A RATIONAL LANDEN TRANSFORMATION. THE CASE OF DEGREE SIX.

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Abstract. We prove the existence of a Landen-type transformation for the integral of a rational function. The convergence of its iterates is established.

1. Introduction

The transformation theory of elliptic integrals was initiated by Landen in [5, 4], wherein he proved the invariance of the function

$$G(a,b) \quad := \quad \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \tag{1.1}$$

under the transformation

$$a_1 = (a+b)/2 \qquad\qquad b_1 = \sqrt{ab}, \tag{1.2}$$

i.e. that

$$G(a_1,b_1) \quad = \quad G(a,b). \tag{1.3}$$

Gauss [3] rediscovered this invariance while numerically calculating the length of a lemniscate. An elegant proof of (1.3) is given by Newman in [7]. Here, the substitution $x = b\tan\theta$ converts $2G(a,b)$ into the integral of $\left[(a^2+x^2)(b^2+x^2)\right]^{-1/2}$ over $\mathbb{R}$; the change of variables $t = (x - ab/x)/2$ completes the proof.

The Gauss-Landen transformation can be iterated to produce a double sequence (a_n, b_n) such that $0 \leq a_n - b_n < 2^{-n}$. It follows that a_n and b_n converge to a common limit, the so-called *arithmetic-geometric* mean of a and b, denoted by $AGM(a,b)$. Passing to the limit in $G(a,b) = G(a_n,b_n)$ produces

$$\frac{\pi}{2AGM(a,b)} \quad = \quad \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{1.4}$$

The reader is referred to [2] and [6] for details.

In this paper we develop a *rational Landen transformation.* These are transformations analogous to (1.3) that preserve the integral of a rational function over the positive real line. We have produced such transformations where the integrand is any even rational function. Here we present the details for degree 6.

Define

$$U_6(a,b;c,d,e) \quad := \quad \int_0^{\infty} \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\,dx. \tag{1.5}$$

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Then our main result is:

Theorem 1.1. *Let $a_0, b_0, c_0, d_0, e_0 \in \mathbb{R}^+$ and define*

$$\begin{aligned} a_{n+1} &= \frac{a_nb_n + 5a_n + 5b_n + 9}{(a_n+b_n+2)^{4/3}} \\ b_{n+1} &= \frac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}} \\ c_{n+1} &= \frac{c_n+d_n+e_n}{(a_n+b_n+2)^{2/3}} \\ d_{n+1} &= \frac{(b_n+3)c_n+2d_n+(a_n+3)e_n}{a_n+b_n+2} \\ e_{n+1} &= \frac{c_n+e_n}{(a_n+b_n+2)^{1/3}}. \end{aligned} \tag{1.6}$$

Then U_6 is invariant under this transformation, i.e.

$$U_6(a_n, b_n; c_n, d_n, e_n) = U_6(a_0, b_0; c_0, d_0, e_0). \tag{1.7}$$

Moreover, $(a_n, b_n) \to (3,3)$ and there exists a number L such that $(c_n, d_n, e_n) \to (1,2,1)\,L$. Passing to the limit in (1.7) produces

$$L = \frac{2}{\pi}\int_0^\infty \frac{c_0x^4+d_0x^2+e_0}{x^6+a_0x^4+b_0x^2+1}\,dx. \tag{1.8}$$

The invariance of U_6 under the transformation (1.6) is shown in Section 2, and the convergence of the sequence $(a_n, b_n, c_n, d_n, e_n)$ is established in Section 4.

