A REMARKABLE SEQUENCE OF INTEGERS

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Abstract. A survey of properties of a sequence of coefficients appearing in the evaluation of a quartic definite integral is presented. These properties are of analytical, combinatorial and number-theoretical nature.

1. A quartic integral

The problem of explicit evaluation of definite integrals has been greatly simplified due to the advances in symbolic languages like Mathematica and Maple. Some years ago the first author described in [26] how he got interested in these topics and the appearance of the sequence of rational numbers

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \tag{1.1}$$

for $0 \leq l \leq m$. These are rational numbers with a simple denominator. The numbers $2^{2m} d_{l,m}$ are the remarkable integers in the title. These rational coefficients $d_{l,m}$ appeared in the evaluation of the *quartic integral*

$$N_{0,4}(a;m) := \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}, \tag{1.2}$$

for $a > -1$, $m \in \mathbb{N}$. The formula

$$N_{0,4}(a;m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}}, \tag{1.3}$$

with

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l \tag{1.4}$$

has been established by a variety of methods, some of which are reviewed in [4]. The symbolic status of (1.2) has not changed much since we last reported on [26]. Mathematica 6.0 is unable to compute it when a and m are entered as parameters. On the other hand, the corresponding indefinite integral is evaluated in terms of the Appell-F1 function defined by

$$F_1(a; b_1, b_2; c; x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m! n! (c)_{m+n}} x^m y^n \tag{1.5}$$

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as

$$\int \frac{dx}{(x^4+2ax^2+1)^{m+1}} = x\,F_1\left[\frac{1}{2}, 1+m, 1+m, \frac{3}{2}, -\frac{x^2}{a_+}, \frac{x^2}{-a_-}\right],$$

where $a_\pm := a \pm \sqrt{-1+a^2}$. Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the ascending factorial.

The coefficients $\{d_{l,m} : 0 \leq l \leq m\}$ have remarkable properties that will be discussed here. Those properties have mainly been discovered by following the methodology of Experimental Mathematics, as presented in [11, 12]. Many of the properties presented here have been *guessed* using a symbolic language and subsequently established by traditional methods. The reader will find in [8] a detailed introduction to the polynomial $P_m(a)$ in (1.4).

2. A triple sum expression for $d_{l,m}$

Our first approach to the evaluation of (1.3) was a byproduct of a new proof of Wallis's formula,

$$J_{2,m} := \int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}, \tag{2.1}$$

where m is a nonnegative integer. Wallis' formula has the equivalent form

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots. \tag{2.2}$$

The reader will find in [8] a proof of the equivalence of these two formulations.

We describe in [9] our first proof of (2.1). Section 3 shows that a simple extension leads naturally to the concept of *rational Landen transformations*. These are transformations on the coefficients of a rational integrand that preserve the value of the integral. It is the rational analog of the well known transformation

$$a \mapsto \frac{a+b}{2}, \quad b \mapsto \sqrt{ab} \tag{2.3}$$

that preserves the elliptic integral

$$G(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{2.4}$$

The reader will find in [13] and [24] details about these topics.

The proof of Wallis' formula begins with the change of variables $x = \tan\theta$. This converts $J_{2,m}$ to its trigonometric form

$$J_{2,m} = \int_0^{\pi/2} \cos^{2m}\theta\,d\theta = \frac{\pi}{2^{2m+1}}\binom{2m}{m}. \tag{2.5}$$

The usual elementary proof of (2.5) presented in textbooks is to produce a recurrence for $J_{2,m}$. Writing $\cos^2\theta = 1-\sin^2\theta$ and using integration by parts yields

$$J_{2,m} = \frac{2m-1}{2m} J_{2,m-1}. \tag{2.6}$$

Now verify that the right side of (2.5) satisfies the same recursion and that both sides give $\pi/2$ for $m=0$.

A second elementary proof of Wallis's formula, also given in [9], is done using a simple *double-angle trick*:

$$J_{2,m} = \int_0^{\pi/2} \cos^{2m}\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^m d\theta.$$

Now introduce the change of variables $\psi = 2\theta$, expand and simplify the result by observing that the odd powers of cosine integrate to zero. Hence (2.5) is reduced to an inductive proof of the binomial recurrence

$$J_{2,m} = 2^{-m} \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{2i} J_{2,i}. \tag{2.7}$$

Note that $J_{2,m}$ is uniquely determined by (2.7) along with the initial value $J_{2,0} = \pi/2$. Thus (2.5) now follows from the identity

$$f(m) := \sum_{i=0}^{\lfloor m/2 \rfloor} 2^{-2i} \binom{m}{2i} \binom{2i}{i} = 2^{-m} \binom{2m}{m} \tag{2.8}$$

since (2.8) can be written as

$$J_{2,m} = 2^{-m} \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{2i} J_{2,i},$$

where

$$J_{2,i} = \frac{\pi}{2^{2i+1}} \binom{2i}{i}.$$

The last step is to verify the identity (2.8). This can be done *mechanically* using the theory developed by Wilf and Zeilberger, which is explained in [27, 28]. The sum in (2.8) is the example used in [28] (page 113) to illustrate their method.

