

The integrals in Gradshteyn and Ryzhik. Part 5: Some trigonometric integrals

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ABSTRACT. We present evaluations and provide proofs of definite integrals involving the function $x^p \cos^n x$. These formulae are generalizations of 3.761.11 and 3.822.1, among others, in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

1. Introduction

The table of integrals [4] contains a large variety of evaluations of the type

$$(1.1) \quad I = \int_a^b A(x)R(\sin x, \cos x) dx$$

where A is an algebraic function, R is rational and $-\infty \leq a < b \leq \infty$. We present a systematic discussion of two families of integrals of this type. This paper is part of a general program started in [9, 10, 11, 12] intended to provide proofs and context to the formulas in [4].

The first class considered here corresponds to the complete integrals

$$(1.2) \quad c(n, p) := \int_0^{\pi/2} x^p \cos^n x dx,$$

and

$$(1.3) \quad s(n, p) := \int_0^{\pi/2} x^p \sin^n x dx,$$

where $n, p \in \mathbb{N}$. In section 2 we present closed-form expressions for these integrals. These expressions involve the sums

$$(1.4) \quad \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \dots k_j^2},$$

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that are closely related to the multiple zeta values

$$(1.5) \quad \zeta(i_1, i_2, \dots, i_s) = \sum_{0 < k_1 < k_2 < \dots < k_s} \frac{1}{k_1^{i_1} k_2^{i_2} \dots k_s^{i_s}}.$$

The reader will find in Section 3.4 of [3] an introduction to these sums.

In general, one does not expect such elementary evaluations to extend to $p \notin \mathbb{N}$. For example, the change of variables $x = \pi t^2/2$ produces

$$(1.6) \quad \int_0^{\pi/2} x^{-1/2} \cos x \, dx = \sqrt{2\pi} \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt.$$

The latter integral is evaluated in terms of the *cosine Fresnel* function

$$(1.7) \quad \text{FresnelC}[x] := \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt,$$

which indeed is not an elementary function.

The second class considered here presents generalizations of the formula **3.822.1** in [4] stated as

$$(1.8) \quad \int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \quad n \in \mathbb{N}.$$

The integral in (1.8) can be transformed via $t = x^2$ to provide the evaluation of

$$(1.9) \quad \int_0^\infty \cos^{2n+1} t^2 dt,$$

that is given as the case $p = 2$ in Theorem 3.2.

Section 3 contains analytic expressions for the generalizations

$$(1.10) \quad C_n(p, b) := \int_0^\infty x^{-p} \cos^{2n+1}(x+b) dx,$$

and

$$(1.11) \quad S_n(p, b) := \int_0^\infty x^{-p} \sin^{2n+1}(x+b) dx.$$

The last section also contains some evaluations obtained by differentiation with respect to parameters. An illustrative example is

$$(1.12) \quad \int_0^\infty \int_0^\infty \frac{\log x \log y}{\sqrt{xy}} \cos(x+y) dx dy = (\gamma + 2 \log 2) \pi^2,$$

that is equivalent to

$$(1.13) \quad \int_0^\infty \int_0^\infty \log x \log y \cos(x^2 + y^2) dx dy = \frac{1}{16} (\gamma + 2 \log 2) \pi^2.$$

A generalization of this evaluation appears as Example 3.3.

The method described in the present work gives impetus to a class of integrals that are closely related to the particular integral computations addressed in this paper.

2. The first example

In this section we present the evaluation in closed-form of the definite integrals

$$(2.1) \quad c(n, p) := \int_0^{\pi/2} x^p \cos^n x \, dx.$$

A special case of this appears as **3.822.1** in [4].

The first step towards the evaluation of $c(n, p)$ is to produce a recurrence.

Theorem 2.1. The integral $c(n, p)$ satisfies the recurrence

$$(2.2) \quad c(n, p) = \frac{n-1}{n} c(n-2, p) - \frac{p(p-1)}{n^2} c(n, p-2),$$

for $n \geq 2, p \geq 2$.

