

The next step in the reduction process is to integrate (13.5) with respect to the internal momenta Q_j. This gives an expression for the integral G in terms of only the external momenta P_j and the masses m_j. This step can be achieved by introducing the *Schwinger parametrizaton* (see [35] and chapter 1, section 4 of [22] for details) and x_j are called the *Schwinger variables*. The final result is the representation

$$G = \frac{(-1)^{-LD/2}}{\prod_{j=1}^{N} \Gamma(a_j)} \int_0^\infty U^{-D/2} \exp\left(\sum_{j=1}^{N} x_j m_j^2 \right) \exp\left(-\frac{F}{U} \right) \mathbf{dx}. \tag{13.6}$$

The function F corresponds to a quadratic structure of the external momentum defined by

$$F = \sum_{i,j=1}^{E} C_{i,j} P_i \cdot P_j. \tag{13.7}$$

The function U and the coefficients $C_{i,j}$ are the *Symanzik polynomials* in the Schwinger parameters x_j. These polynomials are given in terms of determinants of the so-called *matrix of parameters*. The polynomial $C_{i,j}$ are symmetric, that is $C_{i,j} = C_{j,i}$. A systematic algorithm to write down the expression (13.6) directly from the Feynman diagram is presented in [35].

Example 13.1. Figure 2 depicts the interaction of three particles corresponding to the three external lines of momentum P_1, P_2, P_3. In this case the Schwinger parametrization provides the integral

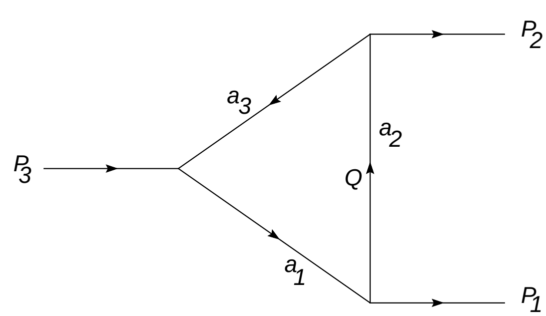

FIGURE 2. The triangle

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1}x_2^{a_2-1}x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}} \\
&\times \exp(x_1m_1^2+x_2m_2^2+x_3m_3^2)\exp\left(-\frac{C_1P_1^2+2C_{12}P_1\cdot P_2+C_{22}P_2^2}{x_1+x_2+x_3}\right) dx_1dx_2dx_3.
\end{aligned}
$$

The algorithm in [35] and [36] gives the coefficients $C_{i,j}$ as

$$C_{11} = x_1(x_2+x_3),\ C_{12} = x_1x_3,\ C_{22} = x_3(x_1+x_2). \tag{13.8}$$

Conservation of momentum gives $P_3 = P_1 + P_2$ and replacing the coefficients $C_{i,j}$ we obtain

$$
\begin{aligned}
G &= \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \int_0^\infty \int_0^\infty \int_0^\infty x^{a_1-1}x^{a_2-1}x^{a_3-1} \times \\
&\times \frac{\exp\left(x_1m_1^2+x_2m_2^2+x_3m_3^2\right)\exp\left(-\frac{x_1x_2P_1^2+x_2x_3P_2^2+x_3x_1P_3^2}{x_1+x_2+x_3}\right)}{(x_1+x_2+x_3)^{D/2}} dx_1dx_2dx_3.
\end{aligned}
$$

To solve the Feynman diagram in Figure 2 it is required to evaluate the integral G as a function of the variables $P_1,\ P_2 \in \mathbb{R}^4$, the masses m_i, the dimension D and the parameters a_i.