Note. The WZ-method is an algorithm in Computational Algebra that, among other things, will produce for a hypergeometric/holonomic sum, such as (3.7), a recurrence like (3.10). The reader will find in [27] and [28] information about this algorithm.

The command

$$ct(binomial(m, 2i)\, binomial(2i, i) 2^{-2i}, 1, i, m, N)$$

produces

$$f(m+1) = \frac{2m+1}{m+1} f(m), \tag{2.9}$$

a recursion satisfied by the sum. One completes the proof by verifying that $2^{-m}\binom{2m}{m}$ satisfies the same recursion. Note that (2.6) and (2.9) are equivalent since $J_{2,m}$ and $f(m)$ differ only by a factor of $\pi/2^{m+1}$.

We have seen that Wallis's formula can be proven by an angle-doubling trick followed by a hypergeometric sum evaluation. Perhaps the most interesting application of the double-angle trick is in the theory of rational Landen transformations. See [24] for an overview.

Now we employ the same ideas in the evaluation of (1.3). The change of variables $x = \tan\theta$ yields

$$N_{0,4}(a;m) = \int_0^{\pi/2} \left(\frac{\cos^4\theta}{\sin^4\theta + 2a\sin^2\theta\cos^2\theta + \cos^4\theta} \right)^{m+1} \times \frac{d\theta}{\cos^2\theta}.$$

Observe first that the denominator of the trigonometric function in the integrand is a polynomial in $u = 2\theta$. In detail,

$$\sin^4\theta + 2a\sin^2\theta\cos^2\theta + \cos^4\theta = 2\left[(1+a) + (1-a)\cos^2 u\right].$$

In terms of the double-angle $u = 2\theta$, the original integral becomes

$$N_{0,4}(a;m) = 2^{-(m+1)} \int_0^{\pi} \left(\frac{(1+\cos u)^2}{(1+a)+(1-a)\cos^2 u} \right)^{m+1} \times \frac{du}{1+\cos u}.$$

Next, expand the binomial $(1+\cos u)^{2m+1}$ and check that

$$\int_0^{\pi} \left[(1+a)+(1-a)\cos^2 u\right]^{-(m+1)} \cos^j u\, du = 0 \tag{2.10}$$

for j odd. The vanishing of half of the terms in the binomial expansion turns out to be a crucial property. The remaining integrals, those with j even, can be simplified by using the double-angle trick one more time. The result is

$$N_{0,4}(a;m) = \sum_{j=0}^{m} 2^{-j} \binom{2m+1}{2j} \int_0^{\pi} \left[(3+a)+(1-a)\cos v\right]^{-(m+1)} (1+\cos v)^j\, dv,$$

where $v = 2u$ and we have used the symmetry of cosine about $v = \pi$ to reduce the integrals form $[0, 2\pi]$ to $[0, \pi]$. The familiar change of variables $z = \tan(v/2)$ produces (1.3) with the complicated formula

$$d_{l,m} = \sum_{j=0}^{l} \sum_{s=0}^{m-l} \sum_{k=s+l}^{m} \frac{(-1)^{k-l-s}}{2^{3k}} \binom{2k}{k} \binom{2m+1}{2s+2j} \binom{m-s-j}{m-k} \binom{s+j}{j} \binom{k-s-j}{l-j}.$$

Note. In spite of its complexity, obtaining this expression was the first step in the mathematical road described in this paper. It was precisely what Kauers and Paule [20] required to clarify some combinatorial properties of $d_{l,m}$. Some arithmetical properties can be read directly from it. For example, we can see that $d_{l,m}$ is a rational number and that $2^{3m} d_{l,m} \in \mathbb{Z}$; that is, its denominator is a power of 2 bounded above by $3m$. Improvements on this bound are outlined in Section 3.

3. A single sum expression for $d_{l,m}$

The idea of doubling the angle that proved productive in Section 2 can be expressed in the realm of rational functions via the change of variables

$$y = R_2(x) := \frac{x^2-1}{2x}. \tag{3.1}$$

The inverse has two branches

$$x = y \pm \sqrt{y^2+1}, \tag{3.2}$$

where the plus sign is valid for $x \in (0, +\infty)$ and the other one on $(-\infty, 0)$. The rational function R_2 arises from the identity

$$\cot 2\theta = R_2(\cot\theta). \tag{3.3}$$

This change of variables gives the proof of the next theorem.

