ON SOME FAMILIES OF INTEGRALS SOLVABLE IN TERMS OF POLYGAMMA AND NEGAPOLYGAMMA FUNCTIONS

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Abstract. Beginning with Hermite's integral representation of the Hurwitz zeta function, we derive explicit expressions in terms of elementary, polygamma, and negapolygamma functions for several families of integrals of the type $\int_0^\infty f(t)K(q,t)dt$ with kernels $K(q,t)$ equal to $\left(e^{2\pi qt}-1\right)^{-1}$, $\left(e^{2\pi qt}+1\right)^{-1}$, and $(\sinh(2\pi qt))^{-1}$.

1. Introduction

The Hurwitz zeta function, defined by

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{1.1}$$

for $z \in \mathbb{C}$, $\operatorname{Re} z > 1$, and $q \neq 0, -1, -2, \cdots$, admits the integral representation

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{e^{-qt}}{1-e^{-t}} t^{z-1} dt, \tag{1.2}$$

where $\Gamma(z)$ is Euler's gamma function, which is valid for $\operatorname{Re} z > 1$ and $\operatorname{Re} q > 0$, and can be used to prove that $\zeta(z,q)$ admits an analytic extension to the whole complex plane except for a simple pole at $z = 1$. Hermite proved an alternate integral representation, which actually provides an explicit realization of this analytic continuation for real $q > 0$:

(1.3)
$$\zeta(z,q) = \frac{1}{2}q^{-z} + \frac{1}{z-1}q^{1-z} + 2q^{1-z}\int_0^\infty \frac{\sin(z\tan^{-1} t)}{(1+t^2)^{z/2}\left(e^{2\pi tq}-1\right)}\, dt.$$

Special cases of $\zeta(z,q)$ include the Bernoulli polynomials,

$$B_m(q) = -m\,\zeta(1-m,q), \quad m \in \mathbb{N}, \tag{1.4}$$

defined by their generating function

$$\frac{te^{qt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(q)\frac{t^m}{m!} \tag{1.5}$$

and given explicitly in terms of the Bernoulli numbers B_k by

$$B_m(q) = \sum_{k=0}^{m} \binom{m}{k} B_k q^{m-k}, \tag{1.6}$$

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and the polygamma functions,

$$\psi^{(m)}(q) = (-1)^{m+1}\, m!\, \zeta(m+1,q), \quad m \in \mathbb{N}, \tag{1.7}$$

defined by

$$\psi^{(m)}(q) \;:=\; \frac{d^{\,m+1}}{dq^{\,m+1}} \ln\Gamma(q), \quad m \in \mathbb{N}. \tag{1.8}$$

The function $\zeta(z,q)$ is analytic for $z \neq 1$, and direct differentiation of (1.3) yields

(1.9)

$$\zeta'(z,q) = -\frac{1}{2}q^{-z}\ln q - \frac{q^{1-z}}{(z-1)^2} - \frac{q^{1-z}}{z-1}\ln q - 2q^{1-z}\ln q \int_0^\infty \frac{\sin(z\tan^{-1}t)\,dt}{(1+t^2)^{z/2}(e^{2\pi qt}-1)}$$
$$+\, 2q^{1-z}\int_0^\infty \frac{\cos(z\tan^{-1}t)\,\tan^{-1}t\;dt}{(1+t^2)^{z/2}(e^{2\pi qt}-1)} - q^{1-z}\int_0^\infty \frac{\sin(z\tan^{-1}t)\,\ln(1+t^2)\,dt}{(1+t^2)^{z/2}(e^{2\pi qt}-1)},$$

where $\zeta'(z,q)$ denotes $\partial\zeta(z,q)/\partial z$.

In this paper we derive, starting from the representations (1.3) and (1.9), several definite integral evaluations of the type

$$F(q) = \int_0^\infty \frac{f(t)}{e^{2\pi qt}-1}\,dt.$$

The main examples considered here are the families

$$\begin{aligned} I_k(q) &= \int_0^\infty \frac{t}{(1+t^2)^{k+1}(e^{2\pi qt}-1)}\,dt, \\ T_k(q) &= \int_0^\infty \frac{t^k\tan^{-1}t}{e^{2\pi qt}-1}\,dt, \\ L_k(q) &= \int_0^\infty \frac{t^k\ln(1+t^2)}{e^{2\pi qt}-1}\,dt, \end{aligned}$$

and the associated integrals obtained by replacing the factor $e^{2\pi qt}-1$ in the denominator of the integrands by $e^{2\pi qt}+1$ and $\sinh(2\pi qt)$. We produce closed-form expressions for $I_k(q)$ in terms of the polygamma functions $\psi^{(m)}(q)$, $1 \le m \le k$, and for $T_{2k}(q)$ and $L_{2k+1}(q)$, in terms of the derivative of the Hurwitz zeta function at negative integers or, equivalently, the *balanced* functions

$$A_m(q) \;:=\; m\,\zeta'(1-m,q), \tag{1.10}$$

or the balanced negapolygamma functions,

$$\psi^{(-m)}(q) := \frac{1}{m!}\left[A_m(q) - H_{m-1}B_m(q)\right], \tag{1.11}$$

defined for $m \in \mathbb{N}$, which were introduced in [3]. H_r is the harmonic number ($H_0 := 0$). We define a function $f(q)$ to be *balanced* (on the unit interval) if it satisfies the properties

$$\int_0^1 f(q)dq = 0, \quad \text{and} \quad f(0) = f(1).$$

For certain particular rational values of q the balanced negapolygamma functions evaluate to rational linear combinations of elementary functions of special constants such as $\ln 2, \ln \pi$, the Euler constant γ, G/π (G is Catalan's constant), $\zeta'(-1)$, etc.

