A PRETTY BINOMIAL IDENTITY

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Abstract. An identity involving binomial coefficients that appeared in the evaluation of a definite integral is established by a variety of methods.

1. Introduction

The evaluation of finite sums involving binomial coefficients appears throughout the undergraduate curriculum. Students are often exposed to identities such as

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \text{ and } \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}. \tag{1.1}$$

Elementary proofs abound: the first identity results from choosing $x = y = 1$ in the binomial expansion of $(x+y)^n$. The second one may be obtained by comparing the coefficient of x^n in the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$. The reader is surely aware of many other proofs, including some combinatorial in nature.

At the end of the previous century, the evaluation of these sums was trivialized by the work of H. Wilf and D. Zeilberger [8]. In the preface to the charming book [8], the authors begin with the phrase

> `You've been up all night working on your new theory, you found the answer, and it is in the form that involves factorials, binomial coefficients, and so on, ...`

and then proceed to introduce the method of *creative telescoping* discussed in Section 3. This technique provides an automatic tool for the verification of these type of identities. The points of view presented in [3] and [10] provide an entertaining comparison of what is admissible as a proof.

In this short note we present a variety of proofs of the identity

$$\sum_{k=0}^{m} 2^{-2k}\binom{2k}{k}\binom{2m-k}{m} = \sum_{k=0}^{m} 2^{-2k}\binom{2k}{k}\binom{2m+1}{2k}. \tag{1.2}$$

2. The origin

The formula (1.2) comes from an unexpected source. Several evaluations of

$$N_{0,4}(a;m) := \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{2.1}$$

Date: July 13, 2010.

2000 *Mathematics Subject Classification.* Primary 11B65, Secondary 33C05, 33F10.

Key words and phrases. binomial sums, hypergeometric functions, WZ method, constant term identities.

for $a > -1$ and $m \in \mathbb{N}$, in the form

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}} P_m(a), \tag{2.2}$$

are given in [1]. Here $P_m(a)$ is a polynomial in a. The first expression obtained for $P_m(a)$ in [6], via elementary methods, is

$$P_m(a) = \sum_{j=0}^{m} \binom{2m+1}{2j} (a+1)^j \sum_{k=0}^{m-j} \binom{m-j}{k} \binom{2(m-k)}{m-k} 2^{-3(m-k)} (a-1)^{m-k-j}.$$

The reader will find the details in [5], page 140. The alternative expression

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k \tag{2.3}$$

appeared first in [4]. The complexity of these expressions lead the authors to evaluate it at $a = 1$. This produced (1.2).

3. A proof using the WZ method

An efficient procedure to prove the identity $a_n = b_n$ is to produce a recurrence satisfied by both sequences and matching the required initial conditions. For example, to prove

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{3.1}$$

it suffices to check that both sides satisfy $x_n = 2x_{n-1}$ and they share the initial condition $x_0 = 1$. The real question is how to produce the recurrence. In this case, it is easier to look for a recurrence for the summand $\binom{n}{k}$ and then sum over k. The basic identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \tag{3.2}$$

after summing over all values of k yields the desired recurrence. The result now follows from checking initial values.

H. Wilf and D. Zeilberger [7, 8, 9] came up with a breakthrough in producing an algorithm that provides the recurrence for the summand $t_{n,k}$ in $s_n = \sum_k t_{n,k}$. The method applies to terms $t_{n,k}$ of hypergeometric type; that is $t_{n+1,k}/t_{n,k}$ and $t_{n,k+1}/t_{n,k}$ are rational functions of the indices. This so-called WZ-method has been implemented in modern symbolic languages. For instance, `Maple` shows that both sides of (1.2) satisfy the recurrence

$$(2m+3)(2m+2)f(m+1) = (4m+5)(4m+3)f(m). \tag{3.3}$$

The identity (1.2) now follows from the fact that both sides reduce to 1 at $m = 0$. Furthermore, iterating (3.3) yields

$$f(m) = 2^{-2m} \binom{4m+1}{2m}. \tag{3.4}$$

4. A CONSTANT TERM APPROACH

The second identity in (1.1) arises from matching the coefficient of x^n in the trivial identity $(1+x)^n\,(1+x)^n = (1+x)^{2n}$. The proof of (1.2) presented in this section is based on producing (Laurent) polynomials whose constant terms give the two sides, respectively.

