THE p-ADIC VALUATION OF STIRLING NUMBERS

ANA BERRIZBEITIA, LUIS A. MEDINA, ALEXANDER C. MOLL, VICTOR H. MOLL, AND LAINE NOBLE

Abstract. Let $p > 2$ be a prime. The p-adic valuation of Stirling numbers of the second kind is analyzed. Two types of tree diagrams that encode this information are introduced. Conditions that describe the infinite branching of these trees, similar to the case $p = 2$, are presented.

1. Introduction

The Stirling numbers of second kind $S(n,k)$, defined for $n \in \mathbb{N}$ and $0 \leq k \leq n$ count the number of ways to partition a set of n elements into exactly k nonempty subsets. They are explicitly given by

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n, \tag{1.1}$$

or, alternatively, by the recurrence

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.2}$$

with the initial conditions $S(0,0) = 1$ and $S(n,0) = 0$ for $n > 0$.

Divisibility properties of integer sequences are expressed in terms of p-adic valuations: given a prime p and a positive integer b, there exist unique integers a, n, with a not divisible by p and $n \geq 0$, such that $b = ap^n$. The number n is called the $p-$*adic valuation* of b. We write $n = \nu_p(b)$. Thus, $\nu_p(b)$ is the highest power of p that divides b.

The 2-adic valuation of the Stirling numbers was discussed in [1]. The main conjecture described there is that the partitions of the positive integers $\mathbb{N}$ in classes of the form

$$C_{m,j} := \{2^m i + j : \, i \in \mathbb{N}\} \tag{1.3}$$

where the index i starts at the point where $2^m i + j \geq k$, leads to a clear pattern for the 2-adic valuations of Stirling numbers. The class $C_{m,j}$ is called *constant* if $\nu_2(C_{m,j})$ consists of a single value. This single value is called the constant of the class $C_{m,j}$. The parameter m in (1.3) is called the *level* of the class. The level are defined inductively as follows: assume that the $(m-1)$-level has been defined and that it consists of the s classes

$$C_{m-1,i_1}, \, C_{m-1,i_2}, \, \ldots, C_{m-1,i_s}. \tag{1.4}$$

Date: August 31, 2009.

2000 *Mathematics Subject Classification.* Primary 11B50, Secondary 05A15.

Key words and phrases. valuations, Stirling numbers.

Each class C_{m-1,i_j} splits into two classes modulo 2^m, namely C_{m,i_j} and $C_{m,i_j+2^{m-1}}$. The m-level is formed by the non-constant classes modulo 2^m. The main conjecture of [1] is now stated:

Conjecture 1.1. *Let $k \in \mathbb{N}$ be fixed. Then we conjecture that*
a) there exists a level $m_0(k)$ and an integer $\mu(k)$ such that for any $m \geq m_0(k)$, the number of nonconstant classes of level m is $\mu(k)$, independently of m;

b) moreover, for each $m \geq m_0(k)$, each of the $\mu(k)$ nonconstant classes splits into one constant and one nonconstant subclass. The latter generates the next level set.

This conjecture was established in [1] only for the case $k = 5$. The proof makes strong use of the recurrence (1.2) and the fact that the 2-adic valuations for $S(n,k)$, for $1 \leq k \leq 4$ are easily determined.

The goal of this paper is to describe a similar behavior for the p-adic valuations of $S(n,k)$ for the case of p an odd prime.

2. Modular and p-adic trees

Consider a sequence of positive integers $\mathbf{a} := \{a_j\}$, with a_j dividing a_{j+1}. The tree associated to $\mathbf{a}$ starts with a root vertex and as first generation has a_1 vertices representing all the residue classes modulo a_1. Each vertex, branches into the second generation into a_2/a_1 siblings corresponding to residues modulo a_2. For instance, the vertex corresponding to 1 in the first generation branches into the vertices corresponding to the classes $1, 1+a_1,\ 1+2a_1, \ldots, 1+(a_2/a_1-1)a_1$. The m-th generation of the tree contains a_m vertices that provide a partition of $\mathbb{N}$ into a collection of residue classes.

We define two types of trees that encode the p-adic valuations of the Stirling numbers. In each case, the sequence a_j described above corresponds to the period of the function $S(n,k)$ modulo powers of the prime p. This periodicity is described first.

Given a prime p and an integer $m \in \mathbb{N}$, the sequence $S(n,k) \bmod p^m$, for k fixed is periodic of period $L_m := (p-1)p^{m+\alpha(p,k)-1}$, where $\alpha(p,k)$ is defined by $p^{\alpha(p,k)} < k \leq p^{\alpha(p,k)+1}$. Observe that L_m divides L_{m+1} with quotient p. For instance, the Stirling numbers $S(n,5)$ modulo 3^2 have period 18, the repeating block being

$$\{1, 6, 5, 6, 3, 0, 4, 6, 8, 6, 6, 0, 7, 6, 2, 6, 0, 0\}. \tag{2.1}$$

The question of the minimal period of $S(n,k)$ modulo p^m has been discussed by Nijenhuis and Wilf [5] and Kwong [4]. This paper is not concerned with minimality and uses only the period L_m defined above. See [2], [3] for more information on this topic.

Indexing. Fix a prime p and $m \in \mathbb{N}$. The residue classes modulo L_{m+1} can be written in the form

$$n = i_0 + i_1L_1 + i_2L_2 + \cdots + i_mL_m, \tag{2.2}$$

where $0 \leq i_0 \leq p-2 = L_1 - 1$ and $0 \leq i_r \leq p-1$ for $1 \leq r \leq m$. The coefficients i_r are uniquely determined by n.

Definition 2.1. The set

$$D_{m,j} := \{n \in \mathbb{N} : \; n \equiv j \mod L_m\}, \tag{2.3}$$

will be called a *class* at the m-th level.

The tree associated to the sequence $\{L_j\}$ is now modified by branching a subset of the nodes at the m-th level. Recall that each vertex at the m-th level corresponds to a class $D_{m,j}$. Two different branching criteria produce the two types of trees mentioned above.

Note. We assume throughout that $k < p$. This has the advantage that the term $k!$ in (1.1) is invertible modulo p, so it may be ignored in the question of divisibility of $S(n,k)$ by p. This assumption also yields $\alpha(p,k) = 0$ and the sequence $S(n,k) \bmod p^m$ is periodic of period $L_m := (p-1)p^{m-1} = \varphi(p^m)$, where φ is the Euler totient function.

Modular trees. A *branching* vertex is one for which

$$S(D_{m,j}, k) \equiv 0 \bmod p^m. \tag{2.4}$$

The remaining vertices will be called *terminal*. The periodicity of $S(n,k)$ modulo p^m shows that (2.4) is independent of the element in the class $D_{m,j}$. Indeed, if $n_1 \equiv n_2 \bmod p-1$, then

$$S(n_1,k) \equiv S(n_2,k) \bmod p. \tag{2.5}$$

In particular, if $S(n_1,k) \not\equiv 0 \bmod p$, then $\nu_p(S(n,k)) = 0$ for all $n \equiv n_1 \bmod (p-1)$. On the other hand, if $S(n_1,k) \equiv 0 \bmod p$, then is not guaranteed that $\nu_p(S(n_1,k))$ and $\nu_p(S(n_2,k))$ are the same.