We note that, in view of Lerch's result [2]

$$\ln \Gamma(q) = \zeta'(0,q) - \zeta'(0), \tag{1.12}$$

$A_1(q) = \zeta'(0,q)$ can be expressed in terms of the gamma function as

$$A_1(q) = \ln \frac{\Gamma(q)}{\sqrt{2\pi}}. \tag{1.13}$$

The problem of closed-form expressions for T_{2k+1} and L_{2k} remains open.

2. A series expansion

All the results presented in this paper are consequences of the Taylor series expansion of the function

$$f(z,t) = \frac{\sin(z \tan^{-1} t)}{(1+t^2)^{z/2}}, \tag{2.1}$$

which appears in the integral representation of the Hurwitz zeta function:

$$\zeta(z,q) = \frac{1}{2}q^{-z} + \frac{1}{z-1}q^{1-z} + 2q^{1-z}\int_0^\infty \frac{f(z,t)}{e^{2\pi t q}-1}\,dt. \tag{2.2}$$

Theorem 2.1. The Taylor series

$$\frac{\sin(z \tan^{-1} t)}{(1+t^2)^{z/2}} = \sum_{k=0}^{\infty} \frac{(-1)^k (z)_{2k+1}}{(2k+1)!} t^{2k+1} \tag{2.3}$$

and

$$\frac{\cos(z \tan^{-1} t)}{(1+t^2)^{z/2}} = \sum_{k=0}^{\infty} \frac{(-1)^k (z)_{2k}}{(2k)!} t^{2k} \tag{2.4}$$

hold for $|t| < 1$.

Proof. Both sides of (2.3) satisfy the equation

$$(1+t^2)\frac{d^2 g}{dt^2} + 2t(z+1)\frac{dg}{dt} + z(z+1)g = 0 \tag{2.5}$$

and the initial conditions $g(0) = 0$, $g'(0) = z$. It is straightforward to show that the series on the right-hand side of (2.3) converges for $|t| < 1$. The proof of (2.4) is similar. □

Corollary 2.2. Let $m \in \mathbb{N}$. Then, for $t \in \mathbb{R}$,

$$\cos(m \tan^{-1} t) = (1+t^2)^{-m/2} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \binom{m}{2k} t^{2k} \tag{2.6}$$

and

$$\sin(m \tan^{-1} t) = (1+t^2)^{-m/2} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^k \binom{m}{2k+1} t^{2k+1}. \tag{2.7}$$

Proof. The expression

$$(-m)_n = (-1)^n n! \binom{m}{n}$$

vanishes for $n > m$, so the series (2.3, 2.4) terminate for $z = -m$. □

Since one can write $t^2 = (1+t^2) - 1$, it is clear that, for $m \in \mathbb{N}$, the functions $t^{-1}(1+t^2)^{m/2} \sin(m \tan^{-1} t)$ and $(1+t^2)^{m/2} \cos(m \tan^{-1} t)$ are also polynomials in $1+t^2$. We now give their explicit forms.

Corollary 2.3. Let $m \in \mathbb{N}$. Then

$$\cos(m \tan^{-1} t) = \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^p \frac{m}{m-p} \binom{m-p}{p} 2^{m-2p-1} (1 + t^2)^{p-m/2} \tag{2.8}$$

and

$$\sin(m \tan^{-1} t) = t \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^p \binom{m-p-1}{p} 2^{m-2p-1} (1 + t^2)^{p-m/2}. \tag{2.9}$$

Proof. Performing the binomial expansion of $t^{2k} = \left[(1+t^2) - 1\right]^k$ in (2.7) we have

$$\begin{aligned} \sin(m \tan^{-1} t) &= (1+t^2)^{-m/2} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^k \binom{m}{2k+1} t^{2k+1} \\ &= t(1+t^2)^{-m/2} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^j \left\{ \sum_{k=j}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} \binom{k}{j} \right\} (1+t^2)^j. \end{aligned}$$

The result now follows from the identity

$$\sum_{k=j}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} \binom{k}{j} = \binom{m-j-1}{j} 2^{m-2j-1}, \tag{2.10}$$

where $0 \leq j \leq \lfloor \frac{m-1}{2} \rfloor$. A similar argument gives the identity for cosine. □

3. A family of integrals derived from Hermite's representation

The Hermite representation (1.3) can be written as

$$\int_0^\infty \frac{\sin(z \tan^{-1} t)}{(1+t^2)^{z/2}(e^{2\pi q t} - 1)} \, dt = \frac{1}{2} q^{z-1} \zeta(z, q) - \frac{1}{4q} - \frac{1}{2(z-1)}. \tag{3.1}$$

A direct consequence of the expansion (2.7), when used in (3.1) with $z \in -\mathbb{N}$, is the following well-known relation (1.4) between the Bernoulli polynomials and the Hurwitz zeta function.