The presence of the binomial coefficient $\binom{2m+1}{2k}$ in (1.2) suggests to consider

$$\begin{aligned}\left(1+\frac{t+t^{-1}}{2}\right)^{2m+1} &= \sum_{r=0}^{2m+1} 2^{-r}\binom{2m+1}{r}(t+t^{-1})^r \\ &= \sum_{r=0}^{2m+1} 2^{-r}\sum_{k=0}^{r}\binom{2m+1}{r}\binom{r}{k}t^k t^{-(r-k)}.\end{aligned} \tag{4.1}$$

Its constant term is

$$\sum_{k=0}^{m} 2^{-2k}\binom{2k}{k}\binom{2m+1}{2k}, \tag{4.2}$$

which is the right hand side of (1.2). From

$$\left(1+\frac{t+t^{-1}}{2}\right)^{2m+1} = \frac{(t+1)^{4m+2}}{(2t)^{2m+1}} \tag{4.3}$$

this constant term evaluates to $2^{-(2m+1)}\binom{4m+2}{2m+1}$.

The left hand side comes from the constant term in the identity $(1+t)^{-1/2}\times(1+t)^{-m-1} = (1+t)^{-m-1/2}$. In this product, employ the expansions

$$(1+t)^{-1/2} = \sum_{k=0}^{\infty} 2^{-2k}\binom{2k}{k}(-t)^k \text{ and } (1+t)^{-m-1} = \sum_{k=0}^{\infty}\binom{m+k}{k}(-t)^k \tag{4.4}$$

to compute the coefficient of t^m of the product. Observe that only terms up to order m contribute to this computation. Thus,

$$\sum_{k=0}^{m}\binom{2m-k}{m}(-t)^{m-k} = \sum_{k=0}^{m}\binom{2m-k}{m-k}(-t)^k \tag{4.5}$$

shows that the constant term of $t^{-m}(1+t)^{-m-1}(1+t)^{-1/2}$ is the left hand side of (1.2). That is,

$$(-1)^m\binom{-m-\frac{3}{2}}{m} = \frac{1}{2^{2m+1}}\binom{4m+2}{2m+1} \tag{4.6}$$

giving the result.

5. AN EXCURSION INTO THE HYPERGEOMETRIC WORLD

A natural setting of binomial sums such as (1.2) is in the context of hypergeometric functions. These are functions defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ where the ratio a_{n+1}/a_n is a rational function of the index n. A classical example is given by

$${}_2F_1\left(\begin{matrix} a \quad b \\ c \end{matrix}\,;x\right) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}. \tag{5.1}$$

Here $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$ is the ascending factorial. Observe that the function ${}_2F_1$ is symmetric in a and b. Moreover, if a is a negative integer, the sum is finite and the function reduces to a polynomial.

Proposition 5.1. The identity (1.2) is equivalent to

$$ {}_2F_1\left(\begin{matrix} -\frac{1}{2}-m & & -m \\ & 1 & \end{matrix} ;1 \right) = \binom{2m}{m} {}_2F_1\left(\begin{matrix} \frac{1}{2} & & -m \\ & -2m & \end{matrix} ;1 \right). \tag{5.2}$$

Proof. The relations $\left(\frac{1}{2}\right)_k = 2^{-2k}(2k)!/k!$ and

$$(-r)_k = \begin{cases} \frac{(-1)^k r!}{(r-k)!} & \text{for } 0 \leq k \leq r \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

show that the right hand side of (1.2) and (5.2) agree. Similarly, the relation

$$\left(-\tfrac{1}{2}-m\right)_k = \frac{(-1)^k(2m+1)!\,(m-k)!}{2^{2k}\,(2m-2k+1)!\,m!} \tag{5.4}$$

shows that both left hand sides also agree. □

The hypergeometric terms in (5.2) can be evaluated using a classical formula of Gauss [2]:

$$ {}_2F_1\left(\begin{matrix} a & & b \\ & c & \end{matrix} ;1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5.5}$$

and the specialization

$$ {}_2F_1\left(\begin{matrix} -n & & b \\ & c & \end{matrix} ;1 \right) = \frac{(c-b)_n}{(c)_n}. \tag{5.6}$$

The identity

$$(-x)_n = (-1)^n(x-n+1)_n \tag{5.7}$$

completes the argument.

6. A CONTOUR INTEGRATION APPROACH

Complex analytic techniques are useful in establishing identities involving binomial coefficients. The representation

$$\binom{m}{n} = \frac{1}{2\pi i}\int_{|z|=r} \frac{(1+z)^m}{z^{n+1}}\,dz \tag{6.1}$$

for $r>0$ provides the relevant connection. This will now be employed to produce a proof of (1.2).