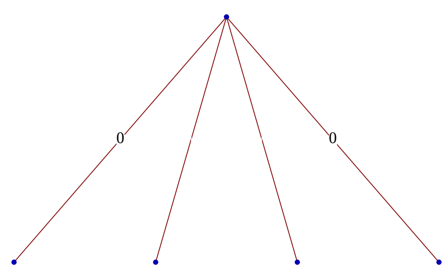

Figure 1. The first level for $p = 5$ and $k = 3$.

The first level consists of $p-1$ vertices corresponding to the residue classes modulo $L_1 = p-1$. The branches that correspond to values with $S(n,k) \not\equiv 0 \bmod p$ are labeled by the valuation, namely 0, they are terminated. The remaining branches are not labeled and are split modulo $(p-1)p$ into the next level.

The p edges coming out of a branching vertex i correspond to the p numbers $n_j := i + (p-1)j$ with $0 \leq j \leq p-1$. All these indices are congruent modulo $p-1$ and distinct modulo $L_2 := p(p-1)$. Their corresponding Stirling numbers

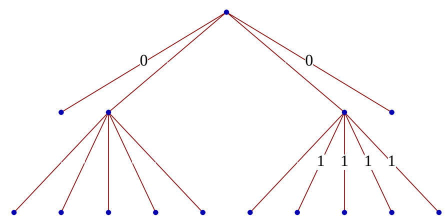

FIGURE 2. The first and second level for $p = 5$ and $k = 3$.

satisfy $\nu_p(S(n_j, k)) \geq 1$. At this point, we consider the values of $S(n_j, k)$ modulo p^2. Using the fact that

$$a \equiv b \bmod L_2 \text{ implies } S(a, k) \equiv S(b, k) \bmod p^2, \tag{2.6}$$

we conclude that, if $S(n_j, k) \not\equiv 0 \bmod p^2$, then $\nu_p(S(n_j, k)) = 1$. Those vertices are labeled 1 and terminated. The remaining vertices, namely those for which $S(n_j, k) \equiv 0 \bmod p$, are split again modulo p. For these classes we have that their valuations are at least 2.

Example 2.2. Figures 1 and 2 illustrate the first two levels for $p = 5$ and $k = 3$. At the first level, the integers are divided in classes modulo $L_1 = 4$ and the corresponding Stirling numbers are considered modulo $p = 5$. Figure 1 shows that two classes, those congruent to 0 and 3 modulo 4, have Stirling numbers not divisible by 5. Thus, $\nu_5(D_{1,0}) = \nu_5(D_{1,3}) = 0$. These branches are labeled by their valuation. The Stirling numbers corresponding to the two remaining classes *do not* have the same valuation. At this point one can only conclude that $\nu_5(n, 3) \geq 1$ for $n \equiv 1$ or $2 \bmod 4$. For example,

$$\begin{aligned} \nu_5(S(13,3)) &= \nu_5(7508501) = 3 \\ \nu_5(S(17,3)) &= \nu_5(5652751651) = 2. \end{aligned}$$

The numbers congruent to 1 modulo 4 split into classes congruent to 1, 5, 9, 13, 17 modulo $L_2 = 20 = 4 \cdot 5$. Similarly, the class 2 modulo 4 becomes the five classes 2, 6, 10, 14, 18 modulo 20. The construction of the tree continues by testing the Stirling numbers corresponding to these indices modulo $5^2 = 25$.

Lemma 2.3. *If a vertex j appears at the level m, then $\nu_p(D_{m,j}) \geq m - 1$.*

The modular tree has the advantage that it is easy to generate: at the m-th level the Stirling numbers are tested modulo p^m. Moreover, if a branch terminates at level m, then the corresponding indices satisfy $\nu_p(n) = m$. Experimentally it is found that they become very large. This is a disadvantage. An alternative to these trees is proposed next.

p-adic trees. This second type of trees differ from the modular trees in the mechanism employed to terminate a branch. Recall that, at level m, each vertex corresponds to a class $D_{m,j}$. In this new type of trees, a vertex is declared *terminal* if the corresponding class has constant p-adic valuation; i.e. for $n \in D_{m,j}$, the valuation $\nu_p(S(n, k))$ is independent of n. In practice, *this is decided in an experimental* *manner*. Given a parameter T_∞, the class $D_{m,j}$ is declared constant if its first T_∞-values have the same valuation. Naturally a class is branched if it contains two elements with distinct valuation. The last section presents an algorithm where this experimental procedure is replaced by a rigorous procedure.

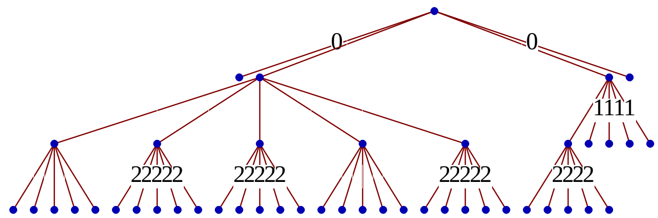

FIGURE 3. The first three levels of the modular tree for $p = 5$ and $k = 3$

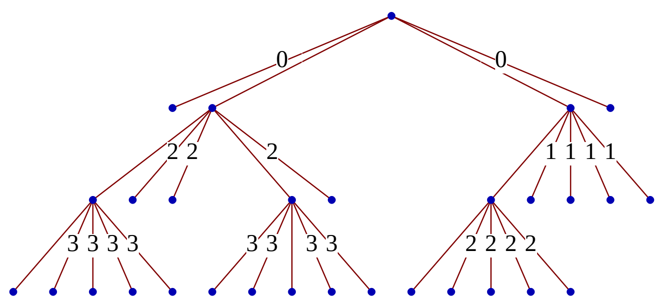

FIGURE 4. The first three levels of the p-adic tree for $p = 5$ and $k = 3$

Figures 3 and 4 present the first three levels of the modular and 5-adic trees for $k = 3$, respectively. Observe that in the modular trees there are vertices that branch to the third level only to discover that each of the siblings has the same constant valuation. This is the case for the second vertex at the second level. The p-adic tree detects this phenomena at the second level. This accounts for the fact that, in general, p-adic trees are smaller that the modular trees.

The study of the p-adic valuations of Stirling numbers begins with an elementary criteria.

Lemma 2.4. *Let i be a residue class modulo L_1 such that $S(i,k) \not\equiv 0 \bmod p$. Then i is a terminal vertex and the class $D_{1,i}$ has p-adic valuation 0.*

Example 2.5. Take $p = 7$ and $k = 4$. The periodicity of the Stirling numbers shows that if $i \equiv 0 \bmod L_1 = 6$, then $S(i,4) \equiv S(6,4) \bmod 7$. In this case $S(6,4) =$

$65 \equiv 2 \bmod 7$, thus $S(6t, 4) \equiv 2 \bmod 7$. Therefore the class $D_{1,0}$ has constant 7-adic valuation 0 and the vertex 0 is terminal.