Lemma 3.1. The Bernoulli polynomials $B_m(q)$, $m \in \mathbb{N}$, are given by

$$B_m(q) = -m \, \zeta(1-m, q). \tag{3.2}$$

Proof. Substitute (2.7) into (3.1) with $z = -m$ and use the well-known result [4] (3.411.2)

$$\int_0^\infty \frac{t^{2k+1}}{e^{2\pi qt}-1}dt = (-1)^k \frac{B_{2k+2}}{4(k+1)q^{2k+2}} \tag{3.3}$$

to obtain

$$\begin{aligned}\zeta(-m,q) &= -\frac{q^{m+1}}{m+1} + \frac{q^m}{2} - 2q^{m+1}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}\frac{B_{2k+2}}{4(k+1)q^{2k+2}} \\ &= -\frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}B_k q^{m+1-k} \\ &= -\frac{1}{m+1}B_{m+1}(q),\end{aligned}$$

in view of (1.6) and the facts that $B_0 = 1, B_1 = -1/2$, and $B_{2k+1} = 0$ for $k \in \mathbb{N}$. □

Similarly, the alternate expansion (2.9), when substituted into (3.1) with $z = m+1$, $m \in \mathbb{N}$, leads us to consider the following family of integrals.

Theorem 3.2. The integrals

$$I_k(q) := \int_0^\infty \frac{t}{(1+t^2)^{k+1}(e^{2\pi qt}-1)}\,dt, \tag{3.4}$$

$k \in \mathbb{N}$, are given by

$$I_k(q) = -\frac{1}{4k} - \frac{\binom{2k}{k}}{2^{2k+2}q} + \frac{1}{k2^{2k}}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{(j-1)!}\binom{2k-j-1}{k-j}2^{j-1}q^j\psi^{(j)}(q). \tag{3.5}$$

Proof. Use the expansion (2.9) in (3.1) with $z = m+1$ to obtain the recursion

$$\sum_{p=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^p 2^{m-2p+1}\binom{m-p}{p}I_{m-p}(q) = q^m\zeta(m+1,q) - \frac{1}{2q} - \frac{1}{m}$$

for $m \in \mathbb{N}$. Expressing the Hurwitz zeta functions in terms of polygamma functions by inverting (1.7), we find

$$\sum_{p=\lfloor\frac{m+1}{2}\rfloor}^{m}(-1)^p 2^{2p}\binom{p}{m-p}I_p(q) = -2^{m-1}\left[\frac{q^m\psi^{(m)}(q)}{m!} + \frac{(-1)^m}{2q} + \frac{(-1)^m}{m}\right]. \tag{3.6}$$

The recursion (3.6) can be solved in closed form by inverting the sum. We first lower the lower limit to $p = 1$ since the binomial coefficient vanishes when $p < m-p$, and then use the orthogonality formula

$$\sum_{j=1}^{k}(-1)^j j\binom{2k-j-1}{k-j}\binom{p}{j-p} = \begin{cases}(-1)^k k & \text{if } p = k\\ 0 & \text{otherwise}\end{cases}$$

and the evaluations

$$\sum_{j=1}^{k} 2^j \binom{2k-j-1}{k-j} = 2^{2k-1} \quad \text{and} \quad \sum_{j=1}^{k} j\, 2^j \binom{2k-j-1}{k-j} = k\binom{2k}{k}$$

to obtain the explicit formula (3.5).

□

Note. The case $k = 0$ appears in [4] (3.415.1):

$$I_0(q) = \int_0^\infty \frac{t}{(1+t^2)(e^{2\pi q t}-1)}\, dt = \frac{1}{2}\ln q - \frac{1}{4q} - \frac{1}{2}\psi(q), \tag{3.7}$$

where $\psi(q)$ is the digamma function. This result also follows from (3.1) in the limit $z \to 1$, in view of

$$\psi(q) = \lim_{z\to 1}\left[\frac{1}{z-1} - \zeta(z,q)\right]. \tag{3.8}$$

4. Two new families of integrals

As we know from Corollary 2.3, the functions

$$t^{-1}(1+t^2)^{-z/2}\sin(z\tan^{-1} t) \quad \text{and} \quad (1+t^2)^{-z/2}\cos(z\tan^{-1} t)$$

are polynomials in $1+t^2$ when $z \in -\mathbb{N}$, a fact that allows considerable simplification of representation (1.9). This leads us to consider the families of integrals

$$T_k(q) \;=\; \int_0^\infty \frac{t^k \tan^{-1} t}{e^{2\pi q t}-1}\, dt \tag{4.1}$$

and

$$L_k(q) \;=\; \int_0^\infty \frac{t^k \ln(1+t^2)}{e^{2\pi q t}-1}\, dt, \tag{4.2}$$

that appear after differentiating Hermite's representation (1.3) with respect to the parameter z. A direct differentiation of (3.1) yields

$$\int_0^\infty \frac{\cos(z\tan^{-1} t)\tan^{-1} t}{(1+t^2)^{z/2}(e^{2\pi q t}-1)}\, dt - \frac{1}{2}\int_0^\infty \frac{\sin(z\tan^{-1} t)\ln(1+t^2)}{(1+t^2)^{z/2}(e^{2\pi q t}-1)}\, dt = \frac{1}{2}\left[q^{z-1}\zeta(z,q)\ln q + q^{z-1}\zeta'(z,q) + \frac{1}{(z-1)^2}\right]. \tag{4.3}$$

Setting $z = -m$, $m \in \mathbb{N}_0$, we find a recursion for these integrals.

