ITERATED PRIMITIVES OF LOGARITHMIC POWERS

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Abstract. The evaluation of iterated primitives of powers of logarithms is expressed in closed form. The expressions contain polynomials with coefficients given in terms of the harmonic numbers and their generalizations. The logconcavity of these polynomials is established.

1. Introduction

The search for closed forms of definite integrals has the extra appeal of connecting many diverse areas of mathematics. At the beginning of Calculus, the student is usually told of

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}, \tag{1.1}$$

that hints to a relation between exponential and trigonometric functions. The reader will find in Chapter 8 of [3] a collection of different proofs of (1.1). These include the classical ones, presented in most textbooks, as well as one by R. A. Kortram [11] relating this evaluation to the number of representations of a number as a sum of squares.

The second author has begun the project of establishing all the entries in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik [8]. An example of the surprising connections encountered in this process is the subject of this note.

Consider the sequence of functions $\{f_n(x) : n \in \mathbb{N}\}$ defined by the iterated integrals

$$\begin{aligned} f_0(x) &= \ln(1+x) \\ f_n(x) &= \int_0^x f_{n-1}(t)\,dt. \end{aligned} \tag{1.2}$$

The first few examples are given by

$$\begin{aligned} f_1(x) &= -x + (1+x)\ln(1+x) \\ f_2(x) &= -\frac{x}{4}(3x+2) + \frac{1}{2}(1+x)^2\ln(1+x) \\ f_3(x) &= -\frac{x}{36}(11x^2+15x+6) + \frac{1}{6}(1+x)^3\ln(1+x) \\ f_4(x) &= -\frac{x}{288}(25x^3+52x^2+42x+12) + \frac{1}{24}(1+x)^4\ln(1+x). \end{aligned}$$

This data suggests that

$$f_n(x) = -xA_n(x) + B_n(x)\ln(1+x) \tag{1.3}$$

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where A_n and B_n are polynomials. Moreover, the value

$$B_n(x) = \frac{1}{n!}(1+x)^n \tag{1.4}$$

can be guessed from the data above.

In view of the fact that a closed form for the polynomials $A_n(x)$ seems harder to find, we begin by considering

$$\alpha_n := \text{lcm}\{\text{Denominators of } A_n(x)\}, \tag{1.5}$$

the sequence of denominators in the reduced form of $A_n(x)$. This sequence starts with

$$\{1, 1, 4, 36, 288, 7200, 43200, 2116800, 33868800\}. \tag{1.6}$$

A direct search in Neil Sloane's [15] gives no information. On the other hand, the quotient

$$\beta_n := \frac{\alpha_n}{n\alpha_{n-1}} \tag{1.7}$$

gives the values

$$\{1, 2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13\} \tag{1.8}$$

and this is the sequence $A014963$ of `Sloane`. Namely

$$\beta_n = \begin{cases} p & \text{if } n \text{ is a power of } p \\ 1 & \text{otherwise.} \end{cases} \tag{1.9}$$

The sequence β_n is the exponential of the von Mangoldt function

$$\Lambda_n = \begin{cases} \ln p & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise,} \end{cases} \tag{1.10}$$

one of the basic functions in the theory of prime numbers [9].

In this note we provide an explicit formula for the polynomial $A_n(x)$ in terms of harmonic numbers. This establishes the form of β_n discussed above. Section 5 considers the iterated primitives of a power of $\ln(1+x)$. A sequence of polynomials involving the generalized harmonic numbers is given.

2. The recurrences

The first few values of $f_n(x)$ suggest the ansatz

$$f_n(x) = -xA_n(x) + B_n(x)\ln(1+x), \tag{2.1}$$

with A_n and B_n polynomials in x. This is replaced in the relation $f_n'(x) = f_{n-1}(x)$ to produce

$$B_n'(x) = B_{n-1}(x) \tag{2.2}$$

$$xA_n'(x) + A_n(x) = xA_{n-1}(x) + \frac{B_n(x)}{1+x}. \tag{2.3}$$

The expression for $B_n(x)$ in (1.4) is obtained directly from here.

Theorem 2.1. The polynomial A_n is given for $n \geq 0$ by

$$A_n(x) = \frac{1}{n!} \sum_{k=1}^{n} \binom{n}{k} \left(H_n - H_{n-k} \right) x^{k-1}, \tag{2.4}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the harmonic number.