PROOF. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$(2.3) \quad c(n, p) = c(n-2, p) - I(n, p)$$

where

$$I(n, p) := \int_0^{\pi/2} x^p \cos^{n-2} x \sin^2 x \, dx.$$

Now

$$\begin{aligned} I(n, p) &= \int_0^{\pi/2} x^p \sin x \times \frac{d}{dx} \left(-\frac{1}{n-1} \cos^{n-1} x \right) dx \\ &= \frac{1}{2n-1} \int_0^{\pi/2} (x^p \cos x + px^{p-1} \sin x) \cos^{n-1} x \, dx \\ &= \frac{c(n, p)}{n-1} + \frac{p}{n-1} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \, dx. \end{aligned}$$

Moreover

$$\begin{aligned} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \, dx &= \int_0^{\pi/2} x^{p-1} \frac{d}{dx} \left(-\frac{1}{n} \cos^n x \right) dx \\ &= \frac{p-1}{n} c(n, p-2). \end{aligned}$$

□

Strategy: According to (2.2), the integral $c(n, p)$ can be evaluated in terms of the initial values given in the table. The indices m and q have the same parity as n and p respectively and range over $0 \leq m \leq n$ and $0 \leq q \leq p$.

n modulo 2	p modulo 2	initial conditions
0	0	$c(m, 0) \quad c(0, q)$
1	0	$c(m, 0) \quad c(1, q)$
0	1	$c(m, 1) \quad c(0, q)$
1	1	$c(m, 1) \quad c(1, q)$

We now evaluate the initial conditions $c(n, 0)$, $c(n, 1)$, $c(0, p)$ and $c(1, p)$.

The expression for $c(0, p)$.

The computation of the identity

$$(2.4) \quad c(0, p) = \frac{1}{p+1} \left(\frac{\pi}{2}\right)^{p+1}$$

is immediate.

The expression for $c(n, 0)$.

This is classical. The result appears as **3.621.3** and **3.621.4** in [4].

Theorem 2.2. (Wallis' formula and companion). Let $n \in \mathbb{N}_0$. Then

$$(2.5) \quad c(2n, 0) = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

and

$$(2.6) \quad c(2n+1, 0) = \frac{2^{2n}}{(2n+1) \binom{2n}{n}}.$$

The shortest proof of Theorem 2.2 employs the representation

$$(2.7) \quad c(n, 0) = \int_0^{\pi/2} \cos^n x \, dx = 2^{n-1} B\left(\frac{n+1}{2}, \frac{n+1}{2}\right),$$

that appears as **3.621.1** in [4]. Here B is the *Euler's beta function* defined by the integral

$$(2.8) \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

The expression (2.7) follows from the change of variables $t = \cos u$. To express (2.5) and (2.6), in terms of the beta function, employ the standard relation

$$(2.9) \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and the special values

$$(2.10) \quad \Gamma(n) = (n-1)! \text{ and } \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi} (2n)!}{2^{2n} n!}$$

that are valid for $n \in \mathbb{N}$.

The identity in Theorem 2.2, in the case n is even, that is,

$$(2.11) \quad c(2n, 0) = \int_0^{\pi/2} \cos^{2n} \theta \, d\theta = \frac{\pi}{2^{2n+1}} \binom{2n}{n},$$

is Wallis's formula and sometimes found in calculus books (see e.g. [6], page 492). To prove it, first write $\cos^2 \theta = 1 - \sin^2 \theta$ and use integration by parts to obtain the recursion

$$(2.12) \quad c(2n, 0) = \frac{2n-1}{2n} c(2n-2, 0).$$

Then verify that the right side of (2.11) satisfies the same recurrence together with the initial value $\pi/2$ for $n = 0$.

We now present a new proof of Wallis's formula (2.11) in the context of rational integrals. Extensions of the ideas in this proof have produced *rational Landen transformations*. The reader will find in [1, 2, 5, 7, 8] details on these transformations.

Start with

$$c(2n, 0) = \int_0^{\pi/2} \cos^{2n} \theta \, d\theta = \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^n \, d\theta.$$

Now introduce $\psi = 2\theta$ and expand and simplify the result by observing that the odd powers of cosine integrate to zero. The inductive proof of (2.11) requires

$$(2.13) \quad c(2n, 0) = 2^{-n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} c(2i, 0).$$

Note that $c(2n, 0)$ is uniquely determined by (2.13) along with the initial value $c(0, 0) = \pi/2$. Thus (2.11) now follows from the identity

$$(2.14) \quad f(n) := \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-2i} \binom{n}{2i} \binom{2i}{i} = 2^{-n} \binom{2n}{n}.$$

We now provide a mechanical proof of (2.14) using the theory developed by Wilf and Zeilberger, which is explained in [13, 14]; the sum in (2.14) is the example used in [14] (page 113) to illustrate their method. The command

$$ct(\text{binomial}(n, 2i) \text{binomial}(2i, i) 2^{-2i}, 1, i, n, N)$$

produces

$$(2.15) \quad f(n+1) = \frac{2n+1}{n+1} f(n),$$

and one checks that $2^{-n} \binom{2n}{n}$ satisfies this recursion. Note that (2.12) and (2.15) are equivalent under

$$c(2n, 0) = \frac{\pi}{2^{n+1}} f(n).$$

The proof is complete.