Proposition 4.1. For $m \in \mathbb{N}_0$, the integrals $T_{2k}(q)$ and $L_{2k+1}(q)$ satisfy the relation

$$2\sum_{k=0}^{\lfloor \frac{m}{2}\rfloor} (-1)^k \binom{m}{2k} T_{2k}(q) + \sum_{k=0}^{\lfloor \frac{m-1}{2}\rfloor} (-1)^k \binom{m}{2k+1} L_{2k+1}(q) = \frac{1}{(m+1)q^{m+1}}\left[A_{m+1}(q) - B_{m+1}(q)\,\ln q\right] + \frac{1}{(m+1)^2}. \tag{4.4}$$

Proof. This is derived directly from (4.3) using the expansions (2.6) and (2.7), and the relations (1.4) and (1.10). □

Equation (4.4) can be used iteratively to find explicit expressions for the integrals $T_{2k}(q)$ and $L_{2k+1}(q)$ in terms of the functions $A_m(q)$ and $B_m(q)$.

Example 4.2. The value $m = 0$ in (4.4) yields

$$T_0(q) = \frac{1}{2q} A_1(q) - \frac{1}{2q} B_1(q) \ln q + \frac{1}{2}.$$

Using (1.12) and $B_1(q) = q - \frac{1}{2}$, we have

$$T_0(q) = \int_0^\infty \frac{\tan^{-1} t}{e^{2\pi q t} - 1}\, dt = \frac{1}{2} - \frac{1}{2}\ln q + \frac{\ln q}{4q} + \frac{\ln \Gamma(q)}{2q} - \frac{\ln \sqrt{2\pi}}{2q}. \tag{4.5}$$

We note that this result corresponds to Binet's second expression for $\ln \Gamma(q)$ [5].

Some particular evaluations of $T_0(q)$ are

$$\int_0^\infty \frac{\tan^{-1} t}{e^{2\pi t} - 1}\, dt = \frac{1}{2} - \frac{\ln \sqrt{2\pi}}{2},$$
$$\int_0^\infty \frac{\tan^{-1} t}{e^{\pi t} - 1}\, dt = \frac{1}{2} - \frac{\ln 2}{2},$$
$$\int_0^\infty \frac{\tan^{-1} t}{e^{\pi t/2} - 1}\, dt = \frac{1}{2} - \ln \pi - 2\ln 2 + 2\ln \Gamma(\tfrac{1}{4}).$$

Example 4.3. The case $m = 1$ in (4.4) yields

$$2T_0(q) + L_1(q) = \frac{A_2(q)}{2q^2} - \frac{B_2(q)\ln q}{2q^2} + \frac{1}{4}, \tag{4.6}$$

and using $A_2(q) = 2\zeta'(-1, q)$, $B_2(q) = q^2 - q + 1/6$, and the expression for $T_0(q)$ obtained in (4.5), we get

$$\begin{aligned} L_1(q) &= \int_0^\infty \frac{t\, \ln(1 + t^2)}{e^{2\pi q t} - 1}\, dt \\ &= \frac{1}{q^2}\zeta'(-1, q) - \frac{\ln \Gamma(q)}{q} + \frac{\ln \sqrt{2\pi}}{q} - \left(\frac{1}{12q^2} - \frac{1}{2}\right)\ln q - \frac{3}{4}. \end{aligned} \tag{4.7}$$

We see that $L_1(q)$ will evaluate to special values whenever $\zeta'(-1, q)$ does. Particular examples of the latter are

$$\zeta'(-1, \tfrac{1}{2}) = -\tfrac{1}{2}\zeta'(-1) - \tfrac{1}{24}\ln 2$$

and

$$\zeta'(-1, \tfrac{1}{4}) = -\tfrac{1}{8}\zeta'(-1) + G/4\pi,$$

and particular evaluations of $L_1(q)$ are

$$\int_0^\infty \frac{t\ln(1+t^2)}{e^{2\pi t}-1}\,dt = \zeta'(-1) + \ln\sqrt{2\pi} - \frac{3}{4},$$
$$\int_0^\infty \frac{t\ln(1+t^2)}{e^{\pi t}-1}\,dt = -2\zeta'(-1) + \frac{2}{3}\ln 2 - \frac{3}{4},$$
$$\int_0^\infty \frac{t\ln(1+t^2)}{e^{\pi t/2}-1}\,dt = -2\zeta'(-1) + \frac{5}{3}\ln 2 - \frac{3}{4} + \frac{4G}{\pi} - 4\ln\Gamma(\tfrac{1}{4}) + 4\ln\sqrt{2\pi}.$$

We now evaluate the integrals $T_{2k}(q)$ and $L_{2k+1}(q)$ in terms of elementary functions and the balanced negapolygamma functions (1.11).