Proof. The formula is proved by induction. Write

$$A_n(x) = \sum_{k=1}^{n} a_{n,k} x^{k-1} \tag{2.5}$$

and use the recurrence (2.3) with the explicit expression for B_n to conclude that $a_{n,1} = 1/n!$ and for $2 \leq k \leq n$

$$k a_{n,k} = a_{n-1,k-1} + \frac{1}{n!} \binom{n-1}{k-1}. \tag{2.6}$$

This recurrence now gives (2.4). □

Note 2.2. Iterated integrals of $\ln(1-x)$ were considered by Mathar [13, section 4.3], who obtained a similar recurrence.

Theorem 2.3. The polynomials $A_n(x)$ are given by

$$A_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=1}^{n-k} \frac{(-x)^{m-1}}{m}. \tag{2.7}$$

Proof. The definition of $f_n(x)$ implies that

$$f_n(x) = -x^n \sum_{j=1}^{\infty} \frac{(-x)^j}{j(j+1)\cdots(j+n)}. \tag{2.8}$$

Using the partial fraction decomposition

$$\frac{1}{j(j+1)\cdots(j+n)} = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k}{j+k} \binom{n}{k} \tag{2.9}$$

produces

$$\begin{aligned}
n! f_n(x) &= -x^n \sum_{j=1}^{\infty} (-x)^j \sum_{k=0}^{n} \frac{(-1)^k}{j+k} \binom{n}{k} \\
&= -x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (-x)^{-k} \sum_{j=1}^{\infty} \frac{(-x)^{j+k}}{j+k} \\
&= -\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} \sum_{m=k+1}^{\infty} \frac{(-x)^m}{m} \\
&= -\sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left(-\ln(1+x) - \sum_{m=1}^{k} \frac{(-x)^m}{m} \right) \\
&= (1+x)^n \ln(1+x) + \sum_{k=0}^{n} \sum_{m=1}^{k} (-1)^m \binom{n}{k} \frac{x^{n+m-k}}{m} \\
&= (1+x)^n \ln(1+x) + \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=1}^{n-k} \frac{(-x)^m}{m}.
\end{aligned}$$

Dividing this sum by $-x$ gives the result. □

Comparing the two expressions produced for the polynomials A_n yields the next identity.

Corollary 2.4. Let $n \in \mathbb{N}$. Then

$$-\sum_{k=0}^{n} \binom{n}{k} x^k \sum_{m=1}^{n-k} \frac{(-x)^m}{m} = \sum_{k=1}^{n} \binom{n}{k} x^k \sum_{m=1}^{k} \frac{1}{m+n-k}. \tag{2.10}$$

3. The denominators of $A_n(x)$

In this section we analyze the polynomial $A_n(x)$ and compute its denominator when written in reduced form. The proof employs some elementary number theory.

Theorem 3.1. For $n \geq 1$, the common denominator for $A_n(x)$ is given by

$$\alpha_n = n! \operatorname{lcm}(1, 2, \ldots, n). \tag{3.1}$$

The proof of the theorem employs a preliminary divisibility result.

Lemma 3.2. Let p be a prime, $n \geq 1$, and $k \geq 0$ such that $p^k \leq n$. Then the power of p dividing the denominator of $\binom{n}{p^k} (H_n - H_{n-p^k})$ is p^k.

Proof. Observe that

$$H_n - H_{n-p^k} = \sum_{i=0}^{p^k-1} \frac{1}{n-i}. \tag{3.2}$$

In (3.2), the index i ranges over a set of length p^k; therefore there is a unique index i_0 such that p^k divides $n - i_0$, namely $i_0 = n - p^k \lfloor n/p^k \rfloor$. For $i \neq i_0$ write

$n-i=p^{\alpha_i}\beta_i$ and $n-i_0=p^{\alpha}\beta$, with $\alpha \geq k$, $\alpha_i \leq k-1$ and p not dividing β, β_i. Therefore

$$H_n - H_{n-p^k} = \sum_{i \neq i_0} \frac{1}{p^{\alpha_i}\beta_i} + \frac{1}{p^{\alpha}\beta}$$

can be expressed as

$$H_n - H_{n-p^k} = \frac{A}{p^{\gamma}B} + \frac{1}{p^{\alpha}\beta}$$

for some $\gamma \leq k-1$ and B not divisible by p. This gives

$$H_n - H_{n-p^k} = \frac{p^{\alpha-\gamma}A\beta + B}{p^{\alpha}B\beta} \tag{3.3}$$

and it follows that the denominator of $H_n - H_{n-p^k}$ is divisible exactly by p^{α}.