There exist similar higher-order Landen transformations when the integrand is a rational function of *any* even degree. For example:

Theorem 1.2. *Let $a_0, b_0, c_0, d_0, e_0, f_0, g_0$ be positive real numbers, and define*

$$\begin{aligned} a_{n+1} &= \frac{b_n(a_n+c_n)+4a_nc_n+10(a_n+c_n)+8(b_n+2)}{(a_n+b_n+c_n+2)^{3/2}} \\ b_{n+1} &= \frac{a_nc_n+6(a_n+c_n)+2(b_n+10)}{a_n+b_n+c_n+2} \\ c_{n+1} &= \frac{a_n+c_n+8}{(a_n+b_n+c_n+2)^{1/2}} \\ d_{n+1} &= \frac{d_n+e_n+f_n+g_n}{(a_n+b_n+c_n+2)^{3/4}} \\ e_{n+1} &= \frac{g_n(3a_n+b_n+6)+f_n(a_n+4)+e_n(c_n+4)+d_n(3c_n+b_n+6)}{(a_n+b_n+c_n+2)^{5/4}} \\ f_{n+1} &= \frac{g_n(a_n+5)+f_n+e_n+d_n(c_n+5)}{(a_n+b_n+c_n+2)^{3/4}} \\ g_{n+1} &= \frac{g_n+d_n}{(a_n+b_n+c_n+2)^{1/4}}. \end{aligned}$$

Then

$$U_8(a,b,c;d,e,f,g) := \int_0^\infty \frac{dx^6+ex^4+fx^2+g}{x^8+ax^6+bx^4+cx^2+1}\,dx \tag{1.9}$$

is invariant under this transformation.

Numerical calculations show that $(a_n, b_n, c_n) \to (4, 6, 4)$ and that $(d_n, e_n, f_n, g_n) \to (1, 3, 3, 1)\, L$, with similar patterns involving binomial coefficients for higher-order cases.

The case $a = b$ in the integral U_6 deserves special attention. Here

$$x^6 + ax^4 + ax^2 + 1 = (x^2 + 1)(x^4 + (a - 1)x^2 + 1)$$

so the integral can be evaluated by partial fractions. From (1.6) define the map

$$\Phi_6(a, b) \quad = \quad \left(\frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}}, \frac{a + b + 6}{(a + b + 2)^{2/3}} \right)$$

that transforms (a, b) to (a_1, b_1). Then the preimages of the diagonal $\Delta = \{(a, b) \in \mathbb{R}^+ : a = b\}$ under Φ_6 form a sequence of real algebraic curves $\mathbb{X}_n = \Phi_6^{-n}(\Delta)$ containing the point $(3, 3)$. The first curve $\mathbb{X}_1$ is discussed in Section 3; its defining equation, derived from $a_1 = b_1$, is

$$(ab + 5a + 5b + 9)^3 \quad = \quad (a + b + 2)^2(a + b + 6)^3, \tag{1.10}$$

so that $(3, 3)$ is a cusp. The curves $\mathbb{X}_n$ correspond to the points in the first quadrant for which the integral U_6 can be evaluated in a finite number of steps *without computing the poles of the integrand.* The complexity of the curves $\mathbb{X}_n$ increases dramatically with n. For example, $\mathbb{X}_2$ is of degree 90 with leading term

$$T_2(x, y) = 2^{121}3^{35}(x - y)^{18} \left[-163(x^4 + y^4) + 668xy(x^2 + y^2) - 1074x^2y^2 \right]$$

when written with coordinates $x = a - 3$ and $y = b - 3$ centered at the cusp.

2. THE TRANSFORMATION OF U_6

A polynomial $P_d(x)$ of degree d is called *symmetric* if $P_d(1/x) = x^{-d}P(x)$. A symmetric polynomial $P_d(x)$ is said to be *normalized* if it is monic. For example, the normalized polynomial of degree 6 is $P_6(x) = x^6 + a(x^4 + x^2) + 1$. Similarly, $P_{12}(x) = (x^{12} + 1) + \alpha_3(x^{10} + x^2) + \alpha_2(x^8 + x^4) + 2\alpha_1 x^6$.