The left hand side becomes, for an appropiate choice of radii r_1 and r_2,

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{-2k}\binom{2k}{k}\binom{2m+1}{2k} &= \frac{1}{(2\pi i)^2}\sum_{k\geq 0} 2^{-2k}\int_{|z|=r_1}\frac{(1+z)^{2k}}{z^{k+1}}\,dz \int_{|w|=r_2}\frac{(1+w)^{2m+1}}{w^{2k+1}}\,dw \\ &= \frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\frac{(1+w)^{2m+1}}{zw}\sum_{k=0}^{\infty}\left[\frac{(1+z)^2}{4zw^2}\right]^k dw\,dz \\ &= \frac{1}{(2\pi i)^2}\int_{|z|=r_1}\int_{|w|=r_2}\frac{4w(1+w)^{2m+1}}{4zw^2-(1+z)^2}\,dw\,dz. \end{aligned}$$

The choice of $r_2 = 1$ and Cauchy's residue theorem imply

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{(1+w)^{2m+1}}{z(w-(1+z)/2z^{1/2})}\,dw = \frac{(1+z^{1/2})^{4m+2}}{2^{2m+1}z^{\frac{2m+3}{2}}}. \tag{6.2}$$

Then take $r_1 = 5$ to obtain

$$\frac{1}{2\pi i}\int_{|z|=5} \frac{(1+z^{1/2})^{4m+2}}{2^{2m+1}z^{(2m+3)/2}}\,dz = 2^{-2m}\binom{4m+1}{2m}, \tag{6.3}$$

the value of the right hand side of (1.2).

The left hand side of (1.2) is

$$\begin{aligned}
\sum_{k\geq 0} 2^{-2k}\binom{2k}{k}\binom{2m-k}{m-k} &= \frac{1}{(2\pi i)^2}\sum_{k\geq 0} 2^{-2k}\int_{|z|=\frac{1}{2}} \frac{(1+z)^{2k}}{z^{k+1}}\,dz\int_{|w|=\frac{1}{5}} \frac{(1+w)^{2m-k}}{w^{m-k+1}}\,dw \\
&= \frac{1}{(2\pi i)^2}\int_{|w|=\frac{1}{5}} \frac{(1+w)^{2m}}{w^{m+1}}\int_{|z|=\frac{1}{2}} \frac{1}{z}\sum_{k\geq 0}\left(\frac{w(1+z)^2}{4z(1+w)}\right)^k dz\,dw \\
&= \frac{1}{(2\pi i)^2}\int_{|w|=\frac{1}{5}} \frac{(1+w)^{2m+1}}{w^{m+1}}\int_{|z|=\frac{1}{2}} \frac{4\,dz}{4z(1+w)-w(1+z)^2}\,dw.
\end{aligned}$$

The only pole of the integrand, inside the contour, is $z = \frac{1}{w}(2+w-2\sqrt{1+w})$. Thus

$$\int_{|z|=\frac{1}{2}} \frac{4\,dz}{-wz^2+(4+2w)z-w} = \frac{1}{\sqrt{1+w}}. \tag{6.4}$$

This shows that the right hand side is

$$\frac{1}{2\pi i}\int_{|w|=\frac{1}{5}} \frac{(1+w)^{2m+\frac{1}{2}}}{w^{m+1}} = \binom{2m+\frac{1}{2}}{m}. \tag{6.5}$$

The identity

$$\binom{2m+\frac{1}{2}}{m} = 2^{-2m}\binom{4m+1}{2m} \tag{6.6}$$

completes the proof.

7. A PARTIAL COMBINATORIAL PROOF

Consider the set X of lattice paths in the plane that start at $(0,0)$, taking unit steps $N=(0,1)$, $S=(0,-1), E=(1,0)$ and $W=(-1,0)$, of odd length $2m+1$, and ending at the y-axis. It is clear that the number of E steps is the same as the W steps, call it j. Choose the steps that are either E or W in $\binom{2m+1}{2j}$ ways. Then choose which is E and which is W in $\binom{2j}{j}$ ways. Finally choose the remaining $2m+1-2j$ steps to be either N or S in $2^{2m+1-2j}$ ways. This gives

$$|X| = \sum_{j=0}^{m}\binom{2m+1}{2j}\binom{2j}{j}2^{2m+1-2j}. \tag{7.1}$$

Thus, the right hand side of (1.2) gives the cardinality of the set X, aside from the factor 2^{2m}.

The size of X is now evaluated by exhibiting a bijection between X and Y that makes counting simpler. The new set Y is formed by all paths on the x-axis that start and end at 0, take steps $e=+1$ and $w=-1$, and have length $4m+2$. It will be shown that there is a bijection from Y to X. Clearly there must be $2m+1$ of each kind of steps, and so the size of Y is

$$|Y| = \binom{4m+2}{2m+1}. \tag{7.2}$$

The proof is completed by noticing that there is a simple bijection between X and Y given by

$$E \mapsto ee,\ W \mapsto ww,\ N \mapsto ew,\ S \mapsto we. \tag{7.3}$$

It remains to produce a combinatorial interpretation of the left hand side of (1.2). The reader is invited to produce one: the authors have been unable to do so.

Acknowledgements. The work of the fourth author was partially supported by NSF-DMS 0070567.

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