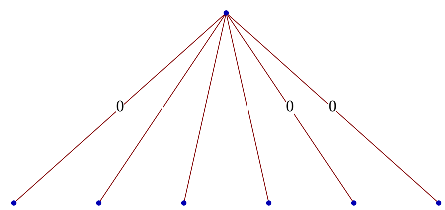

FIGURE 5. The first level for $p = 7$ and $k = 4$

Figure 5 shows that the vertices 0, 4 and 5 are terminal and the classes $D_{1,0}$, $D_{1,4}$ and $D_{1,5}$ have constant valuation 0. The remaining classes $D_{1,1}$, $D_{1,2}$ and $D_{1,3}$ are now tested further. The description is given for the class $D_{1,1}$, the others are treated using the same ideas. Details are given in section 5. For level $m = 1$ and if $n \equiv 1 \bmod L_1 = 6$, we have

$$S(1 + 6t, 4) \equiv S(L_1 + 1, 4) = S(7, 4) \bmod 7. \tag{2.7}$$

The value $S(7, 4) = 350$, shows that $S(7, 4) \equiv 0 \bmod 7$. We conclude that the valuation $\nu_7(S(1+6t, 4)) \geq 1$. The question of whether the class $D_{1,1}$ is constant is decided in an experimental manner. The required number of values to declare the branching of a class is usually small. For instance, for $D_{1,1}$, the value $T_\infty = 7$ suffices in practice. Indeed, $S(43, 4) = 7^2 \times N$, with $N = 6579076481931927346175 0 \not\equiv 0 \bmod 7$. Therefore, the class $D_{1,1}$ is not constant. The value $T_\infty = 7$ is also sufficient to determine that the two remaining classes, $D_{1,2}$ and $D_{1,3}$ are non-constant.

The analysis of the branched class $D_{1,1}$ is now considered at level 2, that is, the sequence $n = 1 + 6t$ is considered modulo $L_2 = (p-1)p = 42$. The class $D_{1,1}$ splits into the seven classes

$$D_{2,1},\ D_{2,7},\ D_{2,13},\ D_{2,19},\ D_{2,25},\ D_{2,31},\ D_{2,37}. \tag{2.8}$$

The branches of the tree depicted in Figure 6 show the branching from the non-constant class $D_{1,1}$. Recall the indexing for residue classes modulo $L_2 = 42$ in the form $n = i_0 + i_1 L_1$ described in (2.2). The subtree depicted in Figure 6 corresponds to indices with $i_0 = 1$ and the label in each branch is the residue of $S(i_0 + i_1 L_1 + t L_2, 4)$ modulo 7^2.

Note. Figure 6 shows that six of the seven classes in (2.8) are constant. This leaves only the leftmost vertex, with label $i_0 = 1$, $i_1 = 0$, for possible branching. This is the class

$$D_{2,1} = \{n \in \mathbb{N} : n \equiv 1 \bmod 42\} = \{1 + 42t :\ t \in \mathbb{N}\}. \tag{2.9}$$

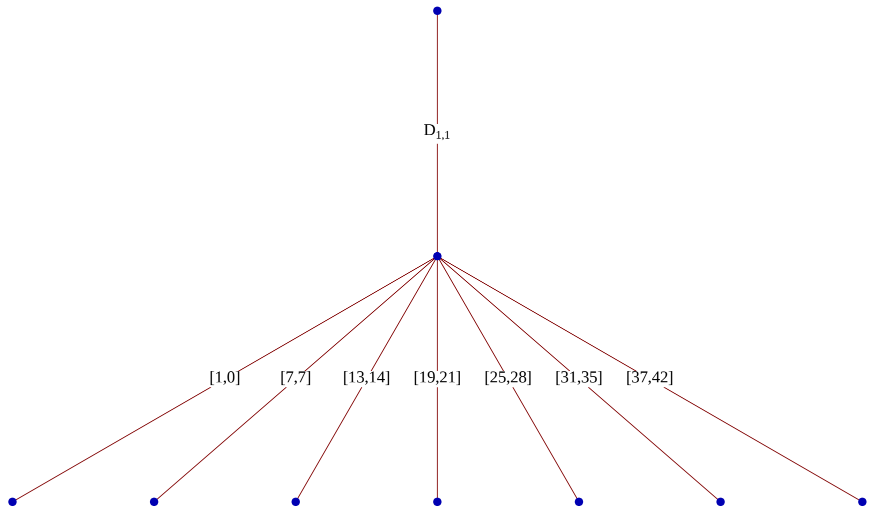

FIGURE 6. Branching into the second level

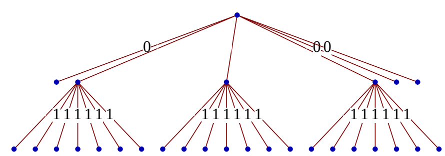

FIGURE 7. The second level for $p = 7$ and $k = 4$

Then one checks that $D_{2,1}$ branches to level 3 by computing the values of $\nu_7(S(1+42t, 4))$ for $0 \leq t \leq T_\infty$. Once again, $T_\infty = 7$ is sufficient to verify that $D_{2,1}$ is a non-constant class.

Note. In order to declare a class $D_{m,j}$ constant, we compute the first T_∞ values of $\nu_p(S(j + tL_m, k))$ for a precribed length T_∞. The class $D_{m,j}$ is considered constant if these values are the same. Experiments have shown that a modest value T_∞ suffices in each case.

3. Criteria for branching on modular trees

The goal of this section is to discuss the branching mechanism on modular trees. The assumption $k < p$ and the labeling described in (2.2) are imposed throughout.

Recall that each vertex at level m represents a class $D_{m,j} = \{n \in \mathbb{N} : n \equiv j \bmod L_m\}$. Using the value $L_1 = p - 1$, it follows that level 1 is represented by the numbers $\{0, 1, 2, \ldots, p-2\}$ and a vertex i_0 branches to level 2 if

$$S(i_0 + tL_1, k) \equiv 0 \bmod p. \tag{3.1}$$

The terminal (= non-branching) vertices stop at level 1 and they are labeled by the valuation ($= 0$).

The branched vertices are now considered modulo L_2 and they branch to level 3 if

$$S(i_0 + i_1 L_1 + tL_2, k) \equiv 0 \bmod p^2. \tag{3.2}$$

Continuing with this process, the vertices appearing at level m correspond to branches that have not been terminated up to this point. Therefore the corresponding classes satisfy $\nu_p(D_{m,j}) \geq m-1$.

Lemma 3.1. *Assume the a vertex i at level m satisfies*

$$S(i, k) \not\equiv 0 \bmod p^m. \tag{3.3}$$

Then $\nu_p(D_{m,i}) = m-1$ and the vertex is terminal.

The first result states that the left most vertex always terminates.

Lemma 3.2. *Assume $n \equiv 0 \bmod L_1$. Then $S(n,k) \not\equiv 0 \bmod p$ and $\nu_p(S(n,k)) = 0$. Therefore the vertex $D_{1,0}$ is terminal.*

Proof. Write $n = i(p-1)$ and then

$$S(n,k) = \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i(p-1)}. \tag{3.4}$$

Fermat's little theorem shows that $r^{p-1} \equiv 1 \bmod p$, thus

$$S(n,k) \equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} \bmod p. \tag{3.5}$$

The result now follows from

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} = -1. \tag{3.6}$$

□

Definition 3.3. Assume $1 \leq r \leq p-1$, so that $r^{L_1} \equiv 1 \bmod p$. Define β_r by the identity $r^{L_1} \equiv 1 + p\beta_r \bmod p^2$.

The distribution of β_r is complicated. Figure 8 shows the values of β_r for $2 \leq r \leq 7918$. The prime 7919 is the 1000-th prime.

The expression β_r is now employed to compute the residue of $S(n,k)$ modulo p^2.