We now describe the evaluation of the integral G in the special massless situation: $m_1 = m_2 = m_3 = 0$. Moreover we assume that $P_1^2 = P_2^2 = 0$. The integral to be evaluated is then

$$G_1 = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_{\mathbb{R}_+^3} x_1^{a_1-1}x_2^{a_2-1}x_3^{a_3-1} \frac{\exp\left(-\frac{x_1x_3}{x_1+x_2+x_3}P_3^2\right)}{(x_1+x_2+x_3)^{D/2}}\, dx_1\, dx_2\, dx_3.$$

The method of brackets gives

$$G_1 \doteq \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4} \phi_{1234}(P_3^2)^{n_1} \frac{\Delta_1\Delta_2\Delta_3\Delta_4}{\Gamma(D/2+n_1)}, \tag{13.9}$$

where the brackets Δ_j are given by

$$
\begin{aligned}
\Delta_1 &= \langle D/2+n_1+n_2+n_3+n_4\rangle, \\
\Delta_2 &= \langle a_1+n_1+n_2\rangle, \\
\Delta_3 &= \langle a_2+n_3\rangle, \\
\Delta_4 &= \langle a_3+n_1+n_4\rangle.
\end{aligned}
$$

The solution contains no free indices: there are four sums and the linear system corresponding to the vanishing of the brackets eliminates all of them:

$$n_1^* = \tfrac{D}{2} - a_1 - a_2 - a_3,\ n_2^* = -\tfrac{D}{2} + a_2 + a_3,\ n_3^* = -a_2,\ n_4^* = -\tfrac{D}{2} + a_1 + a_2.$$

We conclude that

$$
\begin{aligned}
G_1 &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} (P_3^2)^{D/2-a_1-a_2-a_3} \times \\
&\times \frac{\Gamma(a_1+a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2-a_3)\Gamma(a_2)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-a_1-a_2)}{\Gamma(D-a_1-a_2-a_3)}.
\end{aligned}
$$

Example 13.2. The second example considers the diagram depicted in Figure 3. The resolution of this diagram is well-known and it appears in [12], [24] and [25]. The diagram contains two external lines and two internal lines (propagators) with the same mass m. These propagators are marked 1 and 2.

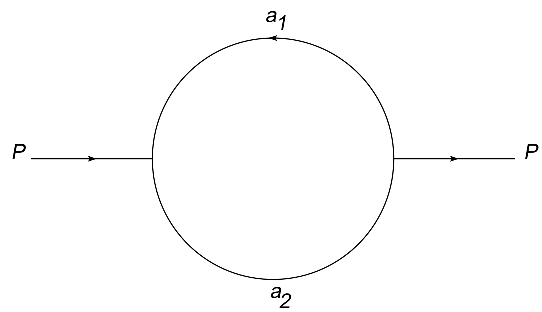

FIGURE 3. The bubble

In momentum variables, the integral representation of this diagram is given by

$$G = \int_{\mathbb{R}^D} \frac{d^DQ}{i\pi^{D/2}} \frac{1}{(Q^2-m^2)^{a_1}\,((P-Q)^2-m^2)^{a_2}}. \tag{13.10}$$

For the diagram considered here, we have $U = x_1 + x_2$ and $F = x_1x_2P^2$. According to (13.6), the Schwinger representation is given by

$$\begin{aligned} G &= \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \\ &\times \int_0^\infty \int_0^\infty \frac{x^{a_1-1}x^{a_2-1}}{(x_1+x_2)^{D/2}} \exp\left(m^2(x_1+x_2)\right) \exp\left(-\frac{x_1x_2}{x_1+x_2}P^2\right) dx_1\,dx_2. \end{aligned}$$

In order to generate the bracket series for G, we expand first the exponential function to obtain

$$G \doteq \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_1,n_2} \phi_{1,2} (P^2)^{n_1} (-m^2)^{n_2} \int_{R_+^2} \frac{x_1^{n_1}x_2^{n_2}\,dx_1\,dx_2}{(x_1+x_2)^{D/2+n_1-n_2}}. \tag{13.11}$$