The first step in the derivation of the transformation (1.6) is to symmetrize the denominator of the integrand, producing an integral in which the degree of the denominator is double that of the original. We then employ a sequence of elementary substitutions to transform the new integral back to one with denominator the same degree as the original. The explicit formulae (1.6) can be iterated; the convergence of the sequence $(a_n, b_n, c_n, d_n, e_n)$ is discussed in Section 4.

Proposition 2.1. Let $R_4(x) = cx^4 + dx^2 + e$, $Q_6(x) = x^6 + ax^4 + bx^2 + 1$, $R_{10}(x) = R_4(x)(x^6 + bx^4 + ax^2 + 1)$, and let $P_{12}(x)$ be the normalized polynomial of degree 12 with parameters $\alpha_1 = \frac{1}{2}(2 + a^2 + b^2)$, $\alpha_2 = a + b + ab$, and $\alpha_3 = a + b$. Then

$$\int_0^\infty \frac{R_4(x)}{Q_6(x)}\, dx \quad = \quad \int_0^\infty \frac{R_{10}(x)}{P_{12}(x)}\, dx. \tag{2.1}$$

Proof. Observe that $P_{12}(x) = x^6 Q_6(x) Q_6(1/x)$ and $R_{10}(x) = x^6 R_4(x) Q_6(1/x)$. □

Now transform the integral (2.1) using the change of variables $x = \tan\theta$ to produce

$$U_6 \quad = \quad \int_0^{\pi/2} \frac{\sum_{k=0}^{5} r_k \cos^k 2\theta}{\sum_{k=0}^{3} s_{2k} \cos^{2k} 2\theta}\, 2\, d\theta,$$

where $r_0, \cdots, r_5$ and $s_0, \cdots, s_6$ are functions of the parameters $a, \cdots, e$. For example, $r_0 = 2c + ac + bc + 2d + ad + bd + 2e + ae + be$, with similar expressions for the rest of them. Observe that the denominator is an even function of cosine, so the odd powers in the numerator have vanishing integral. Therefore, with $\psi = 2\theta$, we have

$$U_6 \quad = \quad 2\int_0^{\pi/2} \frac{r_4 \cos^4\psi + r_2\cos^2\psi + r_0}{s_6\cos^6\psi + s_4\cos^4\psi + s_2\cos^2\psi + s_0}\, d\psi.$$

Letting $\theta = 2\psi$, we obtain

$$U_6 \quad = \quad \int_0^{\pi} \frac{t_2\cos^2\theta + t_1\cos\theta + t_0}{u_3\cos^3\theta + u_2\cos^2\theta + u_1\cos\theta + u_0}\, d\theta,$$

where $t_2, \cdots, t_0$ and $u_3, \cdots, u_0$ are again functions of the parameters. Finally, the change of variables $y = \tan(\theta/2)$ yields

$$U_6 \quad = \quad \int_0^{\infty} \frac{v_4y^4 + v_2y^2 + v_0}{w_6y^6 + w_4y^4 + w_2y^2 + w_0}\, dy,$$

with $v_4, \cdots, v_0$ and $w_6, \cdots, w_0$ dependent upon $a, \cdots, e$. The last step in the proof of (1.6) is to factor out w_0 and scale y to produce a monic polynomial.

3. A SEQUENCE OF REAL ALGEBRAIC CURVES

In the previous section we showed that the integral

$$U_6(a,b;c,d,e) \quad = \quad \int_0^{\infty} \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx \tag{3.1}$$

can be tranformed into a new integral of the same type with denominator

$$x^6 + \frac{ab + 5a + 5b + 9}{(a+b+2)^{4/3}}x^4 + \frac{a+b+6}{(a+b+2)^{2/3}}x^2 + 1.$$

If the denominator of the transformed integral is symmetric, it factors and so the integral can be evaluated by partial fractions. We therefore have:

Proposition 3.1. Suppose (a,b) is a point in $\mathbb{R}_+^2$ such that $\Phi_6^{(i)}(a,b)$ is on the diagonal $\Delta = \{(x,y) \in \mathbb{R}_+^2 : x = y\}$ for some integer i. Then

$$U_6(a,b;c,d,e) \quad = \quad \int_0^{\infty} \frac{cx^4 + dx^2 + 1}{x^6 + ax^4 + bx^2 + 1}\, dx$$

can be evaluated in a finite number of steps.