Theorem 3.4. *Let $n \equiv i_0 + i_1 L_1 \bmod L_2$. Then*

$$S(n,k) \equiv \frac{(-1)^k}{k!} \left[\sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} + i_1 p \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \beta_r \right] \bmod p^2. \tag{3.7}$$

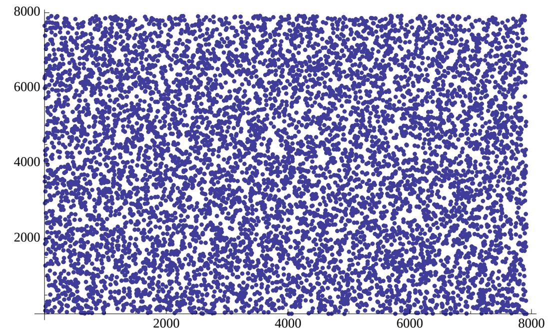

FIGURE 8. The values of β_r for the prime $p = 7919$.

Proof. The relation $r^{L_2} \equiv 1 \bmod p^2$ yields

$$S(n,k) \equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} r^{i_1 L_1} \bmod p^2. \tag{3.8}$$

Therefore

$$\begin{aligned} S(n,k) &\equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} r^{i_1 L_1} \bmod p^2 \\ &\equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} (1+p\beta_r)^{i_1} \bmod p^2 \\ &\equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} (1+i_1 p\beta_r) \bmod p^2 \\ &\equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} + \frac{(-1)^k}{k!} i_1 p \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \beta_r \bmod p^2. \end{aligned}$$

This gives the result. □

Corollary 3.5. *Let $n \equiv i_0 \bmod L_1$. Then*

$$S(n,k) \equiv \frac{(-1)^k}{k!} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \bmod p. \tag{3.9}$$

Therefore the class D_{1,i_0} branches to level 2 precisely when

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \equiv 0 \bmod p. \tag{3.10}$$

Corollary 3.6. *Let $n \equiv i_0 \bmod L_1$ with $1 \leq i_0 \leq k-1$. Then the class D_{1,i_0} branches to level 2.*

Proof. The result follows from the previous corollary and

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} = (-1)^k k! S(i_0,k) = 0 \tag{3.11}$$

in view of the assumption $1 \leq i_0 \leq k-1$. □

Corollary 3.7. *Let $D_{2,j}$ be a class at level 2 with $j \equiv i_0 + i_1 L_1 \bmod L_2$. Then $D_{2,j}$ branches to level 3 precisely when*

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \equiv 0 \bmod p^2 \tag{3.12}$$

and

$$i_1 \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \beta_r \equiv 0 \bmod p. \tag{3.13}$$

Corollary 3.8. *Let $D_{2,j}$ be a class at level 2 with $j \equiv i_0 + i_1 L_1 \bmod L_2$. Assume $1 \leq i_0 \leq k-1$. Then $D_{2,j}$ branches to level 3 precisely when*

$$i_1 \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{i_0} \beta_r \equiv 0 \bmod p. \tag{3.14}$$

In particular, the class corresponding to $i_1 = 0$ always branches.

The remainder of this section addresses patterns of branching behavior that continue through increasing m-levels.

Lemma 3.9. *For given p and $1 \leq r < p$,*

$$r^{\varphi(p^m)} \equiv 1 + \beta_r p^m \bmod p^{m+1}$$

for all m, with β_r independent of m.

Proof. Suppose $r^{\varphi(p^{m-1})} \equiv 1 + \beta_r p^{m-1} \bmod p^m$ for some m. Then

$$\begin{aligned} r^{\varphi(p^m)} &= r^{\varphi(p^{m-1})p} \\ &\equiv (1 + \beta_r p^{m-1} + a p^m)^p \bmod p^{m+1} \\ &\equiv 1 + \beta_r p^m \bmod p^{m+1}. \end{aligned}$$

This is the result. □

Now consider a branching class $D_{m,j}$ at level m. Its siblings consist of the p classes $D_{m+1,j+iL_m}$ with $0 \leq i \leq p-1$. The next result states a condition that guarantees the preservation of the pairwise incongruent property. In particular, from some point on, *exactly* one of them will branch.

Theorem 3.10. *Let $m \geq 1$. Assume that for some branching class $D_{m,j}$ all vertices are incongruent, that is,*

$$S(D_{m+1,j+i_1 L_m}, k) \not\equiv S(D_{m+1,j+i_2 L_m}, k) \bmod p^{m+1} \tag{3.15}$$

for $0 \leq i_1 < i_2 \leq p-1$. Then, every subsequent branching class $D_{M,J}$ with $J \equiv j \bmod L_m$ and $M \geq m$,

$$S(D_{M+1,J+i_1 L_M}, k) \not\equiv S(D_{M+1,J+i_2 L_M}, k) \bmod p^{M+1} \tag{3.16}$$

for $0 \leq i_1 < i_2 \leq p-1$.

Proof. Let $d_1, d_2 \in D_{m,j}$ such that $d_1 \not\equiv d_2 \bmod L_{m+1}$. Index these values by

$$\begin{aligned} d_1 &= j + i_1 L_m + x_1 L_{m+1}, \\ d_2 &= j + i_2 L_m + x_2 L_{m+1}, \end{aligned}$$

with $x_1,\, x_2 \in \mathbb{Z}$. By assumption, $S(d_2,k) - S(d_1,k) \not\equiv 0 \bmod p^{m+1}$. Then

$$\begin{aligned} (-1)^k k!\,[S(d_2,k) - S(d_1,k)] &\equiv \sum_{r=1}^{k} (-1)^r \binom{k}{r} [r^{d_2} - r^{d_1}] \bmod p^{m+1} \\ &\equiv \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^j (r^{L_m})^{i_1} [(r^{L_m})^{i_2 - i_1} - 1] \bmod p^{m+1} \\ &\equiv \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^j (1 + \beta_r p^m)^{i_1} [(1 + \beta_r p^m)^{i_2 - i_1} - 1] \bmod p^{m+1} \\ &\equiv (i_2 - i_1) p^m \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^j \beta_r \bmod p^{m+1}. \end{aligned}$$

Observe that the hypothesis of pairwise incongruence imply that

$$\sum_{r=1}^{k} (-1)^r \binom{k}{r} r^j \beta_r \not\equiv 0 \bmod p$$

and conversely.

Since the Stirling numbers associated with the p vertices emanating from $D_{m,j}$ are by hypothesis pairwise incongruent modulo p^{m+1}, exactly one of these vertices branches. Suppose that this branching vertex corresponds to the class $D_{m+1,j+i^* L_m}$. Now write

$$\begin{aligned} d_1 &= j + i^* L_m + i_1^* L_{m+1} + x_1 L_{m+2} \\ d_2 &= j + i^* L_m + i_2^* L_{m+1} + x_2 L_{m+2} \end{aligned}$$

where again $0 \leq i_1^* < i_2^* \leq p-1$. To verify that the pairwise incongruence condition holds at this new level, it is required to show that

$$S(d_2,k) - S(d_1,k) \not\equiv 0 \bmod p^{m+2}. \tag{3.14}$$

Proceeding as before, we have

$$(-1)^k k!\,[S(d_2,k) - S(d_1,k)] \equiv \sum_{r=1}^{k} (-1)^r \binom{k}{r} \left[r^{d_2} - r^{d_1} \right] \quad \bmod p^{m+2}.$$

Expanding as before, it follows that

$$(-1)^k k!\,[S(d_2,k) - S(d_1,k)] \equiv (i_2^* - i_1^*) p^{m+1} \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{j + i^* L_m} \beta_r \bmod p^{m+2}.$$

The assumption (3.13) proves the pairwise incongruence of the Stirling numbers at level $m+2$. $\square$

Corollary 3.11. *Assume the class $D_{m,j}$ has satisfies the pairwise incongruence condition at level m. Then it satisfies at level $m+1$ if and only if*

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^j\beta_r \not\equiv 0 \bmod p. \tag{3.13}$$

holds. Observe that this condition is independent of m.