Theorem 4.4. Let $k \in \mathbb{N}$. Then,

$$\begin{aligned}(-1)^k T_{2k}(q) &= (-1)^k \int_0^\infty \frac{t^{2k}\tan^{-1} t}{e^{2\pi q t}-1}\,dt \\ &= \frac{1}{2(2k+1)^2} - \frac{\ln q}{2(2k+1)} + \frac{1}{8kq} \\ &\quad + \frac{1}{4}\sum_{j=0}^{k-1} \frac{B_{2j+2}}{(j+1)(2k-2j-1)}\frac{1}{q^{2j+2}} \\ &\quad + \frac{1}{2}\sum_{j=0}^{2k} (-1)^j \frac{(2k)!}{(2k-j)!}\frac{\psi^{(-1-j)}(q)}{q^{j+1}}\end{aligned} \tag{4.8}$$

and

$$T_0(q) = \frac{1}{2} - \frac{\ln q}{2} + \frac{\ln q}{4q} + \frac{\ln\Gamma(q)}{2q} - \frac{\ln\sqrt{2\pi}}{2q}.$$

Similarly, for $k \geq 0$,

(4.9)

$$\begin{aligned}(-1)^{k+1}L_{2k+1}(q) &= (-1)^{k+1}\int_0^\infty \frac{t^{2k+1}\ln(1+t^2)}{e^{2\pi q t}-1}\,dt \\ &= \frac{1}{(2k+2)^2} - \frac{\ln q}{2k+2} + \frac{1}{2q(2k+1)} \\ &\quad + \frac{1}{2}\sum_{j=0}^{k-1} \frac{B_{2j+2}}{(j+1)(2k-2j)}\frac{1}{q^{2j+2}} + \frac{B_{2k+2}}{(2k+2)q^{2k+2}}(\ln q - H_{2k+1}) \\ &\quad + \sum_{j=0}^{2k+1} (-1)^j \frac{(2k+1)!}{(2k-j+1)!}\frac{\psi^{(-1-j)}(q)}{q^{j+1}}.\end{aligned}$$

Proof. We first derive the expression for $T_{2k}(q)$. For any differentiable function f we have

$$q\frac{\partial}{\partial q} f(qt) = t\frac{\partial}{\partial t} f(qt),$$

so

$$\begin{aligned} q\frac{d}{dq}T_{2k}(q) &= \int_0^\infty \left(t^{2k}\tan^{-1}t\right) q\frac{\partial}{\partial q}\left(\frac{1}{e^{2\pi qt}-1}\right) dt \\ &= \int_0^\infty \left(t^{2k+1}\tan^{-1}t\right) \frac{\partial}{\partial t}\left(\frac{1}{e^{2\pi qt}-1}\right) dt \\ &= -(2k+1)T_{2k}(q) - \int_0^\infty \frac{t^{2k+1}\,dt}{(e^{2\pi qt}-1)(1+t^2)}. \end{aligned}$$

Thus

$$\frac{d}{dq}\left(q^{2k+1}T_{2k}(q)\right) = -q^{2k}\int_0^\infty \frac{t^{2k+1}\,dt}{(e^{2\pi qt}-1)(1+t^2)},$$

and since

$$t^{2k} = (-1)^k + (-1)^{k+1}(1+t^2)\sum_{j=0}^{k-1}(-1)^j t^{2j},$$

we have

$$\begin{aligned} (-1)^{k+1}\frac{d}{dq}\left(q^{2k+1}T_{2k}(q)\right) = q^{2k}\int_0^\infty \frac{t\,dt}{(e^{2\pi qt}-1)(1+t^2)} \\ - q^{2k}\sum_{j=0}^{k-1}(-1)^j\int_0^\infty \frac{t^{2j+1}\,dt}{(e^{2\pi qt}-1)}. \end{aligned}$$

Using (3.3) and (3.7) we obtain

$$(-1)^{k+1}\frac{d}{dq}\left(q^{2k+1}T_{2k}(q)\right) = \frac{1}{2}q^{2k}\ln q - \frac{1}{4}q^{2k-1} - \frac{1}{2}q^{2k}\psi(q) - \frac{1}{4}\sum_{j=0}^{k-1}\frac{B_{2j+2}}{j+1}q^{2k-2j-2}.$$

Now, for $k \geq 1$ each of the terms on the right-hand side is integrable at $q = 0$, so that

(4.10)

$$\begin{aligned} (-1)^{k+1}T_{2k}(q) &= \frac{-1}{2(2k+1)^2} + \frac{\ln q}{2(2k+1)} - \frac{1}{8kq} - \frac{1}{2q^{2k+1}}\int_0^q r^{2k}\psi(r)dr \\ &\quad -\frac{1}{4}\sum_{j=0}^{k-1}\frac{B_{2j+2}}{j+1}\frac{q^{-2j-2}}{2k-2j-1} + \frac{c_{2k}}{q^{2k+1}}, \end{aligned}$$

where c_{2k} is a constant of integration which can be determined by studying the behavior of $q^{2k+1}T_{2k}(q)$ as $q \to 0$:

$$\begin{aligned} c_{2k} &= (-1)^{k+1}\lim_{q\to 0} q^{2k+1}T_{2k}(q) \\ &= (-1)^{k+1}\frac{(2k)!}{4(2\pi)^{2k}}\zeta(2k+1) = -\frac{1}{2}\zeta'(-2k). \end{aligned}$$

The evaluation of the limit above is obtained by replacing $\tan^{-1}(x/q)$ by $\pi/2$ in

$$q^{2k+1}T_{2k}(q) = \int_0^\infty \frac{x^{2k}\tan^{-1}(x/q)}{e^{2\pi x}-1}dx$$

and employing formula [4](3.411.1):

$$\int_0^\infty \frac{x^{\nu-1}}{e^{\mu x}-1}dx = \frac{1}{\mu^\nu}\Gamma(\nu)\zeta(\nu), \quad \text{Re}\,\mu > 0, \text{Re}\,\nu > 1. \tag{4.11}$$