We now determine the exponent of the highest power of p dividing $\binom{n}{p^k}$, which according to Kummer's theorem [12] is the number of borrows involved in subtracting p^k from n in base p. Let $n = n_l \cdots n_{\alpha} \cdots n_k \cdots n_0$ be the standard base-p representation of n; then $n - i_0 = n_l \cdots n_{\alpha} 0 \cdots 0$. Since $n_{\alpha} \neq 0$ and $n_j = 0$ for $k-1 < j < \alpha$, there are $\alpha - k$ borrows when subtracting p^k from n in base p. Therefore the power of p dividing $\binom{n}{p^k}$ is $p^{\alpha-k}$, and the power of p in $\binom{n}{p^k}(H_n - H_{n-p^k})$ is $p^{\alpha-k} \cdot p^{-\alpha} = p^{-k}$. $\square$

Proof of Theorem. Only the terms $1, 2, \ldots, n$ appear as part of the denominators of coefficients of the polynomial $n!A_n(x)$. Thus, the common denominator is a divisor of $\text{lcm}(1, 2, \ldots, n)$. The previous Lemma shows that every prime power $p^k \leq n$ appears. It follows that the denominator of $n!A_n(x)$ is $\text{lcm}(1, 2, \ldots, n)$.

Since every prime dividing $n!$ also divides $\text{lcm}(1, 2, \ldots, n)$, there is no cancellation with the numerator of $\sum_{j=1}^{n} \binom{n}{j}(H_n - H_{n-j})x^{j-1}$ when we divide by $n!$, so the denominator of $A_n(x)$ is $n!\,\text{lcm}(1, 2, \ldots, n)$.

Corollary 3.3. The expression β_n in (1.7) is now given by

$$\beta_n = \frac{\text{lcm}(1, 2, \ldots, n)}{\text{lcm}(1, 2, \ldots, n-1)}. \tag{3.4}$$

This gives (1.9).

4. The polynomial A_n is logconcave

A sequence of coefficients $\{a_0, a_1, \ldots, a_n\}$ is *unimodal* is there is an index j_* such that $a_0 \leq a_1 \leq \cdots \leq a_{j^*}$ and $a_{j_*} \geq a_{j^*+1} \geq \cdots \geq a_n$. A polynomial is called unimodal if its sequence of coefficients is unimodal. The polynomial is called *logconcave* if its coefficients satisfy $a_j^2 \geq a_{j-1}a_{j+1}$. An elementary argument shows that logconcavity implies unimodality [18]. Unimodal and logconcave sequences appear frequently in algebra and combinatorics. The reader will find in [5, 16] a survey of these results.

The logconcavity of the polynomial $B_n(x)$ is elementary; it simply corresponds to that of the binomial coefficients. The argument for $A_n(x)$ is established next.

Theorem 4.1. The polynomial $A_n(x)$ is logconcave.

Proof. The result is equivalent to the inequality

$$\binom{n}{j}^2 (H_n - H_{n-j})^2 \geq \binom{n}{j-1}\binom{n}{j+1} (H_n - H_{n-j+1})(H_n - H_{n-j-1}). \tag{4.1}$$

Introduce the notation

$$f_n(j) = H_n - H_{n-j} = \sum_{i=0}^{j-1} \frac{1}{n-i}; \tag{4.2}$$

then (4.1) is equivalent to

$$\frac{(n+1)(n-j+1)}{j} f_n^2(j) - f_n(j) + 1 \geq 0. \tag{4.3}$$

This is a quadratic inequality with discriminant

$$\frac{j(4n+5) - 4(1+n)^2}{j} \leq \frac{n(4n+5) - 4(1+n)^2}{j} = -\frac{3n+4}{j} < 0, \tag{4.4}$$

that gives the logconcavity of A_n. □

Note 4.2. There are many instances where polynomials appearing in connection with the evaluation of integrals are logconcave. For instance, the polynomial

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l \tag{4.5}$$

with

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}, \tag{4.6}$$

appears in the formula

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}}. \tag{4.7}$$

The reader will find in [1] different proofs of these formulas. The unimodality of $P_m(a)$ was established in [2] and its logconcavity, in [10]. A direct proof of this result also appears in [6].