Note. A branching class $D_{m,j}$ satisfying condition (3.13) separates at level $m+1$ into $p-1$ terminal vertices (with Stirling numbers of constant p-adic valuation m) and exactly one branching vertex. Observe that this is exactly the analog of the main conjecture of [1].

Note. Let j be an index such that the hypothesis of Theorem 3.10 is satisfied for $D_{m,j}$ at level m. Write

$$j = i_0 + i_1L_1 + \cdots + i_{m-1}L_{m-1}, \tag{3.14}$$

and observe that

$$r^j = r^{i_0}\prod_{t=1}^{m-1} r^{L_t i_t} \equiv r^{i_0} \bmod p.$$

Thus, the condition (3.13) depends only on j modulo L_1.

Corollary 3.12. *Using the expansion*

$$j = i_0 + i_1L_1 + \cdots + i_{m-1}L_{m-1}, \tag{3.15}$$

the hypothesis of Theorem 3.10 is satisfied for $D_{m,j}$ at level m if and only if it is satisfied for D_{1,i_0}.

Corollary 3.13. *If (3.13) is not satisfied, that is,*

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^j\beta_r \equiv 0 \bmod p, \tag{3.16}$$

for j, then for d_1, d_2 such that $d_1 \equiv d_2 \equiv j \bmod L_1$ and $d_1 \equiv d_2 \bmod L_m$,

$$S(d_1,k) \equiv S(d_2,k) \bmod p^{m+1}.$$

Example 3.14. This example continues Example 2.2 for $p=5$ and $k=3$. Congruence (3.16) is satisfied for the class $D_{1,1}$, that is, for $n \equiv 1 \bmod L_1 = 4$. Since $29 \equiv 49 \equiv 1 \bmod 4$ and $29 \equiv 49 \bmod 20$, Corollary 3.13 shows that $S(29,3) \equiv S(49,3) \bmod 5^3$. This is depicted in Figure 9. Moreover, the class $D_{2,9}$ corresponds to a terminal vertex: for all n such that $n \equiv 9 \bmod 20$, $\nu_5(S(n,3)) = 2$.

4. Refined branching criteria for modular trees

In this section we describe a refinement of the criteria developed in the previous section. Examples illustrating the use of this criteria are given in Section 5. The goal is to provide a generalization of Theorem 3.10. We start by generalizing Definition 3.3.

Definition 4.1. Fix a prime p. Given $1 \leq r < p$, define the sequence $\beta_{1,r}, \beta_{2,r}, \ldots, \beta_{\omega+1,r}$ by

$$r^{\varphi(p^{\omega+1})} \equiv 1 + \beta_{1,r}p^{\omega+1} + \beta_{2,r}p^{\omega+2} + \ldots + \beta_{\omega+1,r}p^{2\omega+1} \bmod p^{2\omega+2}. \tag{4.1}$$

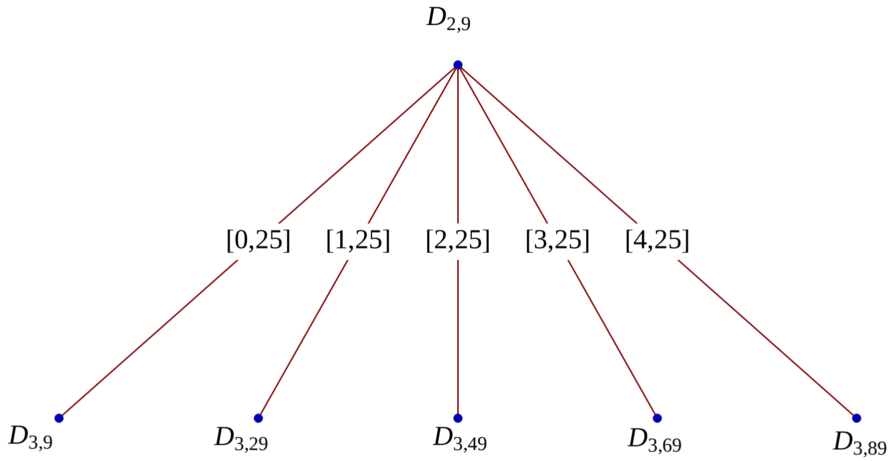

FIGURE 9. Branching from the class $D_{2,9}$. $S(9+i_1\cdot 20,k)\equiv S(9+i_2\cdot 20,k) \bmod 5^3$, for all i_1,i_2.

Next a generalization of Lemma 3.9.

Lemma 4.2. *The coefficients $\beta_{1,r},\beta_{2,r},\ldots,\beta_{\omega,r}$ are independent on ω i.e. if $m>\omega$ then*

$$r^{\varphi(m)} \equiv 1+\beta_{1,r}p^m+\beta_{2,r}p^{m+1}+\cdots+\beta_{\omega,r}p^{\omega+m-1} \textit{ mod } p^{\omega+m} \tag{4.2}$$

Proof. The proof is similar to the one for Lemma 3.9. □

Definition 4.3. Define the condition

$$T_{\omega+1,j}:=\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^j(\beta_{1,r}+\beta_{2,r}p+\ldots+\beta_{\omega+1,r}p^{\omega})\not\equiv 0 \bmod p^{\omega+1} \tag{4.3}$$

for j modulo $L_{\omega+1}$, $\omega\geq 0$. Observe that $T_{1,j}$ coincides with (3.13).

Proposition 4.4. *$T_{\omega+1,j}$ true (or false) implies $T_{\omega+1,j+i_mL_m}$ true (or false) for all $m\geq\omega+1$.*

Proof. For $1\leq r<p$, $r^{L_m}\equiv 1 \bmod p^m$, so $r^{L_m}\equiv 1 \bmod p^{\omega+1}$ for $m\geq\omega+1$. □

Just as the condition $T_{1,j}$ served as a test for a particular branching behavior continuing through m-levels, the collection of conditions $T_{1,j},T_{2,j},\ldots,T_{\omega,j}$ now serves as a test. The implication of these conditions is presented in the following proposition and theorem.