Closed form expression for $c(1, p)$.

We now consider the evaluation of

$$(2.16) \quad c(1, p) := \int_0^{\pi/2} x^p \cos x \, dx.$$

The following evaluation appears as **3.761.11** in [4].

Theorem 2.3. Let $p \in \mathbb{N}$ and $\delta_{\text{odd}, p}$ be Kronecker's delta function at the odd integers. Then

$$(2.17) \quad c(1, p) = \sum_{k=0}^{\xi_p} (-1)^k \frac{p!}{(p-2k)!} \left(\frac{\pi}{2} \right)^{p-2k} - (-1)^{\xi_p} \delta_{\text{odd}, p} p!$$

where $\xi_p = \lfloor \frac{p}{2} \rfloor$.

PROOF. Both sides of the equation (2.17) satisfy the initial value problem

$$(2.18) \quad u_p - p(p-1)u_{p-2} = \left(\frac{\pi}{2}\right)^p \quad \text{and} \quad u_0 = 1, \quad u_1 = \frac{\pi-2}{2}.$$

Actually the recurrence (2.18) is obtained using integration by parts in (2.16). Iterating this recurrence yields the right hand side of (2.17). \square

Note 2.4. The result in Theorem 2.3 can be expressed in terms of the Taylor polynomial for $\cos x$:

$$(2.19) \quad f_p(x) = (-1)^{\xi_p} p! \left(-1 + \sum_{k=0}^{\xi_p} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right).$$

The formula (2.17) can be restated

$$(2.20) \quad c(1, p) = \begin{cases} f_p(\pi/2), & \text{for } p \text{ odd,} \\ f'_p(\pi/2), & \text{for } p \text{ even.} \end{cases}$$

Closed form expression for $c(n, 1)$: in fact, this would be the last initial condition we require to execute the strategy outlined at the beginning of this section.

Theorem 2.5. The integral $c(n, 1)$ satisfies the recurrence

$$(2.21) \quad c(n, 1) = \frac{n-1}{n} c(n-2, 1) - \frac{1}{n^2}.$$

PROOF. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$(2.22) \quad c(n, 1) = c(n-2, 1) - J,$$

where

$$(2.23) \quad J = \int_0^{\pi/2} x \sin^2 x \cos^{n-2} x \, dx.$$

Integration by parts leads to

$$(2.24) \quad J = \frac{1}{n-1} \int_0^{\pi/2} (\sin x + x \cos x) \cos^{n-1} x \, dx.$$

This produces (2.21). \square

The solution of (2.21) yields a closed-form formula for $c(n, 1)$.

Theorem 2.6. The integral $c(n, 1)$ is given according to the parity of n , by

$$(2.25) \quad c(2n, 1) = \frac{\binom{2n}{n}}{2^{2n+2}} \left(\frac{\pi^2}{2} - \sum_{k=1}^n \frac{2^{2k}}{k^2 \binom{2k}{k}} \right),$$

for even indices. For odd indices, we have

$$(2.26) \quad c(2n+1, 1) = \frac{2^{2n}}{(2n+1) \binom{2n}{n}} \left(\frac{\pi}{2} - \sum_{k=0}^n \frac{\binom{2k}{k}}{2^{2k} (2k+1)} \right).$$

To establish this result we solve a more general recurrence than (2.21).