Expanding now the term

$$\frac{1}{(x_1+x_2)^{D/2+n_1-n_2}} \doteq \sum_{n_3,n_4} \phi_{3,4} \frac{x_1^{n_3}x_2^{n_4}}{\Gamma(D/2+n_1-n_2)} \Delta_1, \tag{13.12}$$

with $\Delta_1 = \langle \frac{D}{2} + n_1 - n_2 + n_3 + n_4 \rangle$, and replacing in (13.11) yields

$$G \doteq \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_1,\cdots,n_4} \phi_{1,2,3,4} \frac{(P^2)^{n_1}(-m^2)^{n_2}}{\Gamma(\frac{D}{2}+n_1-n_2)} \Delta_1\Delta_2\Delta_3, \tag{13.13}$$

where

$$\begin{aligned} \Delta_1 &= \langle \tfrac{D}{2} + n_1 - n_2 + n_3 + n_4 \rangle, \\ \Delta_2 &= \langle a_1 + n_1 + n_3 \rangle, \\ \Delta_3 &= \langle a_2 + n_1 + n_4 \rangle. \end{aligned}$$

The expression for G contains 4 indices and the vanishing of the brackets allows us to express all of them in terms of a single index. We will denote by G_j the expression for G where the index n_j is free.

The sum G_1: in this case the solution of the corresponding linear system is

$$n_2^* = \tfrac{D}{2} - a_1 - a_2 - n_1,\ n_3^* = -a_1 - n_1,\ n_4^* = -a_2 - n_1, \tag{13.14}$$

and the sum G_1 becomes

$$\begin{aligned} G_1 &= (-1)^{-D/2} \frac{(-m^2)^{D/2-a_1+a_2}}{\Gamma(a_1)\Gamma(a_2)} \times \\ &\times \sum_{n_1=0}^{\infty} \frac{\Gamma(a_1+a_2-D/2+n_1)\,\Gamma(a_1+n_1)\,\Gamma(a_2+n_1)}{\Gamma(a_1+a_2+2n_1)} \frac{\left(\frac{P^2}{m^2}\right)^{n_1}}{n_1!}. \end{aligned}$$

This can be expressed as

$$G_1 = \lambda_1 (-m^2)^{D/2-a_1+a_2} {}_3F_2 \left(\left. \begin{matrix} a_1+a_2-\frac{D}{2}, & a_1, & a_2 \\ \frac{1}{2}(a_1+a_2+1), & \frac{1}{2}(a_1+a_2) & \end{matrix} \right| \frac{P^2}{4m^2}, \right) \tag{13.15}$$

where

$$\lambda_1 = (-1)^{-D/2} \frac{\Gamma(a_1+a_2-D/2)}{\Gamma(a_1+a_2)}. \tag{13.16}$$

The sum G_2: keeping n_2 as the free index gives

$$n_1^* = \tfrac{D}{2} - a_1 - a_2 - n_2,\ n_3^* = a_2 - \tfrac{D}{2} + n_2,\ n_4^* = a_1 - \tfrac{D}{2} + n_2,$$

which leads to

$$G_2 = \lambda_2 (P_1^2)^{D/2-a_1+a_2} {}_3F_2 \left(\left. \begin{matrix} a_1+a_2-\frac{D}{2}, & \frac{1}{2}(1+a_1+a_2-D), & \frac{1}{2}(2+a_1+a_2-D) \\ 1+a_1-\frac{D}{2}, & 1+a_2-\frac{D}{2} & \end{matrix} \right| \frac{4m^2}{P^2} \right)$$

where the prefactor λ_2 is given by

$$\lambda_2 = (-1)^{-D/2} \frac{\Gamma(a_1+a_2-D/2)\Gamma(\frac{D}{2}-a_1)\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_1-a_2)}.$$