Proposition 4.5. *Consider a branching class $D_{j,m}$. Suppose that $T_{j,m}$ is false, then p^{2m} divides $S(j+iL_m,k)$ for all $i\in\mathbb{N}$ precisely when*

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^j\equiv 0 \textit{ mod } p^{2m}. \tag{4.4}$$

Proof. Note that

$$\begin{aligned}(-1)^k k! S(j+aL_m,k) &\equiv \sum_{r=1}^{k}(-1)^r\binom{k}{r} r^j \bmod p^{2m}\\ &\quad + a\sum_{r=1}^{k}(-1)^r\binom{k}{r} r^j\left(\beta_{1,r}p^m+\beta_{2,r}p^{m+1}+\cdots+\beta_{n,r}p^{2m-1}\right) \bmod p^{2m}.\end{aligned}$$

Since $T_{j,m}$ is false, the second sum is congruent to zero modulo p^{2m}. The result is established. □

Theorem 4.6. *Suppose that $D_{m,j}$ is a branching class. If $T_{1,j}, T_{2,j}, \ldots, T_{m-1,j}$ are false and $T_{m,j}$ is true, then all siblings of $D_{m,j}$ at level $m+1$ are incongruent modulo p^{2m}. Moreover, all siblings of every subsequent branching class $D_{M,J}$ with $J \equiv j \bmod L_m$ and $M > m$ are incongruent modulo p^{m+M}.*

Proof. The proof of this theorem is similar to the one of Theorem 3.10. Let $d_1, d_2 \in D_{m,j}$ such that $d_1 \not\equiv d_2 \bmod L_{m+1}$. Index these values by

$$\begin{aligned} d_1 &= j + i_1 L_m + x_1 L_{m+1},\\ d_2 &= j + i_2 L_m + x_2 L_{m+1}\end{aligned}$$

where $i_1 < i_2$ and $x_1, x_2 \in \mathbb{Z}$. Note that

$$(-1)^k k![S(d_1,k) - S(d_2,k)] \equiv (i_2 - i_1)p^m T_{m,j} \bmod p^{2m}. \tag{4.5}$$

Here we are abusing the notation and letting

$$T_{m,j} = \sum_{r=1}^{k}(-1)^r\binom{k}{r} r^j\left(\beta_{1,r} + \beta_{2,r}p^1 + \cdots + \beta_{n,r}p^{m-1}\right) \tag{4.6}$$

rather than the test itself. By assumption $T_{m,j} \not\equiv 0 \bmod p^m$, hence (4.5) is not congruent to zero modulo p^{2m}. The rest of the proof follows by induction as in Theorem 3.10. □

Example 4.7. Theorem 4.6 describes the behavior of the class $D_{2,11}$ for $p = 5$ and $k = 4$. $T_{1,11}$ is false, but $T_{2,11}$ is true. Then since $231 \equiv 391 \equiv 11 \bmod 20$ and $231 \not\equiv 391 \bmod 100$,

$$S(231,4) \not\equiv S(391,4) \bmod 5^4.$$

This is depicted in Figure 10.

5. The p-adic valuation of Stirling numbers

In this section we present three illustrative examples of the algorithm developed in this paper. We present the data using p-adic trees, since they do not grow as fast as modular trees. A conjecture on the structure of these valuations is also presented. This conjecture extends the results presented in [1].

Recall the construction of trees described here: fix a prime p and an index k satisfying $0 \leq k < p$. The Stirling numbers $S(n,k)$ are periodic modulo p^m, of period $L_m = (p-1)p^{m-1}$. The description of the patterns of the valuations $\nu_p(S(n,k))$ is given in terms of the classes

$$D_{m,j} = \{n \in \mathbb{N} : n \equiv j \bmod L_m\}. \tag{5.1}$$

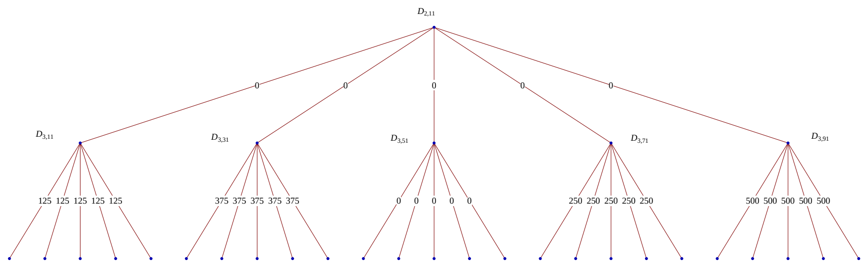

FIGURE 10. Branching from the class $D_{2,11}$, for $p = 5$ and $k = 4$.

Each class corresponds to a vertex of a tree and, for each $m \in \mathbb{N}$, the collection of all classes $D_{m,j}$, is called the m-th level of the tree. The fundamental connection between classes and divisibility properties is given by

$$a,\, b \in D_{m,j} \text{ if and only if } S(a,k) \equiv S(b,k) \bmod p^m. \tag{5.2}$$

Example 5.1. Let $p = 7$ and $k = 3$. Figure 11 presents the p-adic tree corresponding to these numbers.

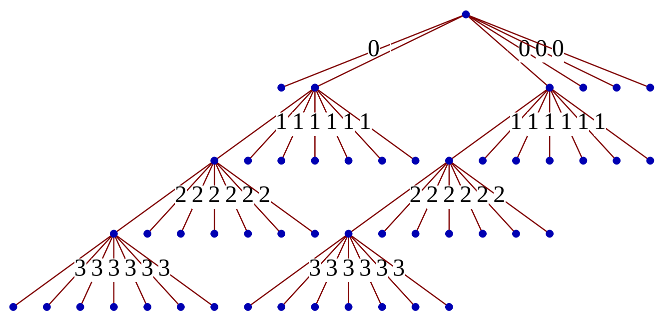

FIGURE 11. The 7-adic tree of $S(n, 3)$

The first level of the tree associated with this case corresponds to the residues modulo $L_1 = 6$. This gives the six classes $D_{1,0}$, $D_{1,1}$, $D_{1,2}$, $D_{1,3}$, $D_{1,4}$ and $D_{1,5}$ that form the first level.

Lemma 3.2 guarantees that the class

$$D_{1,0} = \{n \in \mathbb{N} : \, n \equiv 0 \bmod 6\} \tag{5.3}$$

terminates and it has valuation 0. On the other hand, the classes

$$D_{1,3} = \{n \in \mathbb{N} : \, n \equiv 3 \bmod 6\} \tag{5.4}$$

$$D_{1,4} = \{n \in \mathbb{N} : \, n \equiv 4 \bmod 6\} \tag{5.5}$$

$$D_{1,5} = \{n \in \mathbb{N} : \, n \equiv 5 \bmod 6\} \tag{5.6}$$

also terminate with valuation 0. The reason is simple: the class $D_{1,3}$ contains 3 and $S(3,3) = 1$. Periodicity of $S(n,3)$ implies that every index in $D_{1,3}$ satisfies $S(4t+3,3) \equiv 1 \bmod 7$. Therefore $\nu_5(D_{1,3}) = 0$. Similarly, $D_{1,4}$ contains 4 and $S(4,3) = 6$ and $D_{1,5}$ contains 5 and $S(5,3) = 25$.

The remaining two classes, $D_{1,1}$ and $D_{1,2}$, satisfy $S(D_{1,1},k) \equiv S(D_{1,2},k) \equiv 0 \bmod 7$, hence testing is required.