Theorem 3.1. *Let f be a rational function and assume that the integral of f over $\mathbb{R}$ is finite. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \left[f(y+\sqrt{y^2+1}) + f(y-\sqrt{y^2+1}) \right] dy + \tag{3.4}$$
$$+ \int_{-\infty}^{\infty} \left[f(y+\sqrt{y^2+1}) - f(y-\sqrt{y^2+1}) \right] \frac{y\,dy}{\sqrt{y^2+1}}.$$

Moreover, if f is an even *rational function, the identity (3.4) remains valid if one replaces each interval of integration by $\mathbb{R}^+$.*

Theorem 3.2. *For $m \in \mathbb{N}$, let*

$$Q(x) = \frac{1}{(x^4+2ax^2+1)^{m+1}}. \tag{3.5}$$

Define

$$\begin{aligned} Q_1(y) &:= \left[Q(y+\sqrt{y^2+1}) + Q(y-\sqrt{y^2+1}) \right] + \\ &+ \frac{y}{\sqrt{y^2+1}} \left[Q(y+\sqrt{y^2+1}) - Q(y-\sqrt{y^2+1}) \right]. \end{aligned}$$

Then

$$Q_1(y) = \frac{T_m(2y)}{2^m(1+a+2y^2)^{m+1}}, \tag{3.6}$$

where

$$T_m(y) = \sum_{k=0}^{m} \binom{m+k}{m-k} y^{2k}. \tag{3.7}$$

Proof. Introduce the variable $\phi = y + \sqrt{y^2+1}$. Then $y - \sqrt{y^2+1} = -\phi^{-1}$ and $y = \frac{1}{2}(\phi - \phi^{-1})$. Moreover,

$$\begin{aligned} Q_1(y) &= \left[Q(\phi) + Q(\phi^{-1}) \right] + \frac{\phi^2-1}{\phi^2+1} \left(Q(\phi) - Q(\phi^{-1}) \right) \\ &= \frac{2}{\phi^2+1} \left[\phi^2 Q(\phi) + Q(\phi^{-1}) \right] \\ &:= S_m(\phi). \end{aligned}$$

The result of the theorem is therefore equivalent to

$$2^m \left(1 + a + \tfrac{1}{2}(\phi - \phi^{-1})^2) \right)^{m+1} S_m(\phi) = T_m(\phi - \phi^{-1}). \tag{3.8}$$

A direct simplification of the left hand side of (3.8) shows that this identity is equivalent to proving

$$\frac{\phi^{2m+1} + \phi^{-(2m+1)}}{\phi + \phi^{-1}} = T_m(\phi - \phi^{-1}). \tag{3.9}$$

To establish this, one simply checks that both sides of (3.9) satisfy the second order recurrence

$$c_{m+2} - (\phi^2 + \phi^{-2}) c_{m+1} + c_m = 0, \tag{3.10}$$

and the values for $m = 0$ and $m = 1$ match. This is straight-forward for the expression on the left hand side, while the WZ-method settles the right hand side. $\square$

We now prove (1.3). The identity in Theorem 3.1 shows that

$$\int_0^\infty Q(x)\,dx = \int_0^\infty Q_1(y)\,dy, \tag{3.11}$$

and this last integral can be evaluated in elementary terms. Indeed,

$$\begin{aligned}\int_0^\infty Q_1(y)\,dy &= \int_0^\infty \frac{T_m(2y)\,dy}{2^m(1+2y^2)^{m+1}} \\ &= \frac{1}{2^m}\sum_{k=0}^m \binom{m+k}{m-k}\int_0^\infty \frac{(2y)^{2k}\,dy}{(1+a+2y^2)^{m+1}}.\end{aligned}$$

The change of variables $y = t\sqrt{1+a}/\sqrt{2}$ gives

$$\int_0^\infty Q_1(y)\,dy = \frac{1}{[2(1+a)]^{m+1/2}}\sum_{k=0}^m \binom{m+k}{m-k} 2^k(1+a)^k \int_0^\infty \frac{t^{2k}\,dt}{(1+t^2)^{m+1}},$$

and the elementary identity

$$\int_0^\infty \frac{t^{2k}\,dt}{(1+t^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2k}{k}\binom{2m-2k}{m-k}\binom{m}{k}^{-1}$$

gives

$$\int_0^\infty Q_1(y)\,dy = \frac{\pi}{2^{2m+1}}\frac{1}{[2(1+a)]^{m+1/2}}\sum_{k=0}^m \binom{m+k}{m-k}2^k\binom{2k}{k}\binom{2m-2k}{m-k}\binom{m}{k}^{-1}(1+a)^k.$$

This can be simplified further using

$$\binom{m+k}{m-k}\binom{2k}{k} = \binom{m+k}{m}\binom{m}{k} \tag{3.12}$$

and the equality (3.11) to produce

$$\int_0^\infty Q(y)\,dy = \frac{\pi}{2^{2m+1}}\frac{1}{[2(1+a)]^{m+1/2}}\sum_{k=0}^m 2^k\binom{m+k}{m}\binom{2m-2k}{m-k}(1+a)^k. \tag{3.13}$$

This completes the proof of (1.3). The coefficients $d_{l,m}$ are given by

$$d_{l,m} = 2^{-2m}\sum_{k=l}^m 2^k\binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}. \tag{3.14}$$

This is clearly an improvement over the expression for $d_{l,m}$ given in the previous section.