Lemma 2.7. Let a_n, b_n and r_n be sequences with $a_n, b_n \neq 0$. Assume that z_n satisfies

$$(2.27) \quad a_n z_n = b_n z_{n-1} + r_n, \quad n \geq 1$$

with initial condition z_0 . Then

$$(2.28) \quad z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left(z_0 + \sum_{k=1}^n \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right).$$

PROOF. Introduce the integrating factor d_n with the property that $d_n b_n = d_{n-1} a_{n-1}$. The recurrence (2.27) becomes

$$(2.29) \quad D_n - D_{n-1} = d_n r_n,$$

where $D_n = d_n a_n z_n$. Therefore, by telescoping,

$$(2.30) \quad D_n = D_0 + \sum_{k=1}^n d_k r_k,$$

with $D_0 = d_0 a_0 z_0$. To find the integrating factor, observe that

$$(2.31) \quad \frac{d_n}{d_{n-1}} = \frac{a_{n-1}}{b_n}.$$

Thus

$$(2.32) \quad d_n = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n} d_0.$$

Replacing in (2.30) yields (2.28). \square

Corollary 2.8. Let $n \in \mathbb{N}$ and assume that z_n satisfies

$$(2.33) \quad 2n z_n = (2n-1) z_{n-1} + r_n, \quad n \geq 1,$$

with the initial condition z_0 . Let $\lambda_j = 2^{2j} \binom{2j}{j}^{-1}$, then

$$(2.34) \quad z_n = \frac{1}{\lambda_n} \left(z_0 + \sum_{k=1}^n \frac{\lambda_k r_k}{2k} \right).$$

PROOF. Use $a_n = 2n$ and $b_n = 2n-1$ in Lemma 2.7. \square

We now apply Lemma 2.7 on the recurrence (2.21), repeated here for convenience to the reader,

$$c(n, 1) = \frac{n-1}{n} c(n-2, 1) - \frac{1}{n^2}.$$

Observe that this recurrence splits naturally into even and odd branches. The value of $c(2n, 1)$ is determined completely by $c(0, 1)$, and $c(2n+1, 1)$ by $c(1, 1)$. Hence, there is no computational interaction between $c(2n, 1)$ and $c(2n+1, 1)$. Let $x_n = c(2n, 1)$ so that x_n satisfies

$$(2.35) \quad 2n x_n = (2n-1) x_{n-1} - \frac{1}{4n},$$

with the initial condition

$$(2.36) \quad x_0 = c(0, 1) = \frac{\pi^2}{8}.$$

Similarly, $y_n = c(2n + 1, 1)$, the odd component of $c(n, 1)$, satisfies

$$(2.37) \quad (2n + 1)y_n = 2ny_{n-1} - \frac{1}{2n + 1}$$

and the initial condition

$$(2.38) \quad y_0 = c(1, 1) = \frac{\pi}{2} - 1.$$

The expressions for z_n in Lemma 2.7 yield the formulas for $c(2n, 1)$ and also $c(2n + 1, 1)$ in Theorem 2.6. The proof is complete.

Note 2.9. The finite sums in (2.25) and (2.26) do not have closed-form, but it is a classical result that, in the limit,

$$(2.39) \quad \sum_{k=1}^{\infty} \frac{2^{2k}}{k^2 \binom{2k}{k}} = \frac{\pi^2}{2}$$

and

$$(2.40) \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}(2k+1)} = \frac{\pi}{2}.$$

Note 2.10. Formula **3.821.3** in [4] gives formulas equivalent to (2.25) and (2.26), respectively.

Finally, we conclude this section by presenting the sought for closed form expression for the integral $c(n, p)$, for arbitrary $n, p \in \mathbb{N}$. The recurrence (2.2), in the case of even indices n , becomes

$$(2.41) \quad 2nX_n(p) = (2n - 1)X_{n-1}(p) - \frac{p(p-1)}{2n}X_n(p-2)$$

where $X_n(p) = c(2n, p)$. The initial value

$$(2.42) \quad X_0(p) = \frac{1}{(p+1)2^{p+1}}\pi^{p+1}$$

given in (2.4) and the recurrence (2.41) show the existence of rational numbers $a_{n,p,p+1-2j}$ such that

$$(2.43) \quad X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j},$$

with $\xi_p = \lfloor \frac{p}{2} \rfloor$. The recurrence (2.41) is now expanded as

$$(2.44) \quad 2n \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} = (2n-1) \sum_{j=0}^{\xi_p} a_{n-1,p,p+1-2j} \pi^{p+1-2j} \\ - \frac{p(p-1)}{2n} \sum_{j=0}^{\xi_p-1} a_{n,p-2,p-1-2j} \pi^{p-1-2j}.$$