The logconcavity of a sequence $\{a_j : 1 \leq j \leq n\}$ can be expressed in terms of the operator $\mathfrak{L}$ defined by $\mathfrak{L}(\{a_j\}) = \{a_j^2 - a_{j-1}a_{j+1}\}$, where $a_j = 0$ if $j < 0$ or $j > n$. Thus, a positive sequence $\mathbf{a}$ is logconcave if $\mathfrak{L}(\mathbf{a})$ is positive. A sequence is called r-logconcave if $\mathfrak{L}^{(s)}(\mathbf{a})$ is positive for $0 \leq s \leq r$. It is called *infinite logconcave* if it is r-logconcave for any $r \in \mathbb{N}$.

The next conjecture is based on extensive symbolic calculations.

Conjecture 4.3. For every $n \in \mathbb{N}$, the polynomial $A_n(x)$ is infinite logconcave.

The question of logconcavity of a sequence $\{a_k\}$ is intimately connected with the location of the zeros of its generating polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Newton showed that if the zeros of P are real and negative, then $\{a_k\}$ is logconcave. It turns out that the zeros of the polynomial $P_m(a)$, generated by the sequence $\{d_{l,m}\}$ in (4.6), do not satisfy this condition. The quest for a proof of the logconcavity of $\{d_{l,m}\}$ lead one of the authors to study the operator $\mathfrak{L}$.

Experimental data suggested that this sequence was infinite logconcave. A similar conjecture on the binomial coefficients was proposed as a testing problem.

The operator $\mathfrak{L}$ *does not* preserve logconcavity. In order to bypass this difficulty, McNamara and Sagan [14] introduce a remarkable method that gives infinite logconcavity. Given $r \in \mathbb{R}$ a sequence $\{a_k\}$ is called r-factor logconcave if $a_k^2 \geq r a_{k+1} a_{k-1}$. The next lemma appears in [14].

Lemma 4.4. Let $\{a_k\}$ be a non-negative sequence and let $r_0 = (3+\sqrt{5})/2$. Then $\{a_k\}$ being r_0-factor logconcave implies that $\mathfrak{L}(\{a_k\})$ is too. So in this case $\{a_k\}$ is infinite logconcave.

This result can be used to check that a sequence $a_{n,j}$ is infinite logconcave for a specific fixed value of n. It was used in [14] to show that the sequence of binomial coefficients $\{\binom{n}{k} : 0 \leq k \leq n\}$ is infinite logconcave for $n \leq 1450$. Their approach is remarkably simple: calculate $\mathfrak{L}^i(a_{n,j})$ for i up to some bound M. If the sequences $\mathfrak{L}(a_{n,j})$, $\mathfrak{L}^2(a_{n,j}), \ldots, \mathfrak{L}^M(a_{n,j})$ are non-negative and $\mathfrak{L}^M(a_{n,j})$ is r_0-factor logconcave, then Lemma 4.4 implies that $\{a_{n,j}\}_{j=0}^n$ is infinite logconcave. Employing this technique we have verified the following result.

Theorem 4.5. The polynomial $A_n(x)$ is infinite logconcave for all $n \leq 300$.

Extending the conjecture on the infinite logconcavity of the binomial coefficients, Stanley [17], McNamara-Sagan [14] and Fisk [7] conjectured the next result. This was established by Petter Brändén in [4].

Theorem 4.6. Suppose that the polynomial $\displaystyle\sum_{k=0}^{n} a_k x^k$ has only real and negative zeros. Then so does the polynomial

$$\sum_{k=0}^{n} \left(a_k^2 - a_{k-1}a_{k+1}\right) z^k, \quad \text{where } a_{-1} = a_{n+1} = 0. \tag{4.8}$$

In particular, the sequence $\{a_k : 0 \leq k \leq n\}$ is infinite logconcave.

The theorem established the fact that the binomial coefficients $\{\binom{n}{k} : 0 \leq k \leq n\}$ are infinite logconcave for every $n \in \mathbb{N}$. On the other hand, examples of sequences conjectured to be infinite logconcave have generating polynomials with non-real zeros. This holds for $P_m(a)$ in (4.5) as well as the polynomial $A_n(x)$ considered here.