We now see that $d_{l,m}$ is a *positive* rational number. The bound on the denominator is now improved to $2m-1$. This comes directly from (3.14) and the familiar fact that the central binomial coefficients $\binom{2m}{m}$ are even.

4. A finite sum

The previous two sections have provided two expressions for the polynomial $P_m(a)$. The elementary evaluation in Section 2 gives

$$P_m(a) = \sum_{j=0}^{m} \binom{2m+1}{2j} (a+1)^j \sum_{k=0}^{m-j} \binom{m-j}{k} \binom{2(m-k)}{m-k} 2^{-3(m-k)} (a-1)^{m-k-j} \tag{4.1}$$

and the results described in Section 3 provide the alternative expression

$$P_m(a) = 2^{-m} \sum_{k=0}^{m} 2^{-k} \binom{2k}{k} \binom{2m-k}{m} (a+1)^{m-k}. \tag{4.2}$$

The reader will find details in [9]. Comparing the values at $a = 1$ given by both expressions leads to

$$\sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m+1}{2k} = \sum_{k=0}^{m} 2^{-2k} \binom{2k}{k} \binom{2m-k}{m}. \tag{4.3}$$

The identity (4.3) can be verified using D. Zeilberger's package EKHAD [28]. Indeed, EKHAD tells us that both sides of (4.3) satisfy the recursion

$$(2m+3)(2m+2)f(m+1) = (4m+5)(4m+3)f(m).$$

To conclude the proof by recursion, we check that they agree at $m = 1$. A symbolic evaluation of both sides of (4.3) leads to

$$\frac{2^{2m+1}\Gamma(2m+3/2)}{\sqrt{\pi}\,\Gamma(2m+2)} = -\frac{2^{2m+1}\sqrt{\pi}}{\Gamma(-2m-1/2)\Gamma(2m+2)}. \tag{4.4}$$

The identity (4.3) now follows from

$$\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m}} \frac{(2m)!}{m!} \text{ for } m \in \mathbb{N}. \tag{4.5}$$

An elementary proof of (4.3) would be desirable.

The left hand sum admits a combinatorial interpretation: multiply by 2^{2m+1} to produce

$$S_1(m) := \sum_{j=0}^{m} \binom{2m+1}{2j} \binom{2j}{j} 2^{2m+1-2j}. \tag{4.6}$$

Consider the set X of all paths in the plane that start at $(0,0)$ and take $2m+1$ steps in any of the four compass directions ($N = (0,1)$, $S = (0,-1)$, $E = (1,0)$ and $W = (-1,0)$) so that the path ends on the y-axis. Clearly there must be the same number of $E's$ and $W's$, say j of them. Then to produce one of these paths, choose which is E and which is W in $\binom{2j}{j}$ ways. Finally, choose the remaining $2m+1-2j$ steps to be either N or S, in $2^{2m+1-2j}$ ways. This shows that the set X has $S_1(m)$ elements.

Now let Y be the set of all paths of the x-axis that start and end at 0, take steps $e = 1$ and $w = -1$, and have length $4m+2$. The cardinality of Y is clearly $\binom{4m+2}{2m+1}$.

There is a simple bijection between the sets X and Y given by $E \to ee$, $W \to ww$, $N \to ew$, $S \to we$. Therefore,

$$S_1(m) = \binom{4m+2}{2m+1}. \tag{4.7}$$

We have been unable to produce a combinatorial proof for the right hand side of (4.3).

5. A related family of polynomials

The expression (3.14) provides an efficient formula for the evaluation of $d_{l,m}$ when l is close to m. For example,

$$d_{m,m} = 2^{-m}\binom{2m}{m} \text{ and } d_{m-1,m} = (2m+1)2^{-(m+1)}\binom{2m}{m}. \tag{5.1}$$

Our attempt to produce a similar formula for small l led us into a surprising family of polynomials.