Note 1. The curve $\mathbb{X}_1 := \Phi_6^{-1}(\Delta)$ is a *real algebraic curve* containing the point $(3,3)$. The equation for $\mathbb{X}_1$ is

$$(ab + 5a + 5b + 9)^3 \quad = \quad (a+b+2)^2(a+b+6)^3, \tag{3.2}$$

which follows directly from $a_1 = b_1$. When written with coordinates $x = a - 3$ and $y = b - 3$, the leading term of $\mathbb{X}_1$ is $T_1(x,y) = -1728(x-y)^2$, so the point $(x,y) = (0,0)$ corresponding to $(a,b) = (3,3)$ is a cusp.

Proposition 3.2. The curve $\mathbb{X}_1$ is parametrized by

$$\begin{aligned} a(t) &= t^{-2}\left(t^5 - t^4 + 2t^3 - t^2 + t + 1\right) \\ b(t) &= t^{-3}\left(t^5 + t^4 - t^3 + 2t^2 - t + 1\right). \end{aligned} \tag{3.3}$$

Proof. Let $p = ab + 5a + 5b + 9$, $q = a + b + 6$ and $r = a + b + 2$. Then (3.2) can be written as $p = qR^2$ with $R^3 = r$. Thus $a + b = R^3 - 2$ and $ab = R^5 - 5R^3 + 4R^2 + 1$, so that

$$a^2 - (R^3 - 2)a + (R^5 - 5R^3 + 4R^2 + 1) = 0. \tag{3.4}$$

The discriminant of (3.4) is $[TR(R-2)]^2$ with $T = \sqrt{R^2 - 4}$, and the equation $T^2 = R^2 - 4$ can be parametrized by $R(t) = t + t^{-1}$ and $T(t) = t - t^{-1}$. The expressions for a and b in terms of t now follow from solving (3.4).

□

Note 2. The parametrization of $\mathbb{X}_1$ yields the factorization

$$x^6 + ax^4 + bx^2 + 1 = (1 + t^2x^2)(t^{-2}x^4 + t^{-3}(1 + t^2)(1 - t + t^2)x^2 + 1)$$

for $(a, b) \in \mathbb{X}_1$, and the integral U_6 can then be evaluated by partial fractions. The determination of the parameter t from (3.3) for given a and b is, in general, not a solvable problem.

Note 3. The points on $\mathbb{X}_1$ with rational coordinates are obtained from (3.3) with $t \in \mathbb{Q}$. For example, $a(1) = b(1) = 3$ produces the cusp. This point is fixed by the map Φ_6, so it is contained in all the curves $\mathbb{X}_i = \Phi_6^{-i}(\Delta)$.

Note 4. The curves $\mathbb{X}_n$ do not exist in the case of an integrand of degree 8 since the equation $a_1 = c_1$ in Theorem 1.2 yields $a = c$. Thus the transformation of degree 8 cannot be employed to produce symmetric integrands from non-symmetric ones.

4. Analysis of convergence

In this section we discuss the convergence of the recurrence (1.6). We first prove that (a_n, b_n) converges to $(3, 3)$, and then that (c_n, d_n, e_n) converges to limits in proportion to $(1, 2, 1)$.