5. An extension

The expression for the iterated integrals described here extends to iterated integrals obtained by

$$\begin{aligned} f_{0,j}(x) &= \ln^j(1+x) \\ f_{n,j}(x) &= \int_0^x f_{n-1,j}(t)\,dt. \end{aligned} \tag{5.1}$$

Symbolic computation gives

$$\begin{aligned} f_{1,2}(x) &= 2x - 2(x+1)u + (x+1)u^2 \\ f_{2,2}(x) &= \frac{1}{4}x(7x+6) - \frac{3}{2}(x+1)^2u + \frac{1}{2}(x+1)^2u^2 \\ f_{3,2}(x) &= \frac{1}{108}x(85x^2+147x+66) - \frac{11}{18}(x+1)^3u + \frac{1}{6}(x+1)^3u^2 \end{aligned}$$

where $u = \ln(1+x)$. The structure of $f_{n,j}(x)$ is provided below.

Theorem 5.1. For $n \geq 0$ and $j \geq 1$, there are polynomials $A_{n,j}(x)$ and $B_{n,k,j}(x)$ such that

$$f_{n,j}(x) = A_{n,j}(x) + \sum_{k=1}^{j} B_{n,k,j}(x) \ln^k(1+x). \tag{5.2}$$

Moreover, $B_{n,k,j}(x) = b_{n,k,j}(1+x)^n$, for some $b_{n,k,j} \in \mathbb{Q}$.

Proof. In the first step, it is shown that $f_{n,j}$ has the stated form, for some polynomials $A_{n,j}, B_{n,k,j}$. This is elementary: integration by parts shows that, for $r, k \in \mathbb{N}$,

$$\int s^r \ln^k s \, ds = P(\ln s) s^{r+1}, \tag{5.3}$$

where P is a polynomial of degree k. Induction on n and (5.1), give the result.

The next is to prove that $B_{n,k,j}(x)$ is a multiple of $(1+x)^n$. In order to achieve this, replace (5.2) in (5.1) to obtain

$$\begin{aligned}(1+x)A'_{n,j}(x) + B_{n,1,j}(x) &= (1+x)A_{n-1,j}(x) \\ (1+x)B'_{n,k,j}(x) + (k+1)B_{n,k+1,j}(x) &= (1+x)B_{n-1,k,j}(x) \quad \text{for } 1 \leq k \leq j-1, \\ B'_{n,j,j}(x) &= B_{n-1,j,j}(x).\end{aligned} \tag{5.4}$$

Fix n and j and use induction on k to show $B_{n,k,j}(-1) = 0$. The first equation in (5.4) gives $B_{n,1,j}(-1) = 0$ and the second one gives $B_{n,k,j}(-1) = 0$, using induction on k. The fact that $B_{n,k,j}(x)$ is a power of $(x+1)^n$ now follows directly from the recurrence (5.4). Indeed,

$$B_{n,k+1,j}(x) = \frac{1+x}{1+k} B_{n-1,k,j}(x) - \frac{1+x}{1+k} B'_{n,k,j}(x) \tag{5.5}$$

gives the result for $1 \leq k \leq j-1$. The case $k = j$ is settled with $B'_{n,j,j}(x) = B_{n-1,j,j}(x)$, where the vanishing of $B_{n,j,j}(x)$ at $x = -1$ adjusts the constant of integration. Therefore

$$B_{n,k,j}(x) = b_{n,k,j}(1+x)^n \tag{5.6}$$

for some $b_{n,k,j}$ to be determined.

For $n \geq 1$ and $j \geq 1$, the coefficients $b_{n,k,j}$ satisfy the equations

$$\begin{aligned} b_{0,j,j} &= 1 \\ b_{0,k,j} &= 0 \text{ for } 0 \leq k \leq j-1 \\ b_{n,j,j} &= \frac{1}{n} b_{n-1,j,j} \\ b_{n,k,j} &= -\frac{k+1}{n} b_{n,k+1,j} + \frac{1}{n} b_{n-1,k,j} \text{ for } 0 \leq k \leq j-1. \end{aligned} \tag{5.7}$$

The third recurrence in (5.7) can be solved directly to obtain $b_{n,j,j} = 1/n!$. □

To write $b_{n,k,j}$ more explicitly, let

$$H_{n,m} = \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_m \leq n} \frac{1}{a_1 a_2 \cdots a_m} \tag{5.8}$$

for $n \geq 0$ and $m \geq 0$. In other words, the sum is over all nondecreasing m-tuples with entries from $\{1, 2, \ldots, n\}$. $H_{n,m}$ generalizes the harmonic number $H_n = H_{n,1}$.