The original idea is very simple: start with

$$P_m(a) = \frac{2}{\pi}\left[2(a+1)\right]^{m+\frac{1}{2}} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}, \tag{5.2}$$

and compute $d_{l,m}$ as coming from the Taylor expansion at $a=0$ of the right hand side. This yields

$$d_{l,m} = \frac{1}{l!m!2^{m+l}}\left(\alpha_l(m)\prod_{k=1}^{m}(4k-1) - \beta_l(m)\prod_{k=1}^{m}(4k+1)\right), \tag{5.3}$$

where α_l and β_l are polynomial in m of degrees l and $l-1$, respectively. The explicit expressions

$$\alpha_l(m) = \sum_{t=0}^{\lfloor l/2\rfloor}\binom{l}{2t}\prod_{\nu=m+1}^{m+t}(4\nu-1)\prod_{\nu=m-l+2t+1}^{m}(2\nu+1)\prod_{\nu=1}^{t-1}(4\nu+1), \tag{5.4}$$

and

$$\beta_l(m) = \sum_{t=1}^{\lfloor (l+1)/2\rfloor}\binom{l}{2t-1}\prod_{\nu=m+1}^{m+t-1}(4\nu+1)\prod_{\nu=m-l+2t}^{m}(2\nu+1)\prod_{\nu=1}^{t-1}(4\nu-1), \tag{5.5}$$

are given in [10].

Trying to obtain more information about α_l and β_l directly from (5.4, 5.5) proved difficult. One uninspired day, we decided to compute their roots numerically. We were pleasantly surprised to discover the following property.

Theorem 5.1. *For all $l \geq 1$, all the roots of $\alpha_l(m)=0$ lie on the line $\operatorname{Re} m = -\frac{1}{2}$. Similarly, the roots of $\beta_l(m)=0$ for $l \geq 2$ lie on the same vertical line.*

The proof of this theorem, due to J. Little [23], starts by writing

$$A_l(s) := \alpha_l((s-1)/2) \text{ and } B_l(s) := \beta_l((s-1)/2) \tag{5.6}$$

and proving that A_l is equal to $l!$ times the coefficient of u^l in $f(s,u)g(s,u)$, where $f(s,u) = (1+2u)^{s/2}$ and $g(s,u)$ is the hypergeometric series

$$g(s,u) = {}_2F_1\left(\frac{s}{2}+\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; 4u^2\right). \tag{5.7}$$

A similar expression is obtained for $B_l(s)$. From here it follows that A_l and B_l each satisfy the three-term recurrence

$$x_{l+1}(s) = 2sx_l(s) - (s^2-(2l-1)^2)x_{l-1}(s). \tag{5.8}$$

Little then establishes a version of Sturm's theorem to prove the final result.

The location of the zeros of $\alpha_l(m)$ now suggest to study the behavior of this family as $l \to \infty$. In the best of all worlds, one will obtain an analytic function of m with all the zeros on a vertical line. Perhaps some Number Theory will enter and ... *one never knows.*

6. Arithmetical properties

The expression (5.3) gives

$$m!2^{m+1}\, d_{1,m} = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1), \tag{6.1}$$

from where it follows that the right hand side is an even number. This led naturally to the problem of determining the 2-adic valuation of

$$A_{l,m} := l!m!2^{m+l}d_{l,m} = \alpha_l(m)\prod_{k=1}^{m}(4k-1) - \beta_l(m)\prod_{k=1}^{m}(4k+1) \tag{6.2}$$

$$= \frac{l!m!}{2^{m-l}}\sum_{k=l}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{l}. \tag{6.3}$$

Recall that, for $x \in \mathbb{N}$, the 2-adic valuation $\nu_2(x)$ is the highest power of 2 that divides x. This is extended to $x = a/b \in \mathbb{Q}$ via $\nu_2(x) = \nu_2(a) - \nu_2(b)$, leaving $\nu_2(0)$ as undefined. It follows from (6.3) that

$$A_{m,m} = 2^m(2m)! \text{ and } A_{m-1,m} = 2^{m-1}(2m-1)!(2m+1), \tag{6.4}$$

so these 2-adic valuations can be computed directly from Legendre's classical formula

$$\nu_2(x) = x - s_2(x), \tag{6.5}$$

where $s_2(x)$ counts the number of 1's in the binary expansion of x.

At the other end of the l-axis,

$$A_{0,m} = \prod_{k=1}^{m}(4k-1) \tag{6.6}$$

is clearly odd, so $\nu_2(A_{0,m}) = 0$. The first interesting case is $l = 1$:

$$A_{1,m} = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1). \tag{6.7}$$

The main result of [10] is that

$$\nu_2(A_{l,m}) = \nu_2(m(m+1)) + 1. \tag{6.8}$$

This was extended in [2].