The fact that the coefficients $a_{n,p,j} \in \mathbb{Q}$ allows us to match the corresponding powers of π in (2.44). The highest order term is π^{p+1} . Only two of the sums contain this power, therefore

$$(2.45) \quad 2na_{n,p,p+1} = (2n-1)a_{n-1,p,p+1}.$$

The initial condition

$$(2.46) \quad a_{0,p,p+1} = \frac{1}{(p+1)2^{p+1}}$$

comes from (2.42). The solution to the initial value problem (2.45, 2.46) is then found using Corollary 2.8 (here $r_n = 0$), namely that

$$(2.47) \quad a_{n,p,p+1} = \frac{\binom{2n}{n}}{(p+1)2^{2n+p+1}}.$$

The coefficient of the next highest power π^{p-1} , in (2.44), yields the recurrence

$$(2.48) \quad 2na_{n,p,p-1} = (2n-1)a_{n-1,p,p-1} - \frac{p(p-1)}{2n} a_{n,p-2,p-1}.$$

Observe that the last term in this relation is given by (2.47). Moreover, (2.42) shows that $a_{0,p,p-1} = 0$. The solution to (2.48), following Corollary 2.8, is

$$(2.49) \quad a_{n,p,p-1} = -\frac{p \binom{2n}{n}}{2^{2n+p+1}} \sum_{k_1=1}^n \frac{1}{k_1^2}.$$

The next power of π in (2.44) produces

$$(2.50) \quad 2na_{n,p,p-3} = (2n-1)a_{n-1,p,p-3} + \frac{p(p-1)(p-2)}{n2^{2n+p}} \binom{2n}{n} \sum_{k_1=1}^n \frac{1}{k_1^2},$$

with $a_{0,p,p-3} = 0$. One more use of Corollary 2.8 yields

$$(2.51) \quad a_{n,p,p-3} = \frac{\binom{2n}{n} p!}{2^{2n+p+1} (p-3)!} \sum_{k_2=1}^n \sum_{k_1=1}^{k_2} \frac{1}{k_1^2 k_2^2}.$$

This procedure can be repeated until all descending powers of π have been exhausted, hence a complete closed form for the integrals $c(n,p)$ will be made possible.

Theorem 2.11. Let $n, p \in \mathbb{N}$ and let $\xi_p = \lfloor \frac{p}{2} \rfloor$. Then the even branches $X_n(p) = c(2n, p)$ of the integral

$$(2.52) \quad c(n, p) = \int_0^{\pi/2} x^p \cos^n x \, dx$$

are given by

$$(2.53) \quad X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} + \delta_{\text{odd},p} \cdot a_{n,p}^*,$$

and the value of $a_{n,p,p+1-2j}$ for $p \geq 2$ and $0 \leq j \leq \xi_p$ is given by

$$a_{n,p,p+1-2j} = \frac{(-1)^j \binom{2n}{n} p!}{2^{2n+p+1} (p+1-2j)!} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \dots k_j^2},$$

and

$$(2.54) \quad a_{n,p}^* = \frac{(-1)^{\xi_p} \binom{2n}{n} p!}{2^{2n}} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_p \leq n} \frac{1}{k_1^2 k_2^2 \dots k_p^2} \sum_{j=1}^{k_p} \frac{2^{2j}}{j^2 \binom{2j}{j}}.$$

Similarly, for the odd branches $Y(n, p) = c(2n+1, p)$ we have

$$(2.55) \quad Y_n(p) = \sum_{j=0}^{\xi_p} b_{n,p,p-2j} \pi^{p-2j} + \delta_{\text{odd},p} \cdot b_{n,p}^*,$$

with

$$b_{n,p,p-2j} = \frac{(-1)^j p! 2^{2n+2j-p}}{(2n+1) \binom{2n}{n} (p-2j)!} \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_j \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \dots (2k_j+1)^2},$$

and

$$b_{n,p}^* = \frac{(-1)^{\xi_p} p! 2^{2n}}{(2n+1) \binom{2n}{n}} \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_p \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \dots (2k_p+1)^2} \sum_{j=0}^{k_p} \frac{\binom{2j}{j}}{2^{2j} (2j+1)}.$$

3. Some examples on the halfline

In this section we provide an analytic expression for

$$(3.1) \quad C_n(p, b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) \, dx,$$

and

$$(3.2) \quad S_n(p, b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b) \, dx.$$

In the table [4] the evaluation of the special case $p = \frac{1}{2}$ and $b = 0$:

$$(3.3) \quad \int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} \, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}},$$

and

$$(3.4) \quad \int_0^\infty \frac{\sin^{2n+1} x}{\sqrt{x}} dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}},$$

as **3.822.2** and **3.821.14**.