The cases G_3 **and** G_4 are computed by a similar procedure. The results are

$$G_3 = \lambda_3 (P_1^2)^{-a_1} (-m^2)^{D/2-a_2} {}_3F_2 \left(\left. \begin{matrix} a_1, & \frac{1}{2}(1+a_1-a_2), & \frac{1}{2}(2+a_1-a_2) \\ 1+a_1-a_2, & 1-a_2+\frac{D}{2} & \end{matrix} \right| \frac{4m^2}{P^2} \right)$$

and

$$G_4 = \lambda_4 (P_1^2)^{-a_2} (-m^2)^{D/2-a_1} {}_3F_2 \left(\left. \begin{matrix} a_2, & \frac{1}{2}(1-a_1+a_2), & \frac{1}{2}(2-a_1+a_2) \\ 1-a_1+a_2, & 1-a_1+\frac{D}{2} & \end{matrix} \right| \frac{4m^2}{P^2} \right)$$

where the prefactors λ_3 and λ_4 are given by

$$\lambda_3 = (-1)^{-D/2} \frac{\Gamma(a_2-D/2)}{\Gamma(a_2)} \text{ and } \lambda_4 = (-1)^{-D/2} \frac{\Gamma(a_1-D/2)}{\Gamma(a_1)}. \tag{13.17}$$

The contributions of these four sums are now classified according to their region of convergence. This is determined by the parameter $\rho := |4m^2/P^2|$. In the region

$\rho > 1$, only the sum G_1 converges, therefore $G = G_1$ there. In the region $\rho < 1$ the three remaining sums converge. Therefore, according to Rule 3.4, we have

$$G = \begin{cases} G_1 & \text{for} \quad \rho > 1, \\ G_2 + G_3 + G_4 & \text{for} \quad \rho < 1. \end{cases} \tag{13.18}$$

We have evaluated the Feynman diagram in Figure 3 and expressed its solution in terms of hypergeometric functions that correspond naturally to the two quotient of the two energy scales present in the diagram.

Example 13.3. The next example shows that the method of brackets succeeds in the evaluation of very complicated integrals. We consider a Feynman diagram with four loops as shown in Figure 4.

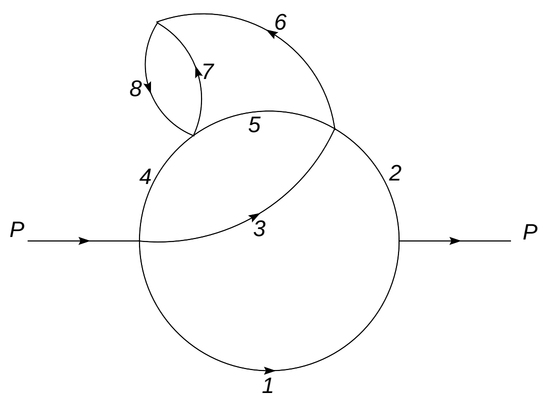

FIGURE 4. A diagram with four loops

The methods described in [35] for the Schwinger representation (13.6) of this diagram, give

$$\begin{aligned} U = \; & x_1x_3x_5x_7 + x_1x_3x_5x_8 + x_1x_3x_6x_7 + x_1x_4x_5x_7 + x_2x_3x_5x_7 + \\ & x_1x_3x_6x_8 + x_1x_4x_5x_8 + x_1x_4x_6x_7 + x_2x_3x_5x_8 + x_2x_3x_6x_7 + \\ & x_2x_4x_5x_7 + x_1x_3x_7x_8 + x_1x_4x_6x_8 + x_1x_5x_6x_7 + x_2x_3x_6x_8 + \\ & x_2x_4x_5x_8 + x_2x_4x_6x_7 + x_3x_4x_5x_7 + x_1x_4x_7x_8 + x_1x_5x_6x_8 + \\ & x_2x_3x_7x_8 + x_2x_4x_6x_8 + x_2x_5x_6x_7 + x_3x_4x_5x_8 + x_3x_4x_6x_7 + \\ & x_1x_5x_7x_8 + x_2x_4x_7x_8 + x_2x_5x_6x_8 + x_3x_4x_6x_8 + x_3x_5x_6x_7 + \\ & x_2x_5x_7x_8 + x_3x_4x_7x_8 + x_3x_5x_6x_8 + x_3x_5x_7x_8 \end{aligned} \tag{13.19}$$