Theorem 6.1. *The* 2*-adic valuation of* $A_{l,m}$ *satisfies*

$$\nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l, \tag{6.9}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ *is the Pochhammer symbol for* $k \geq 1$*. For* $k=0$*, we define* $(a)_0 = 1$*.*

The proof is an elementary application of the WZ-method. Define the numbers

$$B_{l,m} \quad := \quad \frac{A_{l,m}}{2^l(m+1-l)_{2l}}, \tag{6.10}$$

and use the WZ-method to obtain the recurrence

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

Since the initial values $B_{m,m} = 1$ and $B_{m-1,m} = 2m+1$ are odd, it follows inductively that $B_{l,m}$ is an odd integer. The reader will also find in [2] a WZ-free proof of the theorem.

Note. The reader will find in [3] a study of the 2-adic valuation of the Stirling numbers. This study was motivated by the results described in this section. The papers [15, 16, 22, 21, 33, 34] contain information about 2-adic valuations of related sequences.

7. The combinatorics of the valuations

The sequence of valuations $\{\nu_2(A_{l,m}) : \, m \geq l\}$ increase in complexity with l. Some of the combinatorial nature of this sequence is described next. The first feature of this sequence is that it has a block structure, reminiscent of the simple functions of Real Analysis.

Definition 7.1. Let $s \in \mathbb{N}$, $s \geq 2$. We say that a sequence $\{a_j : \, j \in \mathbb{N}\}$ has *block structure* if there is an $s \in \mathbb{N}$ such that each $t \in \{0, 1, 2, \cdots\}$, we have

$$a_{st+1} = a_{st+2} = \cdots = a_{s(t+1)}. \tag{7.1}$$

The sequence is called *s-simple* if s is the largest value for which (7.1) occurs.

Theorem 7.2. *For each* $l \geq 1$*, the set* $X(l) := \{\nu_2(A_{l,m}) : \, m \geq l\}$ *is an* s*-simple sequence, with* $s = 2^{1+\nu_2(l)}$*.*

We now provide a combinatorial interpretation for $X(l)$. This requires the maps

$$\begin{aligned} F(\{a_1, a_2, a_3, \cdots\}) \quad &:= \quad \{a_1, a_1, a_2, a_3, \cdots\}, \\ T(\{a_1, a_2, a_3, \cdots\}) \quad &:= \quad \{a_1, a_3, a_5, a_7, \cdots\}. \end{aligned}$$

We will also employ the notation $c := \{\nu_2(m) : \, m \geq 1\} = \{0, 1, 0, 2, 0, 1, \cdots\}$.

We describe an algorithm that reduces the sequence $X(l)$ to a constant sequence. The algorithm starts with the sequence $X(l) := \{\nu_2(A_{l,l+m-1}) : \quad m \geq 1\}$ and then finds and $n \in \mathbb{N}$ so that $X(l)$ is is 2^n-simple. Define $Y(l) := T^n(X(l))$. At the initial stage, Theorem 7.2 ensures that $n = 1 + \nu_2(l)$. The next step is to introduce the shift $Z(l) := Y(l) - c$ and finally define $W(l) := F(Z(l))$. If $W(l)$ is a constant sequence, then STOP; otherwise repeat the process with W instead of X. Define

$X_k(l)$ as the new sequence at the end of the $(k-1)$th cycle of this process, with $X_1(l) = X(l)$.

This algorithm produces a sequence of integers n_j, so that $X_k(l)$ is 2^{n_k}-simple. The integer vector $\Omega(l) := \left\{ n_1, \, n_2, \, n_3, \cdots, n_{\omega(l)} \right\}$ is called the *reduction sequence* of l. The number ω_l is the number of cycles requires to obtain a constant sequence.

Definition 7.3. Let $l \in \mathbb{N}$. The *composition* of l, denoted by $\Omega_1(l)$, is an integer sequence defined as follows: Write l in binary form. Read the digits from right to left. The first part of $\Omega_1(l)$ is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on until the number has been read completely.

Theorem 7.4. *Let $\{k_1, \cdots, k_n : 0 \leq k_1 < k_2 < \cdots < k_n\}$, be the unique collection of distinct nonnegative integers such that $l = \sum_{i=1}^{n} 2^{k_i}$. Then the reduction sequence $\Omega(l)$ of l is $\{k_1 + 1, \, k_2 - k_1, \cdots, k_n - k_{n-1}\}$.*

It follows that the reduction sequence $\Omega(l)$ is precisely the sequence of compositions of l, that is, $\Omega(l) = \Omega_1(l)$. This is the combinatorial interpretation of the algorithm used to reduce $X(l)$ to a constant sequence.

8. Valuation patterns encoded in binary trees

In this section we describe the precise structure of the graph of the sequence $\{\nu_2(A_{l,m}), m \geq l\}$. The reader is referred to [30] for complete details. In view of the block structure described in the previous section, it suffices to consider the sequences $\{\nu_2(C_{l,m}), m \geq l\}$, which are defined by

$$C_{l,m} =$$

The emerging patterns are still very complicated. For instance, Figure 1 shows the case $l = 13$ and Figure 2 corresponds to $l = 59$. The remarkable fact is that in spite of the complexity of $\nu_2(C_{l,m})$ there is *an exact formula* for it. The rest of this section describes how to find it.