For $m = 0$ the product is empty, so $H_{n,0} = 1$ for $n \geq 0$. For $n = 0$ the sum is empty unless $m = 0$, so $H_{0,m} = 0$ for $m \geq 1$.

For $n \geq 1$ and $m \geq 1$, we have $H_{n,m} = \frac{1}{n}H_{n,m-1} + H_{n-1,m}$ by breaking up the m-tuples in $H_{n,m}$ according to whether $a_m = n$ or not. It follows that the proposed expression for $b_{n,k,j}$ satisfies

$$b_{n,k,j} = -\frac{k+1}{n}b_{n,k+1,j} + \frac{1}{n}b_{n-1,k,j}. \tag{5.9}$$

It follows from here that

$$b_{n,k,j} = \frac{(-1)^{j-k}j!}{n!\,k!}H_{n,j-k} \tag{5.10}$$

satisfies the equations in (5.7).

Theorem 5.2. The polynomial $B_{n,k,j}$ in Theorem 5.1 is given by

$$B_{n,k,j}(x) = \frac{(-1)^{j-k}j!}{n!k!}H_{n,j-k}(1+x)^n. \tag{5.11}$$

Note 5.3. The recursion $H_{n,m} = \frac{1}{n}H_{n,m-1} + H_{n-1,m}$, with n fixed, can be solved explicitly. The initial term is $H_{1,m} = 1$. It may be checked directly that

$$H_{n,m} = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}k^{-m}. \tag{5.12}$$

For $n = 2$, $H_{2,j-k}$ is a geometric series, therefore (5.10) gives

$$b_{2,k,j} = \frac{(-1)^{j-k}j!\,(2^{j-k+1}-1)}{k!\,2^{j-k+1}}. \tag{5.13}$$

The last step in the determination of $f_{n,j}$ is the construction of the polynomial $A_{n,j}(x)$ in (5.2).

The equation for $A_{n,j}(x)$ comes from (5.4):

$$A'_{n,j}(x) = A_{n-1,j}(x) + \frac{(-1)^j j!}{n!}H_{n,j-1}(1+x)^{n-1}, \tag{5.14}$$

with the boundary conditions $A_{0,j}(x) = 0$ and $A_{n,j}(0) = 0$.

The polynomial $A_{n,j}(x)$ is now written as

$$A_{n,j}(x) = \sum_{r=1}^{n}c_{n,r,j}x^r \tag{5.15}$$

for some coefficients $c_{n,r,j}$. The fact that $A_{n,j}$ is of degree n comes directly from (5.14). This recurrence also yields

$$c_{n,1,j} = \alpha_{n,j} \tag{5.16}$$

$$c_{n,r,j} = \frac{1}{r}c_{n-1,r-1,j} + \frac{1}{r}\binom{n-1}{r-1}\alpha_{n,j}, \tag{5.17}$$

where

$$\alpha_{n,j} = \frac{(-1)^j j! H_{n,j-1}}{n!}. \tag{5.18}$$

The claim is that the solution of this recurrence yields the following expression for the polynomial $A_{n,j}(x)$.

Theorem 5.4. The polynomial $A_{n,j}(x)$ in Theorem 5.1 is given by

$$A_{n,j}(x) = \frac{(-1)^j j!}{n!} \sum_{r=1}^{n} \binom{n}{r} \left[\sum_{k=0}^{r-1} \frac{H_{n-k,j-1}}{n-k} \right] x^r. \tag{5.19}$$

Proof. Direct replacement of

$$c_{n,r,j} = \frac{(-1)^j j!}{n!} \binom{n}{r} \sum_{k=0}^{r-1} \frac{H_{n-k,j-1}}{n-k} \tag{5.20}$$

in (5.16) and (5.17). □

Note 5.5. The case $j = 1$ of (5.19) reduces to the expression for $-xA_n(x)$ given in Theorem 2.1.

The final statement deals with an arithmetical property of the coefficients of $B_{n,j}(x)$. This was first observed symbolically and it was one of the motivations for the work discussed here.

Corollary 5.6. Assume p is a prime that divides the denominator of a coefficient of $B_{n,k,j}(x)$ in Theorem 5.2. Then $p \leq \max(n, k)$.

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