The final step in the evaluation of $T_{2k}(q)$ uses the result

$$\int_0^q r^n \psi(r)\,dr \;=\; n!\sum_{j=0}^{n} \frac{(-1)^j}{(n-j)!} q^{n-j}\psi^{(-1-j)}(q) \;-\; n!(-1)^n\psi^{(-1-n)}(0), \tag{4.12}$$

valid for $n \in \mathbb{N}$, which can be obtained from the corresponding indefinite integral given in [3]. Since

$$\psi^{(-1-n)}(0) = \frac{1}{n!}\left[\zeta'(-n) - \frac{H_n B_{n+1}}{n+1}\right],$$

we see that for $n = 2k$ the boundary term above precisely cancels the term proportional to the integration constant c_{2k}, thus leading to the explicit formula (4.8).

The formula for $L_{2k+1}(q)$ is derived in a similar way. We start with

$$\ln(1+t^2) \;=\; \frac{d}{dt}\left[t\ln(1+t^2) - 2t + 2\tan^{-1} t\right]$$

and integrate by parts, observing that

$$\frac{\partial}{\partial t}\left[\frac{t^{2k+1}}{e^{2\pi q t}-1}\right] \;=\; \frac{(2k+1)t^{2k}}{e^{2\pi q t}-1} + t^{2k} q \frac{\partial}{\partial q}\left[\frac{1}{e^{2\pi q t}-1}\right],$$

to conclude that

$$\frac{\partial}{\partial q}\left(q^{2k+2}\, L_{2k+1}(q)\right) \;=\; -2q\frac{\partial}{\partial q}\left(q^{2k+1}T_{2k}(q)\right) + \frac{(-1)^{k+1}B_{2k+2}}{(2k+2)q}.$$

Using the expression (4.10) for $T_{2k}(q)$ we obtain $L_{2k+1}(q)$ up to a constant of integration:

$$\begin{aligned}(-1)^{k+1}L_{2k+1}(q) = {} & \frac{1}{(2k+2)^2} - \frac{\ln q}{2k+2} + \frac{1}{2q(2k+1)} + \frac{1}{q^{2k+2}}\int_0^q r^{2k+1}\psi(r)dr \\ & + \frac{1}{2}\sum_{j=0}^{k-1}\frac{B_{2j+2}}{(j+1)(2k-2j)}\frac{1}{q^{2j+2}} + \frac{B_{2k+2}\,\ln q}{(2k+2)q^{2k+2}} + \frac{c_{2k+1}}{q^{2k+2}}.\end{aligned}$$

As before, the constant c_{2k+1} can be determined by evaluating

$$c_{2k+1} \;=\; \lim_{q\to 0}\left[(-1)^{k+1}q^{2k+2}L_{2k+1}(q) - \frac{B_{2k+2}}{2k+2}\ln q\right].$$

Note that

$$q^{2k+2}L_{2k+1}(q) \;=\; -2\ln q\int_0^\infty \frac{x^{2k+1}\,dx}{e^{2\pi x}-1} + \int_0^\infty \frac{x^{2k+1}\,\ln(x^2+q^2)}{e^{2\pi x}-1}dx,$$

so that, in view of (3.3), the limit above is given simply by

$$\begin{aligned}(-1)^{k+1}c_{2k+1} \;&=\; 2\int_0^\infty \frac{x^{2k+1}\,\ln x}{e^{2\pi x}-1}dx \\ &=\; 2\frac{\partial}{\partial \nu}\left[\frac{\Gamma(\nu)\zeta(\nu)}{(2\pi)^\nu}\right]\Bigg|_{\nu=2k+2} = \frac{\partial}{\partial\nu}\left[\frac{\zeta(1-\nu)}{\cos(\pi\nu/2)}\right]\Bigg|_{\nu=2k+2} \\ &=\; (-1)^k\zeta'(-2k-1),\end{aligned}$$

and thus $c_{2k+1} = -\zeta'(-2k-1)$. In the second line above we used the functional equation for the Riemann zeta function. This time, however, the boundary term from (4.12) and the term containing the integration constant c_{2k+1} cancel only partially, leaving the term proportional to the harmonic number H_{2k+1} that appears in (4.9).

□

Note. Adamchik [1] informed us that he is able to evaluate the same integrals in terms of the Barnes G-function. This function is uniquely defined by the recurrence formula

$$\begin{aligned} G_{n+1}(z+1) &= \frac{G_{n+1}(z)}{G_n(z)}, \\ G_1(z) &= 1/\Gamma(z), \end{aligned} \tag{4.13}$$

and the condition

$$\frac{d^{n+1}}{dx^{n+1}} \{\log G_n(x)\} \geq 0. \tag{4.14}$$

The integrals $T_{2k}(q)$ and $L_{2k+1}(q)$ are reportedly given by

$$\begin{aligned} (-1)^k T_{2k}(q) = -\frac{\ln q - H_{2k+1}}{2(2k+1)} + \frac{1}{2}\sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \zeta'(-j) q^{-j-1} \\ - \frac{1}{2q^{2k+1}} \sum_{j=1}^{2k} (-1)^j j! \left\{ {2k \atop j} \right\} \ln G_{j+1}(q+1) \end{aligned}$$

and

$$\begin{aligned} (-1)^{k+1} L_{2k+1}(q) = \frac{B_{2k+2}}{2k+2} \frac{\ln q}{q^{2k+2}} - \frac{\ln q - H_{2k+2}}{2k+2} \\ + \sum_{j=0}^{2k+1} (-1)^j \binom{2k+1}{j} \zeta'(-j) q^{-j-1} \\ - \frac{1}{q^{2k+2}} \sum_{j=1}^{2k+1} (-1)^j j! \left\{ {2k+1 \atop j} \right\} \ln G_{j+1}(q+1), \end{aligned}$$

where $\left\{ {k \atop j} \right\}$ are the Stirling numbers of the second kind and H_k are the harmonic numbers.