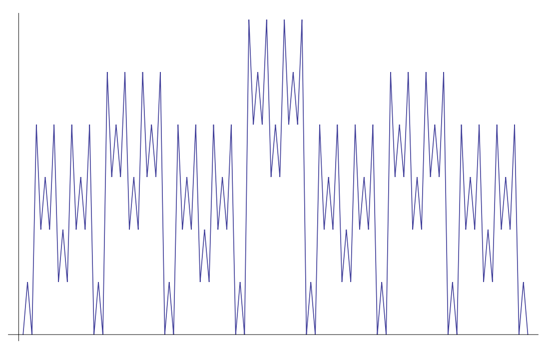

Figure 1. The valuation $\nu_2(C_{13,m})$

We describe now the *decision tree* associated to the index l. Start with a root v_0 at level $k = 0$. To this vertex we attach the sequence $\{\nu_2(C_{l,m}) : m \geq 1\}$ and ask whether $\nu_2(C_{l,m}) - \nu_2(m)$ has a constant value *independent* of m. If the answer is

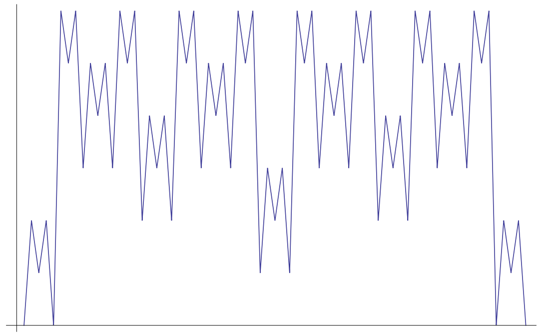

FIGURE 2. The valuation $\nu_2(C_{59,m})$

yes, we say that v_0 is a *terminal vertex* and label it with this constant. The tree is complete. If the answer is negative, we split the integers modulo 2 and produce two new vertices, v_1, v_2, connected to v_0 and attach to the classes $\{\nu_2(C_{l,2m-1}) : m \geq 1\}$ and $\{\nu_2(C_{l,2m}) : m \geq 1\}$ to these vertices. We now ask whether $\nu_2(C_{l,2m-1})-\nu_2(m)$ is independent of m and the same for $\nu_2(C_{l,2m}) - \nu_2(m)$. Each vertex that yields a positive answer is considered terminal and the corresponding constant value is attached to it. Every vertex with a negative answer produces two new ones at the next level.

Assume the vertex v corresponding to the sequence $\{2^k(m-1)+a : m \geq 1\}$ produces a negative answer. Then it splits in the next generation into two vertices corresponding to the sequences $\{2^{k+1}(m-1)+a : m \geq 1\}$ and $\{2^{k+1}(m-1)+2^k+a : m \geq 1\}$. For instance, in Figure 3, the vertex corresponding to $\{4m : m \geq 1\}$, that is not terminal, splits into $\{8m : m \geq 1\}$ and $\{8m-4 : m \geq 1\}$. These two edges lead to terminal vertices. Theorem 8.1 shows that this process ends in a finite number of steps.

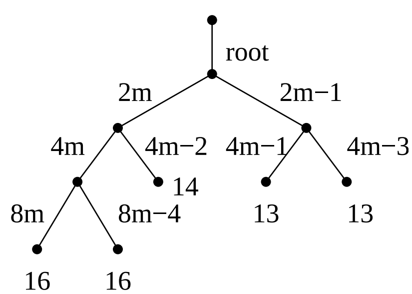

FIGURE 3. The decision tree for $l = 5$

Theorem 8.1. *Let $l \in \mathbb{N}$ and $T(l)$ be its decision tree. Define $k^*(l) := \lfloor \log_2 l \rfloor$. Then*

1) $T(l)$ depends only on the odd part of l; that is, for $r \in \mathbb{N}$, we have $T(l) = T(2^r l)$, up to the labels.

2) The generations of the tree are labelled starting at 0; that is, the root is generation 0. Then, for $0 \leq k \leq k^*(l)$, the k-th generation of $T(l)$ has 2^k vertices. Up to that point, $T(l)$ is a complete binary tree.