and for the function F in (13.6):

$$\begin{aligned} F = \; & (x_1x_2x_3x_5x_7 + x_1x_2x_3x_5x_8 + x_1x_2x_3x_6x_7 + x_1x_2x_4x_5x_7 + \\ & x_1x_2x_3x_6x_8 + x_1x_2x_4x_5x_8 + x_1x_2x_4x_6x_7 + x_1x_3x_4x_5x_7 + \\ & x_1x_2x_3x_7x_8 + x_1x_2x_4x_6x_8 + x_1x_2x_5x_6x_7 + x_1x_3x_4x_5x_8 + \\ & x_1x_3x_4x_6x_7 + x_1x_2x_4x_7x_8 + x_1x_2x_5x_6x_8 + x_1x_3x_4x_6x_8 + \\ & x_1x_3x_5x_6x_7 + x_1x_2x_5x_7x_8 + x_1x_3x_4x_7x_8 + x_1x_3x_5x_6x_8 + \\ & x_1x_3x_5x_7x_8)\, P^2. \end{aligned} \tag{13.20}$$

The large number of terms appearing in the expressions for U and F (34 and 21 respcetively) makes it almost impossible to apply the method of brackets without an apriori factorization of these polynomials. This factorizations minimizes the number of sums and maximizes the number of brackets. In this example, the optimal factorization is given by

$$\begin{aligned} F &= x_1 f_7 \ P^2, \\ U &= x_1 f_6 + f_7, \end{aligned} \tag{13.21}$$

where the functions f_i are given by:

$$\begin{aligned} f_7 &= (x_2 f_6 + f_5), \\ f_6 &= x_3 f_4 + (x_4 f_4 + f_3), \\ f_5 &= x_3(x_4 f_4 + f_3), \\ f_4 &= x_5 f_2 + (x_6 f_2 + f_1), \\ f_3 &= x_5(x_6 f_2 + f_1), \\ f_2 &= (x_7 + x_8), \\ f_1 &= x_7 x_8. \end{aligned} \tag{13.22}$$

To analyze the diagram considered here, we start with the parametric representation

$$G = \frac{(-1)^{-D/2}}{\prod\limits_{j=1}^{8} \Gamma(a_j)} \int\limits_0^\infty \frac{\exp\left(-\dfrac{x_1 f_7}{x_1 f_6 + f_7} p_1^2\right)}{(x_1 f_6 + f_7)^{\frac{D}{2}}} \mathbf{dx}, \tag{13.23}$$

and expand the exponential function first. A systematic expansion associated to the polynomials f_i leads to the order

$$U \longrightarrow f_7 \longrightarrow f_6 \longrightarrow f_5 \longrightarrow f_4 \longrightarrow f_3 \longrightarrow f_2, \tag{13.24}$$

that yields the bracket series

$$G \doteq \frac{(-1)^{-D/2}}{\prod\limits_{j=1}^{8} \Gamma(a_j)} \sum_{n_1,..,n_{15}} \phi_{n_1,..,n_{15}} \frac{(P^2)^{n_1}}{\Gamma(\frac{D}{2} + n_1)} \Omega_{\{n\}} \prod_{j=1}^{15} \Delta_j, \tag{13.25}$$

where we have defined the factor

$$\Omega_{\{n\}} = \frac{1}{\Gamma(-n_1 - n_3)\Gamma(-n_2 - n_4)\Gamma(-n_5 - n_7)\Gamma(-n_6 - n_8)\Gamma(-n_9 - n_{11})\Gamma(-n_{10} - n_{12})},$$