Theorem 4.1. *Let $a_0 \geq 0$ and $b_0 \geq 0$. Then the sequence (a_n, b_n) defined in (1.6) converges to $(3, 3)$.*

Proof. It suffices to prove that

$$(a_1 - 3)^2 + (b_1 - 3)^2 \leq \frac{1}{2}\left[(a_0 - 3)^2 + (b_0 - 3)^2\right], \tag{4.1}$$

since iterating this inequality produces

$$\left[(a_n - 3)^2 + (b_n - 3)^2\right] \leq 2^{-n}\left[(a_0 - 3)^2 + (b_0 - 3)^2\right]$$

and we then have geometric convergence to $(3, 3)$.

The inequality (4.1) is equivalent to

$$\begin{aligned} f(a,b) &= (a+b+2)^{8/3}(a^2+b^2-6a-6b-18)+2(a+b+2)^{4/3}(4ab+18a+18b+18-a^2-b^2)+ \\ &\quad +2(6a^3+6b^3+8a^2b+8ab^2+35a^2+35b^2-a^2b^2+78a+78b+52ab+63)\geq 0, \end{aligned}$$

and we need to prove that $f(a,b)$ has an absolute minimum of 0 at $(3,3)$. Note that $f(a,b) = f(b,a)$, so we may restrict the analysis to the region

$$\Omega \quad = \quad \{(a,b) \in \mathbb{R}^2_+ : a \geq b\}. \tag{4.2}$$

Introduce the new variables $x = (a+b+2)^{1/3}$ and $y = ab$, and write $h(x,y)$ for $f(a,b)$. The region Ω is then transformed into

$$\Omega^* \quad = \quad \{(x,y) \in \mathbb{R}^2_+ : x \geq \sqrt[3]{2} \text{ and } 0 \leq y \leq \left(1 - x^3/2\right)^2\},$$

and in terms of the new variables, we need to prove that

$$\begin{aligned} h(x,y) \quad &= \quad x^{14} - 10x^{11} - 2x^{10} + 12x^9 - 2x^8(y+1) + 44x^7 - \\ &\quad -2x^6 + 4x^4(3y-11) - 20x^3(y-1) - 2(y-1)^2 \geq 0 \end{aligned}$$

for $(x,y) \in \Omega^*$. ☐

Lemma 4.2. *The function h has no critical points in the interior of Ω^*.*

Proof. We have

$$\begin{aligned} h_x(x,y) \quad &= \quad 14x^{13} - 110x^{10} - 20x^9 + 108x^8 - 16x^7(y+1) - \\ &\quad -308x^6 - 12x^5 + 16x^3(3y-11) - 60x^2(y-1), \\ h_y(x,y) \quad &= \quad -2(2y + x^8 - 6x^4 + 10x^3 - 2). \end{aligned}$$

Eliminating y from $h_x(x,y) = 0$, $h_y(x,y) = 0$ yields $2x^3 g(x) = 0$, where

$$g(x) \quad = \quad 4x^{12} + 7x^{10} - 36x^8 - 10x^6 + 54x^5 + 56x^4 - 56x^3 + 144x^2 - 64.$$

The function g has no roots for $x \geq 1$ (in particular for $x \geq \sqrt[3]{2}$), which can immediately be seen by expanding g in terms of $x-1$:

$$\begin{aligned} g(x) \quad &= \quad 4(x-1)^{12} + 48(x-1)^{11} + 271(x-1)^{10} + 950(x-1)^9 + \\ &\quad +2259(x-1)^8 + 3720(x-1)^7 + 4148(x-1)^6 + 2910(x-1)^5 + \\ &\quad +1106(x-1)^4 + 212(x-1)^3 + 273(x-1)^2 + 384(x-1) + 99. \end{aligned}$$