Theorem 3.1. Let $0 < p < 1$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then

$$(3.5) \quad \int_0^\infty x^{-p} \cos^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \sin\left(\frac{\pi p}{2}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}},$$

and

$$(3.6) \quad \int_0^\infty x^{-p} \sin^{2n+1} x dx = \frac{\Gamma(1-p)}{2^{2n}} \cos\left(\frac{\pi p}{2}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}.$$

PROOF. The identity $2 \cos x = e^{ix} + e^{-ix}$ and the binomial theorem yield

$$(3.7) \quad \int_0^\infty x^{-p} \cos^{2n+1} x dx = 2^{-2n-1} \sum_{k=0}^n \binom{2n+1}{k} \int_0^\infty x^{-p} \left(e^{i(2n+1-2k)x} + e^{-i(2n+1-2k)x} \right) dx.$$

Recall the Heaviside step function defined by $H(x) = 1$, if $x > 0$ and $H(x) = 0$ otherwise. Then, each of the integrals in (3.7) is evaluated using the Fourier transform

$$(3.8) \quad \int_{-\infty}^\infty H(x) x^{-p} e^{-i\omega x} dx = \frac{\Gamma(1-p)}{|\omega|^{1-p}} \exp(-i\pi(1-p)\text{sign}(\omega)/2).$$

□

Corollary 3.2. Let $p > 1$ be real and $n \in \mathbb{N}_0$. Then

$$(3.9) \quad \int_0^\infty \cos^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \cos\left(\frac{\pi}{2p}\right) \sum_{k=0}^n \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}},$$

and

$$(3.10) \quad \int_0^\infty \sin^{2n+1} x^p dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \sin\left(\frac{\pi}{2p}\right) \sum_{k=0}^n (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}.$$

PROOF. The change of variables $x \mapsto x^{1/(1-p)}$ in the results of Theorem 3.1 gives the result. □

The last result described here is a further generalization of Theorem 3.1.

Theorem 3.3. Assume $b \in \mathbb{R}$, $0 < p < 1$ and $n \in \mathbb{N}_0$. Define

$$(3.11) \quad C_n(p, b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) dx$$

and

$$(3.12) \quad S_n(p, b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b) dx.$$

Then

$$(3.13) \quad C_n(p, b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}},$$

and

$$(3.14) \quad S_n(p, b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} \frac{\cos(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}.$$

PROOF. Denote the left-hand side of (3.13) and (3.14) by $f_n(b)$ and $g_n(b)$ respectively. Differentiation with respect to the parameter b yields

$$(3.15) \quad \begin{aligned} \frac{\partial g_n}{\partial b} - (-1)^n (2n+1) f_n &= (2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} f_j(b) \\ \frac{\partial f_n}{\partial b} + (-1)^n (2n+1) g_n &= -(2n+1) \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} g_j(b). \end{aligned}$$

Considering b and p fixed, we now show that the right-hand side of (3.13) and (3.14) satisfy the system (3.15) with the same initial conditions. This will establish the result.

In the case of the right-hand side of (3.13), it is required to check the identity

$$\begin{aligned} &2^{-2n} \sum_{k=0}^n (-1)^k \binom{2n+1}{n-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}} = \\ &(2n+1) \sum_{j=0}^n (-1)^j \binom{n}{j} 2^{-2j} \sum_{k=0}^j \binom{2j+1}{j-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2j+1)^{1-p}}. \end{aligned}$$

To verify this we compare the coefficients of the transcendental terms

$$\frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}.$$

It turns out that this question is equivalent to validating the identity

$$(3.16) \quad (-1)^k 2^{-2n} \binom{2n+1}{n-k} (2k+1) = (2n+1) \sum_{j=k}^n (-1)^j 2^{-2j} \binom{n}{j} \binom{2j+1}{j-k}$$

To this end, we employ the WZ-technology as explained in [14]. This method produces the recurrence

$$(3.17) \quad 2(n+k+1)(n+1-k)u(n+1-k) - (n+1)(2n+3)u(n, k) = 0.$$

To prove (3.16) simply check that both sides of (3.16) satisfy the recurrence (3.17) as well as the initial condition $u(0, 0) = 1$.