The $D_{1,1}$ class has a simple branching structure. At level 2 $D_{1,1}$ has seven siblings $D_{2,1}$, $D_{2,7}$, $D_{2,13}$, $D_{2,19}$, $D_{2,25}$, $D_{2,31}$ and $D_{2,37}$. These classes are labeled by the index $i_0 + i_1L_1$ modulo $L_2 = 42$. The label $i_0 = 1$ and $0 \leq i_1 \leq 6$. A direct calculation shows that

$$\begin{aligned}
S(2+0\cdot 6+tL_2,3) &\equiv 0 \bmod 7^2\\
S(2+1\cdot 6+tL_2,3) &\equiv 7 \bmod 7^2\\
S(2+2\cdot 6+tL_2,3) &\equiv 14 \bmod 7^2\\
S(2+3\cdot 6+tL_2,3) &\equiv 21 \bmod 7^2\\
S(2+4\cdot 6+tL_2,3) &\equiv 28 \bmod 7^2.\\
S(2+5\cdot 6+tL_2,3) &\equiv 35 \bmod 7^2\\
S(2+6\cdot 6+tL_2,3) &\equiv 42 \bmod 7^2.
\end{aligned}$$

Observe that these values are incongruent modulo 7^2. An alternative way to see this is by noticing that condition (3.13) holds

$$\sum_{r=1}^{3}(-1)^r\binom{3}{r}r\beta_r \equiv 1 \bmod 7. \tag{5.7}$$

Therefore, Theorem 3.10 states that this pattern of pairwise incongruence will continue through increasing m-levels, as illustrated in Figure 11. The class $D_{1,2}$ also has a simple branching structure since a similar argument produces

$$\sum_{r=1}^{3}(-1)^r\binom{3}{r}r^2\beta_r \equiv 5 \bmod 7. \tag{5.8}$$

It follows that each of the classes $D_{1,1}$ and $D_{1,2}$ split modulo $L_2 = 42$ into six terminal classes, with constant $\nu_7(S(n,3)) = 1$, and one branching (nonconstant) class. This branching class will moreover split modulo $L_3 = 294$ into six terminal classes with constant $\nu_7(S(n,3)) = 2$ and one branching class, and so on, at each m-level. This ends the characterization of this tree.

Example 5.2. The next example considers the valuations $\nu_5(S(n,4)$; that is, $p = 5$ and $k = 4$, see Figure 12.

The first level is formed by the classes $D_{1,0}$, $D_{1,1}, D_{1,2}$ and $D_{1,3}$. Lemma 3.2 states that the class $D_{1,0}$ is terminal with valuation $\nu_5(D_{1,0}) = 0$. The remaining three classes are subject to the *pre-test*: is $S(j+4t,4) \equiv 0 \bmod 5$? If the answer is no, the vertex j is terminal and $\nu_5(D_{1,j}) = 0$. From $j \leq k-1 = 3$, Theorem 3.6 shows that $S(j+4t,4) \equiv 0 \bmod 5$ for $1 \leq j \leq 3$ and $t \in \mathbb{N}$. In order to decide if these vertices branch to level 3, we ask the question: *does* $T_{1,j}$ *hold?*, that is, is

$$T_{1,j} := \sum_{r=1}^{4}(-1)^r\binom{4}{r}r^j\beta_{1,r} \not\equiv 0 \bmod 5\,? \tag{5.9}$$

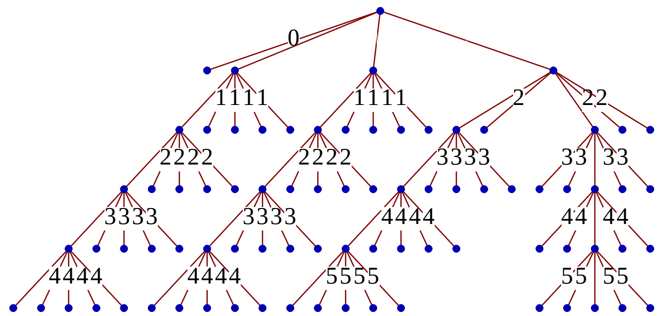

FIGURE 12. The 5-adic tree of $S(n,4)$

Recall that β_r is defined in (3.3).

A direct calculation shows that

$$\beta_{1,1}=0,\ \beta_{1,2}=3,\ \beta_{1,3}=1,\ \beta_{1,4}=1 \tag{5.10}$$

and this gives that $T_{1,1}$ and $T_{1,2}$ are true and $T_{1,3}$ is false.

As in the case of Example 5.1, it follows that each of the classes $D_{2,1}$ and $D_{2,2}$ split modulo 20 into four terminal classes, with constant $\nu_5(S(n,4))=1$, and one branching (nonconstant) class. This branching class will moreover split modulo 100 into four terminal classes with constant $\nu_5(S(n,4))=2$ and one branching class, and so on, at each m-level.

The class $D_{1,3}$ corresponding to the vertex $j=3$ requires further testing. The class $\{n\in\mathbb{N} : n\equiv 3 \bmod 4\}$ at the second level splits into five classes modulo 20: $j=3,\ 7,\ 11,\ 15,\ 19$. Since

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^3\equiv 0 \bmod 5^2, \tag{5.11}$$

then Proposition 4.5 implies that 5^2 divides $S(D_{2,j},k)$ for $j=3,7,11,15,19$. Hence, for each of these, the following test is performed: first, is $S(j+20t,4)\equiv 0 \bmod 5^3$? If not, then the vertex j is terminal, with $\nu_5(S(n,4))=2$ for every $n\equiv j \bmod 20$. Then, does $T_{2,j}$ hold?

The only j for which $S(j+20t,4)\equiv 0 \bmod 5^3$ are 3 and 11, so it is only necessary to test $T_{2,3}$ and $T_{2,11}$. Recall that $\beta_{2,r}$ is defined by

$$r^{20}\equiv 1+5^2\beta_{1,r}+5^3\beta_{2,r} \bmod 5^4 \tag{5.12}$$

so $\beta_{2,r}$, $1\le r\le 4$, is determined by the following calculations:

$$\begin{aligned} 1 &\equiv 1+5^2\cdot 0+5^3\cdot 0 \bmod 5^4\\ 2^{20} &\equiv 1+5^2\cdot 3+5^3\cdot 3 \bmod 5^4\\ 3^{20} &\equiv 1+5^2\cdot 1+5^3\cdot 0 \bmod 5^4\\ 4^{20} &\equiv 1+5^2\cdot 1+5^3\cdot 2 \bmod 5^4 \end{aligned}$$

Thus we have

$$\beta_{2,1}=0,\ \beta_{2,2}=3,\ \beta_{2,3}=0,\ \beta_{2,4}=2.$$

Then

$$\binom{4}{2}2^3(3+3\cdot 5)-\binom{4}{3}3^3+\binom{4}{4}4^3(1+2\cdot 5)=1460\not\equiv 0 \bmod 5^2 \tag{5.13}$$

implies $T_{2,3}$ true, and

$$\binom{4}{2}2^{11}(3+3\cdot 5)-\binom{4}{3}3^{11}+\binom{4}{4}4^{11}(1+2\cdot 5)=45649940\not\equiv 0 \bmod 5^2 \tag{5.14}$$

implies $T_{2,11}$ true. So per Theorem 4.6, each of the classes $j=3, 11$ at the second level splits modulo 100 into four terminal classes with $\nu_5(S(n,4))=3$ and one branching class. This branching class will moreover split modulo 500 into four terminal classes with constant $\nu_5(S(n,4))=4$ and one branching class, and so on, at each m-level. This completes our characterization of this tree.

Example 5.3. The last example described here is $p=11$ and $k=4$, see Figure 13.