and the corresponding brackets by

$$
\begin{aligned}
&\Delta_1 = \left\langle \tfrac{D}{2} + n_1 + n_2 + n_3 \right\rangle, && \Delta_9 = \langle a_2 + n_4 \rangle, \\
&\Delta_2 = \langle -n_1 - n_3 + n_4 + n_5 \rangle, && \Delta_{10} = \langle a_3 + n_5 + n_6 \rangle, \\
&\Delta_3 = \langle -n_2 - n_4 + n_6 + n_7 \rangle, && \Delta_{11} = \langle a_4 + n_8 \rangle, \\
&\Delta_4 = \langle -n_5 - n_7 + n_8 + n_9 \rangle, && \Delta_{12} = \langle a_5 + n_9 + n_{10} \rangle, \\
&\Delta_5 = \langle -n_6 - n_8 + n_{10} + n_{11} \rangle, && \Delta_{13} = \langle a_6 + n_{12} \rangle, \\
&\Delta_6 = \langle -n_9 - n_{11} + n_{12} + n_{13} \rangle, && \Delta_{14} = \langle a_7 + n_{13} + n_{14} \rangle, \\
&\Delta_7 = \langle -n_{10} - n_{12} + n_{14} + n_{15} \rangle, && \Delta_{15} = \langle a_8 + n_{13} + n_{15} \rangle, \\
&\Delta_8 = \langle a_1 + n_1 + n_2 \rangle.
\end{aligned}
\tag{13.26}
$$

There is a unique way to evaluate the series: the numbers of indices is the same as the number of brackets. Solving the corresponding linear system leads to

$$
G = (-1)^{-D/2} \frac{(P^2)^{n_1^*}}{\Gamma(D/2 + n_1^*)} \, \Omega_{\{n^*\}} \, \frac{\prod\limits_{j=1}^{15} \Gamma(-n_j^*)}{\prod\limits_{j=1}^{8} \Gamma(a_j)},
\tag{13.27}
$$

where the values n_i^* are given by

$$
\begin{aligned}
&n_1^* = 2D - a_1 - a_2 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8, && n_9^* = D - a_5 - a_6 - a_7 - a_8, \\
&n_2^* = a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 - 2D, && n_{10}^* = a_6 + a_7 + a_8 - D, \\
&n_3^* = a_1 - \tfrac{D}{2}, && n_{11}^* = a_5 - \tfrac{D}{2}, \\
&n_4^* = -a_2, && n_{12}^* = -a_6, \\
&n_5^* = \tfrac{3D}{2} - a_3 - a_4 - a_5 - a_6 - a_7 - a_8, && n_{13}^* = \tfrac{D}{2} - a_7 - a_8, \\
&n_6^* = a_4 + a_5 + a_6 + a_7 + a_8 - \tfrac{3D}{2}, && n_{14}^* = a_8 - \tfrac{D}{2}, \\
&n_7^* = a_3 - \tfrac{D}{2}, && n_{15}^* = a_7 - \tfrac{D}{2}, \\
&n_8^* = -a_4.
\end{aligned}
$$

Example 13.4. The last example discussed in this paper gives the value of a Feynman diagram as a hypergeometric function of two variables. The diagram shown in Figure 3 contains two external lines and with internal lines (propagators) with distinct masses. The same diagram with equal masses was described in Example 13.2. The integral representation of this diagram in the momentum space is given by

$$
G = \int \frac{d^DQ}{i\pi^{D/2}} \frac{1}{(Q^2 - m_1^2)^{a_1} \, ((P-Q)^2 - m_2^2)^{a_2}}.
\tag{13.28}
$$

On the other hand, the parametric representation of Schwinger is

$$
G = \frac{(-1)^{-\frac{D}{2}}}{\prod\limits_{j=1}^{2} \Gamma(a_j)} \int\limits_0^\infty \frac{\exp\left(x_1 m_1^2\right) \exp\left(x_2 m_2^2\right) \exp\left(-\frac{x_1 x_2}{x_1 + x_2} P^2\right)}{(x_1 + x_2)^{\frac{D}{2}}} \, \mathbf{dx}.
\tag{13.29}
$$