The identities

$$(3.18) \quad \begin{aligned} \int_0^\infty x^{-p} \cos(x+b) dx &= -\Gamma(1-p) \sin(b - \frac{p\pi}{2}) \\ \int_0^\infty x^{-p} \sin(x+b) dx &= \Gamma(1-p) \cos(b - \frac{p\pi}{2}), \end{aligned}$$

which are special cases of

$$(3.19) \quad \begin{aligned} \int_0^\infty x^{-p} \cos(ax + b) dx &= -a^{p-1} \Gamma(1-p) \sin(b - \frac{p\pi}{2}) \\ \int_0^\infty x^{-p} \sin(ax + b) dx &= a^{p-1} \Gamma(1-p) \cos(b - \frac{p\pi}{2}), \end{aligned}$$

show that the corresponding initial values in (3.13) (respectively 3.14) match. The evaluations (3.19) appear as **3.764.1** and **3.764.2** in [4]. To establish (3.18) expand $\cos(x + b)$ as $\cos x \cos b - \sin x \sin b$, use the change of variables $x \mapsto x^p$, and Theorem 3.1. \square

We now discuss some definite integrals that follow from Theorem 3.3.

EXAMPLE 3.1. Differentiating (3.13) with respect to p and setting $p = \frac{1}{2}$ and $b = 0$ gives, after the change of variables $x \mapsto x^2$,

$$(3.20) \quad \begin{aligned} \int_0^\infty \log x \cos^{2n+1} x^2 dx &= -\frac{\sqrt{\pi}}{2^{2n+3}} (\pi + 2\gamma + 4 \log 2) \sum_{k=0}^n \binom{2n+1}{n-k} \frac{1}{\sqrt{4k+2}} \\ &\quad - \frac{\sqrt{\pi}}{2^{2n+2}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\log(2k+1)}{\sqrt{4k+2}}, \end{aligned}$$

where we have used the value $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2 \log 2)$.

EXAMPLE 3.2. Assume $0 < p, q < 1$. Multiplying (3.13) by b^{-q} and integrating over the half-line yields (after replacing b by y)

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{\cos^{2n+1}(x+y)}{x^p y^q} dA &= -\Gamma(1-p)\Gamma(1-q) \cos\left(\frac{\pi(p+q)}{2}\right) \\ &\quad \times \sum_{k=0}^n \binom{2n+1}{n-k} \frac{(2k+1)^{p+q-2}}{2^{2n}}. \end{aligned}$$

In particular, for $n = 0$,

$$(3.21) \quad \int_0^\infty \int_0^\infty \frac{\cos(x+y)}{x^p y^q} dA = -\Gamma(1-p)\Gamma(1-q) \cos\left(\frac{\pi(p+q)}{2}\right).$$

The derivative $\frac{\partial^2}{\partial p \partial q}$ at $p = q = \frac{1}{2}$ produces the evaluation

$$(3.22) \quad \int_0^\infty \int_0^\infty \frac{\log x \log y}{\sqrt{xy}} \cos(x+y) dx dy = (\gamma + 2 \log 2) \pi^2$$

that we promised in the Introduction.

EXAMPLE 3.3. Iterating the method described in the previous example yields

$$\int_{\mathbb{R}_+^n} (\cos \|x\|^2) \cdot \prod_{j=1}^n \log x_j dV = \frac{(-1)^{\Delta_n} \pi^{n/2}}{2^{2n}} \begin{cases} \operatorname{Re} \psi_n & \text{if } n \text{ is even,} \\ \operatorname{Im} \psi_n & \text{if } n \text{ is odd,} \end{cases}$$

with

$$(3.23) \quad \Delta_n = \frac{n(n+1)}{2}, \psi_n = \left(\gamma + 2 \log 2 + \frac{\pi i}{2} \right)^n e^{\pi i n / 4}.$$

Here $\|x\|^2 = x_1^2 + \cdots + x_n^2$ and γ is Euler's constant. For instance, for $n = 3$ we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \log x \log y \log z \cos(x^2 + y^2 + z^2) dx dy dz = \frac{\pi^{3/2}}{8} (-16\xi^3 + 12\xi^2\pi + 6\xi\pi^2 - \pi^3),$$

where $\xi = \gamma + 2 \log 2$.

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