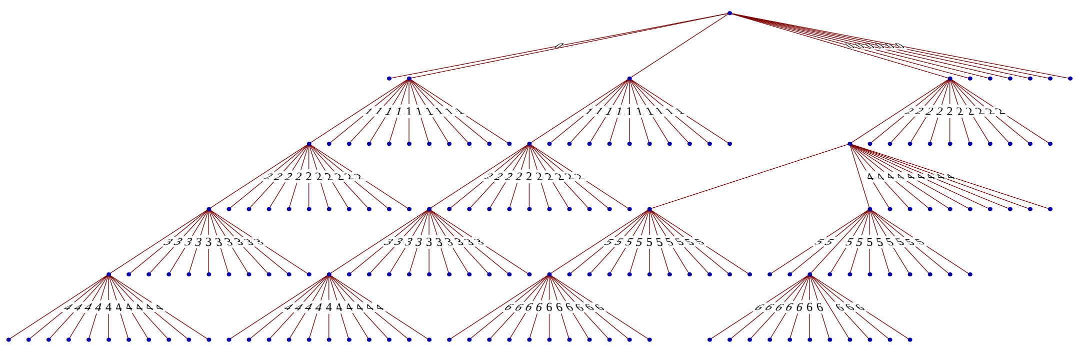

FIGURE 13. The 11-adic tree of $S(n,4)$

As in the previous two examples, the first level of this tree is formed by the classes $D_{1,j},\ j=0,1,2,\cdots,9$. The classes $D_{1,i},\ i=0,4,5,6,7,8,9$ are terminal with valuation $\nu_5(D_{1,i})=0$. The classes $D_{1,j}$ for each $1\le j\le 3$ require test.

Using Definition 3.3 and Example 5.2, the reader can verify that in this case we have $\beta_{1,1}=0$, $\beta_{1,2}=5$, $\beta_{1,3}=0$, and $\beta_{1,4}=10$. This information yields $T_{1,1}$ true, $T_{1,2}$ true, and $T_{1,3}$ false. From here we gather that each of the vertices $j=1,2$ splits modulo 110 into ten terminal classes, with constant $\nu_{11}(S(n,4))=1$, and one branching (nonconstant) class. This branching class will moreover split modulo 1210 into ten terminal classes with constant $\nu_{11}(S(n,4))=2$ and one branching class, and so on, at each m-level.

Since $T_{1,3}$ is false, then the vertex $j=3$ requires further testing. We proceed as in Example 5.2. At the second level, this vertex splits into eleven classes modulo 110. These are

$$j=3,13,23,33,43,53,63,73,83,93,103.$$

Note that

$$\sum_{r=1}^{4}(-1)^r\binom{4}{r}r^3 \equiv 0 \bmod 11^2, \tag{5.15}$$

therefore by Proposition 4.5 we know that 11^2 divides $S(D_{2,j},k)$ for each of these j. The following test is performed at each vertex: first, is $S(j+110t,4) \equiv 0 \bmod 11^3$? If not, then the vertex j is terminal, with $\nu_{11}(S(n,4)) = 2$ for every $n \equiv j \bmod 110$. Then, does $T_{2,j}$ hold?

The only j for which $S(j+110t,4) \equiv 0 \bmod 11^3$ is $j=3$, so it is only necessary to test $T_{2,3}$. Recall that $\beta_{2,r}$ is defined by

$$r^{110} \equiv 1 + 11^2\beta_{1,r} + 11^3\beta_{2,r} \bmod 11^4. \tag{5.16}$$

A direct calculation gives

$$\beta_{2,1}=0,\ \beta_{2,2}=1,\ \beta_{2,3}=4,\ \beta_{2,4}=2.$$

Then

$$\binom{4}{2}2^3(5+1\cdot 11) - \binom{4}{3}3^3(0+4\cdot 11) + \binom{4}{4}4^3(10+2\cdot 11) = -1936 \equiv 0 \bmod 11^2$$

and so $T_{2,3}$ is false. This implies that further testing is required and so we still need to go another level down.

The class $\{n \in \mathbb{N} : n \equiv 3 \bmod 110\}$ splits into eleven classes modulo 1210:

$$j = 3, 113, 223, 333, 443, 553, 663, 773, 883, 993, 1103.$$

Since

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}r^3 \equiv 0 \bmod 11^4, \tag{5.17}$$

then 11^4 divides $S(D_{3,j})$ for each of these j. Once again, for each of these classes, the following two test are performed: first, is $S(j+13310t,4) \equiv 0 \bmod 11^5$? If not, then the vertex j is terminal, with $\nu_{11}(S(n,4)) = 4$ for every $n \equiv j \bmod 1210$. Then, does $T_{3,j}$ hold?

The only j for which $S(j+1210t,4) \equiv 0 \bmod 11^5$ are 3 and 113. The calculation

$$\beta_{3,1}=0,\ \beta_{3,2}=3,\ \beta_{3,3}=0,\ \beta_{3,4}=6 \tag{5.18}$$

implies $T_{3,3}$ and $T_{3,113}$ are true. So by Theorem 4.6, each of the classes $j = 3, 113$ splits modulo 13310 into ten terminal classes with $\nu(S(n,4)) = 5$ and one branching class, and so on, through m-levels. This completes our characterization of this tree.

This process is a systematic and very easily automated way to summarize the structure of the p-adic valuation of $S(n,k)$ for fixed $k < p$.

We conclude with an extension of Conjecture 1.1 for any odd prime.

Conjecture 5.4. *Fix the index k. Given a prime p, its is conjectured that*
a) there exists a level $m_{0,p}(k)$ and an integer $\mu_p(k)$ such that for any $m \geq m_{0,p}(k)$, the number of non-constant classes at the level m is $\mu_p(k)$, independently of m;

b) moreover, for each $m \geq m_{0,p}(k)$, each of the $\mu_p(k)$ nonconstant classes splits into $p-1$ constant and one nonconstant subclass. The latter generates the next level set.

Supporting software. This article is accompanied by the Mathematica package `StirlingTrees.m` available from the webpages:

`http://www.math.rutgers.edu/~lmedina/`

and

`http://www.math.tulane.edu/~vhm/`

Acknowledgements. This project started at MSRI-UP during the summer of 2008. The work was supported by National Security Agency (NSA) grant H98230-08-1-0063, the National Science Foundation (NSF) grant 0754872, a donation from the Gauss Research Foundation in Puerto Rico, and The Mathematical Sciences Research Institute (MSRI). The fourth author acknowledges the partial support of NSF-DMS 0713836.

References

[1] T. Amdeberhan, D. Manna, and V. Moll. The 2-adic valuation of Stirling numbers. *Experimental Mathematics*, 17:69–82, 2008.

[2] H. W. Becker and J. Riordan. The arithmetic of Bell and Stirling numbers. *Amer. J. Math.*, 70:385–394, 1948.

[3] L. Carlitz. Congruences for generalized Bell and Stirling numbers. *Duke Math. J.*, 22:193–205, 1955.

[4] Y. H. H. Kwong. Minimum periods of $S(n,k)$ modulo M. *The Fibonacci Quarterly*, 27(3):217–221, 1989.

[5] A. Nijenhuis and H. S. Wilf. Periodicities of partition functions and Stirling numbers. *J. Number Theory*, 25:308–312, 1987.

University of Wisconsin, Madison, WI 53706-1388
E-mail address: `berrizbe@math.wisc.edu`

Department of Mathematics, Rutgers University, Piscataway, NJ 08854 and Department of Mathematics, University of Puerto Rico, San Juan, PR 00931
E-mail address: `lmedina@math.rutgers.edu`

Columbia University, New York, NY 10027
E-mail address: `acm2141@columbia.edu`

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: `vhm@math.tulane.edu`

Ohio State University, Columbus, OH 43210
E-mail address: `noble@math.ohio-state.edu`