We have been unable to determine the values of $T_{2k+1}(q)$ and $L_{2k}(q)$ using the techniques described here.

5. Some related integrals

The formulas for definite integrals developed in the previous sections involve the kernel $(e^{2\pi q t} - 1)^{-1}$. These evaluations, combined with a simple manipulation, lead to a larger class.

Lemma 5.1. Let

$$F(q) \quad = \quad \int_0^\infty \frac{f(t)}{e^{2\pi qt}-1}\,dt. \tag{5.1}$$

Then

$$G(q) \quad := \quad \int_0^\infty \frac{f(t)}{e^{2\pi qt}+1}\,dt = F(q)-2F(2q), \tag{5.2}$$

$$S(q) \quad := \quad \int_0^\infty \frac{f(t)}{\sinh(2\pi qt)}\,dt = 2F(q)-2F(2q). \tag{5.3}$$

Proof. This is a direct consequence of the identities

$$\begin{aligned}\frac{1}{e^x+1} &= \frac{1}{e^x-1}-\frac{2}{e^{2x}-1},\\ \frac{1}{\sinh x} &= \frac{2}{e^x-1}-\frac{2}{e^{2x}-1}.\end{aligned}$$

□

Example 5.2. The expression (4.5) yields

$$\int_0^\infty \frac{\tan^{-1} t}{e^{2\pi qt}+1}\,dt \quad = \quad \ln 2-\frac{1}{2}+\frac{\ln q}{2}-\frac{\ln 2}{4q}+\frac{\ln\Gamma(q)-\ln\Gamma(2q)}{2q}, \tag{5.4}$$

with special evaluations

$$\begin{aligned}\int_0^\infty \frac{\tan^{-1} t}{e^{2\pi t}+1}\,dt &= \frac{3}{4}\ln 2-\frac{1}{2},\\ \int_0^\infty \frac{\tan^{-1} t}{e^{\pi t}+1}\,dt &= \frac{\ln\pi}{2}-\frac{1}{2},\\ \int_0^\infty \frac{\tan^{-1} t}{e^{\pi t/2}+1}\,dt &= -\frac{1}{2}-\ln 2+2\ln\Gamma(\tfrac{1}{4})-\ln\pi.\end{aligned}$$

Similarly

$$\int_0^\infty \frac{\tan^{-1} t}{\sinh(2\pi qt)}\,dt \quad = \quad \ln 2-\frac{\ln(4\pi)}{4q}+\frac{\ln q}{4q}+\frac{\ln\Gamma(q)}{q}-\frac{\ln\Gamma(2q)}{2q}. \tag{5.5}$$

Some particular values are

$$\begin{aligned}\int_0^\infty \frac{\tan^{-1} t}{\sinh(2\pi t)}\,dt &= \frac{1}{2}\ln 2-\frac{1}{4}\ln\pi,\\ \int_0^\infty \frac{\tan^{-1} t}{\sinh(\pi t)}\,dt &= \frac{1}{2}\ln\pi-\frac{1}{2}\ln 2,\\ \int_0^\infty \frac{\tan^{-1} t}{\sinh(\pi t/2)}\,dt &= 4\ln\Gamma(\tfrac{1}{4})-2\ln\pi-3\ln 2.\end{aligned}$$

Example 5.3. The expression (3.5) yields

$$\begin{aligned}\int_0^\infty \frac{t}{(1+t^2)^{k+1}\,(e^{2\pi qt}+1)}\,dt &= \frac{1}{4k}\\ &+\frac{1}{k2^{2k+1}}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{(j-1)!}\binom{2k-j-1}{k-j}2^jq^j\left[\psi^{(j)}(q)-2^{j+1}\psi^{(j)}(2q)\right]\end{aligned} \tag{5.6}$$

and

$$\int_0^\infty \frac{t}{(1+t^2)^{k+1}\,\sinh(2\pi q t)}\,dt = -\frac{1}{2^{2k+2}q}\binom{2k}{k} + \frac{1}{k2^{2k}}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{(j-1)!}\binom{2k-j-1}{k-j}2^j q^j\left[\psi^{(j)}(q) - 2^j\psi^{(j)}(2q)\right]. \tag{5.7}$$

The function $\psi(q)$ and its derivatives do not satisfy a simple duplication formula. Thus the explicit evaluation of (5.6) and (5.7) requires the values of $\psi^{(j)}$ at q and $2q$.