In order to find the bracket series associated to this integral, we first expand the exponentials

$$G \doteq \frac{(-1)^{-\frac{D}{2}}}{\prod_{j=1}^{2}\Gamma(a_j)} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \left(-m_1^2\right)^{n_1} \left(-m_2^2\right)^{n_2} \left(P^2\right)^{n_3} \int \frac{x_1^{n_1+n_3} x_2^{n_2+n_3}}{(x_1+x_2)^{\frac{D}{2}+n_3}} \mathbf{dx}.$$

and then the denominator

$$\frac{1}{(x_1+x_2)^{\frac{D}{2}+n_3}} \doteq \sum_{n_4,n_5} \phi_{n_4,n_5} \frac{x_1^{n_4}x_2^{n_5}}{\Gamma(\frac{D}{2}+n_3)} \left\langle \tfrac{D}{2}+n_3+n_4+n_5 \right\rangle.$$

We obtain

$$\begin{aligned} G &\doteq \frac{(-1)^{-\frac{D}{2}}}{\prod_{j=1}^{2}\Gamma(a_j)} \sum_{n_1,..,n_5} \phi_{n_1,..,n_5} \frac{\left(-m_1^2\right)^{n_1} \left(-m_2^2\right)^{n_2} \left(P^2\right)^{n_3}}{\Gamma(\frac{D}{2}+n_3)} \Delta_1 \\ &\times \int x_1^{a_1+n_1+n_3+n_4-1}\,dx_1 \int x_2^{a_2+n_2+n_3+n_5-1}\,dx_2. \end{aligned}$$

Using the rules for transforming integrals into brackets yields

$$G \doteq \frac{(-1)^{-\frac{D}{2}}}{\prod_{j=1}^{2}\Gamma(a_j)} \sum_{n_1,..,n_5} \phi_{n_1,..,n_5} \frac{\left(-m_1^2\right)^{n_1} \left(-m_2^2\right)^{n_2} \left(P^2\right)^{n_3}}{\Gamma(\frac{D}{2}+n_3)} \Delta_1\Delta_2\Delta_3, \tag{13.30}$$

with brackets defined by

$$\begin{aligned} \Delta_1 &= \left\langle \tfrac{D}{2}+n_3+n_4+n_5 \right\rangle, \\ \Delta_2 &= \langle a_1+n_1+n_3+n_4 \rangle, \\ \Delta_3 &= \langle a_2+n_2+n_3+n_5 \rangle. \end{aligned} \tag{13.31}$$

We have to choose two free values from $n_1, \cdots, n_5$. The result is a hypergeometric function of multiplicity two. There are 10 such choices and the brackets in (13.31) produce the linear system

$$\begin{aligned} 0 &= \tfrac{D}{2}+n_3+n_4+n_5 \\ 0 &= a_1+n_1+n_3+n_4 \\ 0 &= a_2+n_2+n_3+n_5. \end{aligned} \tag{13.32}$$

We denote by $G_{i,j}$ the solution of this system with free indices n_i and n_j. The series appearing in the solution to the Feynman diagram in Figure 4 is expressed in terms of the Appell function F_4, defined by

$$F_4\left(\begin{array}{cc|c} \alpha & \beta & \\ & & x,\, y \\ \gamma & \delta & \end{array} \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n} \frac{x^m}{m!}\frac{y^n}{n!}. \tag{13.33}$$

Following a procedure similar to the one described in the previous example, we obtain the explicit values of the integral G is given in terms of the functions $G_{i,j}$.

$$G = \begin{cases} G_{1,2} + G_{1,4} + G_{2,5} & \text{when} \quad m_1^2, m_2^2 < P^2 \\ G_{1,3} + G_{3,5} & \text{when} \quad m_1^2, P^2 < m_2^2 \\ G_{2,3} + G_{3,4} & \text{when} \quad m_2^2, P^2 < m_1^2. \end{cases}$$