☐

Lemma 4.3. *The minimum value of h is 0 and occurs at $x = 2$.*

Proof. Along the line $y = 0$, $x \geq \sqrt[3]{2}$, we have

$$h(x,0) \quad = \quad x^{14} - 10x^{11} - 2x^{10} + 12x^9 - 2x^8 + 44x^7 - 2x^6 - 44x^4 + 20x^3 - 2,$$

and expanding in powers of $x-1$ we obtain

$$\begin{aligned} h(x,0) \quad &= \quad (x-1)^{14} + 14(x-1)^{13} + 91(x-1)^{12} + 354(x-1)^{11} + 889(x-1)^{10} + \\ &\quad +1444(x-1)^9 + 1369(x-1)^8 + 352(x-1)^7 - 779(x-1)^6 - 810(x-1)^5 + \\ &\quad +119(x-1)^4 + 714(x-1)^3 + 517(x-1)^2 + 156(x-1) + 15. \end{aligned}$$

Although there are two terms with negative coefficients in this expansion, it is easy to majorize each of them by a higher-power term so that $h(x,0) \geq 15$. Along the curve $y = (1 - x^3/2)^2$, $x \geq \sqrt[3]{2}$, we have

$$\begin{aligned} h(x,(1-x^3/2)^2) \quad &= \quad \tfrac{1}{8}x^4(x-2)^2 \times (4(x-1)^8 + 48(x-1)^7 + 271(x-1)^6 + 902(x-1)^5 + \\ &\quad +1905(x-1)^4 + 2628(x-1)^3 + 2289(x-1)^2 + 1062(x-1) + 107), \end{aligned}$$

which has an absolute minimum of 0 at $x = 2$ as claimed. ☐

This completes the proof of Theorem 4.1.

Theorem 4.4. *Let a_0, b_0, c_0, d_0, e_0 be nonnegative real numbers with $c_0 d_0 e_0 > 0$. Then the sequence (c_n, d_n, e_n) defined in (1.6) converges to a limit (c, d, e) that satisfies $c = e$ and $d = 2c$.*

Proof. Let $A_n = (a_n + b_n + 2)^{1/3}$, and define $\epsilon_1 = A_n - 2$, $\epsilon_2 = A_n^2 - 4$, $\epsilon_3 = a_n - 3$, and $\epsilon_4 = b_n - 3$. Observe that ϵ_i can be of any sign and that $\epsilon_i \to 0$ as $n \to \infty$. ☐

Lemma 4.5. *The sequences c_n, d_n, e_n are bounded from above.*

Proof. The identity

$$I := \int_0^\infty \frac{c_0 x^4 + d_0 x^2 + e_0}{x^6 + a_0 x^4 + b_0 x^2 + 1}\, dx \quad = \quad \int_0^\infty \frac{c_n x^4 + d_n x^2 + e_n}{x^6 + a_n x^4 + b_n x^2 + 1}\, dx \tag{4.3}$$

shows that

$$I \quad \geq \quad c_n \int_0^\infty \frac{x^4}{x^6 + a_n x^4 + b_n x^2 + 1}\, dx,$$

and the integral on the right-hand side is bounded from below because a_n and b_n converge to 3. Thus c_n is bounded from above, and similarly, d_n, e_n are bounded from above. ☐

Lemma 4.6. *There exists $\delta > 0$ such that $6c_n + 2d_n + 6e_n > \delta$.*

Proof. Let $r(x) = x^6 + a_n x^4 + b_n x^2 + 1$ and define

$$\alpha \quad := \quad \max_n \left\{ \int_0^\infty \frac{x^4\, dx}{r(x)}, \int_0^\infty \frac{x^2\, dx}{r(x)}, \int_0^\infty \frac{dx}{r(x)} \right\}.$$

Then $\alpha > 0$ since $a_n, b_n \to 3$, and (4.3) yields $I < 2\alpha(6c_n + 2d_n + 6e_n)$. ☐

Lemma 4.7. *We have* $\lim\limits_{n\to\infty} \frac{c_n + e_n}{d_n} = 1$.