Example 5.4. The expression (4.8) yields

$$\begin{aligned}(-1)^k\int_0^\infty \frac{t^{2k}\,\tan^{-1}t}{e^{2\pi q t}+1}\,dt &= \frac{-1}{2(2k+1)^2} + \frac{\ln 2}{2k+1} + \frac{\ln q}{2(2k+1)} \\ &+ \frac{1}{4}\sum_{j=0}^{k-1}\frac{B_{2j+2}(1-2^{-2j-1})}{(j+1)(2k-2j-1)q^{2j+2}} \\ &+ \frac{1}{2}\sum_{j=0}^{2k}\frac{(-1)^j(2k)!}{(2k-j)!q^{j+1}}\left[\psi^{(-1-j)}(q) - \frac{\psi^{(-1-j)}(2q)}{2^j}\right]\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}(-1)^k\int_0^\infty \frac{t^{2k}\tan^{-1}t}{\sinh(2q\pi t)}\,dt &= \frac{\ln 2}{2k+1} + \frac{1}{8kq} \\ &+ \frac{1}{2}\sum_{j=0}^{k-1}\frac{B_{2j+2}(1-2^{-2j-2})}{(j+1)(2k-2j-1)q^{2j+2}} \\ &+ \sum_{j=0}^{2k}\frac{(-1)^j(2k)!}{(2k-j)!q^{j+1}}\left[\psi^{(-1-j)}(q) - \frac{\psi^{(-1-j)}(2q)}{2^{j+1}}\right].\end{aligned} \tag{5.9}$$

Example 5.5. The expression (4.9) yields

$$\begin{aligned}(-1)^{k+1}\int_0^\infty \frac{t^{2k+1}\,\ln(1+t^2)}{e^{2\pi q t}+1}\,dt &= -\frac{1}{(2k+2)^2} + \frac{2\ln 2}{2k+2} + \frac{\ln q}{2k+2} \\ &+ \frac{1}{2}\sum_{j=0}^{k-1}\frac{B_{2j+2}\,(1-2^{-2j-1})}{(j+1)(2k-2j)q^{2j+2}} \\ &+ \frac{B_{2k+2}}{(2k+2)q^{2k+2}}\left[(1-\frac{1}{2^{2k+1}})\ln q - \frac{\ln 2}{2^{2k+1}} - (1-\frac{1}{2^{2k+1}})H_{2k+1}\right] \\ &+ \sum_{j=0}^{2k+1}\frac{(-1)^j(2k+1)!}{(2k-j+1)!q^{j+1}}\left[\psi^{(-1-j)}(q) - \frac{\psi^{(-1-j)}(2q)}{2^j}\right]\end{aligned} \tag{5.10}$$

and

$$(5.11)\quad (-1)^{k+1}\int_0^\infty \frac{t^{2k+1}\,\ln(1+t^2)}{\sinh(2\pi qt)}\,dt = \frac{2\ln 2}{2k+2} + \frac{1}{2q(2k+1)} + \sum_{j=0}^{k-1}\frac{B_{2j+2}(1-2^{-2j-2})}{(j+1)(2k-2j)q^{2j+2}} + \frac{2B_{2k+2}}{(2k+2)q^{2k+2}}\left[(1-\frac{1}{2^{2k+2}})\ln q - \frac{\ln 2}{2^{2k+2}} - (1-\frac{1}{2^{2k+2}})H_{2k+1}\right] + 2\sum_{j=0}^{2k+1}\frac{(-1)^j(2k+1)!}{(2k-j+1)!q^{j+1}}\left[\psi^{(-1-j)}(q) - \frac{\psi^{(-1-j)}(2q)}{2^{j+1}}\right].$$

For example,

$$(5.12)\quad \int_0^\infty \frac{t\,\ln(1+t^2)}{\sinh(2\pi qt)}\,dt = \frac{1}{q^2}\left[2\zeta'(-1,q) - \frac{1}{2}\zeta'(-1,2q)\right] - \frac{1}{q}\left(2\ln\Gamma(q) - \ln\Gamma(2q)\right) - \ln 2 + \frac{\ln\pi}{2q} - \frac{\ln q}{8q^2} + \frac{\ln 2}{24q^2} + \frac{\ln 2}{2q}.$$

Some special values are

$$\begin{aligned}
\int_0^\infty \frac{t\,\ln(1+t^2)}{\sinh(2\pi t)}\,dt &= -\frac{11}{24}\ln 2 + \frac{1}{2}\ln\pi + \frac{3}{2}\zeta'(-1),\\
\int_0^\infty \frac{t\,\ln(1+t^2)}{\sinh(\pi t)}\,dt &= \frac{1}{3}\ln 2 - \ln\pi - 6\zeta'(-1),\\
\int_0^\infty \frac{t\,\ln(1+t^2)}{\sinh(\pi t/2)}\,dt &= 6\ln 2 + 4\ln\pi + \frac{8G}{\pi} - 8\ln\Gamma(\tfrac{1}{4}).
\end{aligned}$$

Note. Differentiating with respect to the parameter q and evaluating at special values yields many new integrals. For example, the derivative of (4.5) at $q=1$ and $q=2$ yields

$$(5.13)\quad \int_0^\infty \frac{t\,\tan^{-1}t}{\sinh^2\pi t}\,dt = \frac{1}{2\pi} + \frac{\gamma}{\pi} - \frac{\ln\sqrt{2\pi}}{\pi}$$

and

$$(5.14)\quad \int_0^\infty \frac{t\,\tan^{-1}t}{\sinh^2 2\pi t}\,dt = -\frac{1}{8\pi} + \frac{\gamma}{2\pi} - \frac{\ln\pi}{8\pi},$$

respectively.

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