These in turn are expressed in terms of the Appell function:

$$\begin{aligned}
G_{1,2} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_1)\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_1-a_2)} \left(P^2\right)^{\frac{D}{2}-a_1-a_2} \\
&\times \; F_4\left(\begin{matrix} 1+a_1+a_2-D & a_1+a_2-\frac{D}{2} \\ 1+a_1-\frac{D}{2} & 1+a_2-\frac{D}{2} \end{matrix} \middle| \frac{m_1^2}{P^2}, \frac{m_2^2}{P^2} \right) \\
G_{1,4} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_2-\frac{D}{2})}{\Gamma(a_2)} \left(-m_2^2\right)^{\frac{D}{2}-a_2} \left(P^2\right)^{-a_1} \\
&\times \; F_4\left(\begin{matrix} 1+a_1-\frac{D}{2} & a_1 \\ 1+a_1-\frac{D}{2} & 1-a_2+\frac{D}{2} \end{matrix} \middle| \frac{m_1^2}{P^2}, \frac{m_2^2}{P^2} \right) \\
G_{2,5} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_1-\frac{D}{2})}{\Gamma(a_1)} \left(-m_1^2\right)^{\frac{D}{2}-a_1} \left(P^2\right)^{-a_2} \\
&\times \; F_4\left(\begin{matrix} 1+a_2-\frac{D}{2} & a_2 \\ 1-a_1+\frac{D}{2} & 1+a_2-\frac{D}{2} \end{matrix} \middle| \frac{m_1^2}{P^2}, \frac{m_2^2}{P^2} \right) \\
G_{1,3} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_1)}{\Gamma(a_2)\Gamma(\frac{D}{2})} \left(-m_2^2\right)^{\frac{D}{2}-a_1-a_2} \\
&\times \; F_4\left(\begin{matrix} a_1+a_2-\frac{D}{2} & a_1 \\ \frac{D}{2} & 1+a_1-\frac{D}{2} \end{matrix} \middle| \frac{P^2}{m_2^2}, \frac{m_1^2}{m_2^2} \right) \\
G_{3,5} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_1-\frac{D}{2})}{\Gamma(a_1)} \left(-m_1^2\right)^{\frac{D}{2}-a_1} \left(-m_2^2\right)^{-a_2} \\
&\times \; F_4\left(\begin{matrix} \frac{D}{2} & a_2 \\ \frac{D}{2} & 1-a_1+\frac{D}{2} \end{matrix} \middle| \frac{P^2}{m_2^2}, \frac{m_1^2}{m_2^2} \right) \\
G_{2,3} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(\frac{D}{2})} \left(-m_1^2\right)^{\frac{D}{2}-a_1-a_2} \\
&\times \; F_4\left(\begin{matrix} a_1+a_2-\frac{D}{2} & a_2 \\ \frac{D}{2} & 1+a_2-\frac{D}{2} \end{matrix} \middle| \frac{P^2}{m_1^2}, \frac{m_2^2}{m_1^2} \right) \\
G_{3,4} &= (-1)^{-\frac{D}{2}} \frac{\Gamma(a_2-\frac{D}{2})}{\Gamma(a_2)} \left(-m_2^2\right)^{\frac{D}{2}-a_2} \left(-m_1^2\right)^{-a_1} \\
&\times \; F_4\left(\begin{matrix} \frac{D}{2} & a_1 \\ \frac{D}{2} & 1-a_2+\frac{D}{2} \end{matrix} \middle| \frac{P^2}{m_1^2}, \frac{m_2^2}{m_1^2} \right).
\end{aligned}$$

14. Conclusions and future work

The method of brackets provides a very effective procedure to evaluate definite integrals over the interval $[0, \infty)$. The method is based on a heuristic list of rules on the bracket series associated to such integrals. In particular we have provided a variety of examples that illustrate the power of this method. A rigorous validation of these rules as well as a systematic study of integrals from Feynman diagrams is in progress.

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