Proof. Start with

$$\begin{aligned} \frac{c_{n+1} + e_{n+1}}{d_{n+1}} &= \frac{A_n(c_n + d_n + e_n) + A_n^2(c_n + e_n)}{(b_n + 3)c_n + 2d_n + (a_n + 3)e_n} \\ &= \frac{1}{1 + (\epsilon_4 c_n + \epsilon_3 e_n)/(6c_n + 2d_n + 6e_n)} + \\ &\quad + \frac{\epsilon_1(c_n + d_n + e_n) + \epsilon_2(c_n + e_n)}{(6 + \epsilon_4)c_n + 2d_n + (6 + \epsilon_3)e_n}. \end{aligned}$$

Now, since c_n, d_n, e_n are bounded from above,

$$\frac{|\epsilon_4 c_n + \epsilon_3 e_n|}{6c_n + 2d_n + 6e_n} \quad < \quad (|\epsilon_3| + |\epsilon_4|)M/\delta, \tag{4.4}$$

where $M = \max\{c_n, d_n, e_n\}$.
Assuming (without loss of generality) that $\epsilon_3, \epsilon_4 > -1$, we thus have

$$\begin{aligned} \frac{|\epsilon_1(c_n + d_n + e_n) + \epsilon_2(c_n + e_n)|}{|(6 + \epsilon_4)c_n + 2d_n + (6 + \epsilon_3)e_n)|} &< \frac{|\epsilon_1(c_n + d_n + e_n) + \epsilon_2(c_n + d_n)|}{5c_n + 2d_n + 5e_n} \\ &< (|\epsilon_1| + |\epsilon_2|) \times \frac{6M}{\delta}. \end{aligned}$$

□

Lemma 4.8. *We also have* $\lim\limits_{n\to\infty} \frac{c_n}{e_n} = 1$ *and* $\lim\limits_{n\to\infty} \frac{d_n}{e_n} = 2$.

Proof. Since

$$\frac{c_{n+1}}{e_{n+1}} = \frac{c_n + d_n + e_n}{(2+\epsilon_1)(c_n+e_n)} = \frac{1}{2+\epsilon_1} + \frac{d_n}{(2+\epsilon_1)(c_n+e_n)},$$

the conclusion follows from Lemma 4.7. □

It remains to check that the sequence c_n converges, from which the convergence of d_n and e_n follow. Observe that

$$I = \int_0^\infty \frac{c_n x^4 + d_n x^2 + e_n}{x^6 + a_n x^4 + b_n x^2 + 1}\, dx$$

is independent of n. Thus

$$c_n = I \times \left(\int_0^\infty \frac{x^4 + d_n x^2/c_n + e_n/c_n}{x^6 + a_n x^4 + b_n x^2 + 1}\, dx \right)^{-1}$$

converges in view of the lemmas established above. This completes the proof of Theorem 4.4.

Note 5. Numerical calculations with the scheme (1.6) show quadratic convergence. For example, the sequence $(a_n, b_n, c_n, d_n, e_n)$ for the evaluation of

$$\int_0^\infty \frac{45x^4 + 25000x^2 + 1230}{x^6 + x^4 + 3000x^2 + 1}\, dx$$

is shown below:

n	a_n	b_n	c_n	d_n	e_n
0	1	3000	45	25000	1230
1	.415786	14.4465	126.233	63.2884	88.3741
2	2.06562	3.17262	42.2607	156.015	83.6896
3	2.98142	3.00338	75.3541	137.717	65.1111
4	2.99999	3.	69.6338	139.925	70.2771
5	3.	3.	69.9589	139.914	69.9555
6	3.	3.	69.9572	139.914	69.9572
7	3.	3.	69.9572	139.914	69.9572

Therefore $L \sim 69.9572$ and

$$\int_0^\infty \frac{45x^4 + 25000x^2 + 1230}{x^6 + x^4 + 3000x^2 + 1}\, dx \sim 69.9572 \times \frac{\pi}{2} \sim 109.889.$$

5. Conclusions

We have produced a Landen transformation for the integral of a rational function and proved convergence of its iterates.
The bibliography also includes [1].

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