3) The k^*-th generation contains $2^{k^*+1} - l$ terminal vertices. The constants associated with these vertices are given by the following algorithm. Define

$$j_1(l,k,a) := -l + 2(1 + 2^k - a), \tag{8.1}$$

and

$$\gamma_1(l,k,a) = l + k + 1 + \nu_2\left((j_1 + l - 1)!\right) + \nu_2\left((l - j_1)!\right). \tag{8.2}$$

Then, for $1 \leq a \leq 2^{k^*+1} - l$, we have

$$\nu_2\left(C_{l,2^k(m-1)+a}\right) = \nu_2(m) + \gamma_1(l,k,a). \tag{8.3}$$

Thus, the vertices at the k^*-th generation have constants given by $\gamma_1(l,k,a)$.

4) The remaining terminal vertices of the tree $T(l)$ appear in the next generation. There are $2(l - 2^{k^*(l)})$ of them. The constants attached to these vertices are defined as follows: let

$$j_2(l,k,a) := -l + 2(1 + 2^{k+1} - a), \tag{8.4}$$

and

$$j_3(l,k,a) := j_2(l,k,a + 2^k). \tag{8.5}$$

Define

$$\gamma_2(l,k,a) := l + k + 2 + \nu_2\left((j_2 + l - 1)!\right) + \nu_2\left((l - j_2)!\right), \tag{8.6}$$

and

$$\gamma_3(l,k,a) := l + k + 2 + \nu_2\left((j_3 + l - 1)!\right) + \nu_2\left((l - j_3)!\right). \tag{8.7}$$

Then, for $2^{k^*(l)+1} - l + 1 \leq a \leq 2^{k^*(l)}$, we have

$$\nu_2\left(C_{l,2^{k^*(l)+1}(m-1)+a}\right) = \nu_2(m) + \gamma_2(l, k^*(l), a), \tag{8.8}$$

and

$$\nu_2\left(C_{l,2^{k^*(l)+1}(m-1)+a+2^{k^*(l)}}\right) = \nu_2(m) + \gamma_3(l, k^*(l), a), \tag{8.9}$$

give the constants attached to these remaining terminal vertices.

We now use the theorem to produce a formula for $\nu_2(C_{3,m})$. The value $k^*(3) = 1$ shows that the first level contains $2^{1+1} - 3 = 1$ terminal vertex. This corresponds to the sequence $2m - 1$ and has constant value 7, thus,

$$\nu_2\left(C_{3,2m-1}\right) = 7. \tag{8.10}$$

The next level has $2(3 - 2^1) = 2$ terminal vertices. These correspond to the sequences $4m$ and $4m-2$, with constant values 9 for both of them. This tree produces

$$\nu_2\left(C_{3,m}\right) = \begin{cases} 7 + \nu_2\left(\frac{m+1}{2}\right) & \text{if } m \equiv 1 \bmod 2, \\ 9 + \nu_2\left(\frac{m}{4}\right) & \text{if } m \equiv 0 \bmod 4, \\ 9 + \nu_2\left(\frac{m+2}{4}\right) & \text{if } m \equiv 2 \bmod 4. \end{cases} \tag{8.11}$$

The complexity of the graph for $l = 13$ is reflected in the analytic formula for this valuation. The theorem yields

$$\nu_2\left(C_{13,m}\right) = \begin{cases} 36 + \nu_2\left(\frac{m+7}{8}\right) & \text{if } m \equiv 1 \bmod 8, \\ 37 + \nu_2\left(\frac{m+6}{8}\right) & \text{if } m \equiv 2 \bmod 8, \\ 36 + \nu_2\left(\frac{m+5}{8}\right) & \text{if } m \equiv 3 \bmod 8, \\ 40 + \nu_2\left(\frac{m+12}{16}\right) & \text{if } m \equiv 4 \bmod 16, \\ 38 + \nu_2\left(\frac{m+11}{16}\right) & \text{if } m \equiv 5 \bmod 16, \\ 39 + \nu_2\left(\frac{m+10}{16}\right) & \text{if } m \equiv 6 \bmod 16, \\ 38 + \nu_2\left(\frac{m+9}{16}\right) & \text{if } m \equiv 7 \bmod 16, \\ 40 + \nu_2\left(\frac{m+8}{16}\right) & \text{if } m \equiv 8 \bmod 16, \\ 40 + \nu_2\left(\frac{m+4}{16}\right) & \text{if } m \equiv 12 \bmod 16, \\ 38 + \nu_2\left(\frac{m+3}{16}\right) & \text{if } m \equiv 13 \bmod 16, \\ 39 + \nu_2\left(\frac{m+2}{16}\right) & \text{if } m \equiv 14 \bmod 16, \\ 38 + \nu_2\left(\frac{m+1}{16}\right) & \text{if } m \equiv 15 \bmod 16, \\ 40 + \nu_2\left(\frac{m}{16}\right) & \text{if } m \equiv 16 \bmod 16. \end{cases} \tag{8.12}$$

The details for Theorem 8.1 are given in [30].

Note. The p-adic valuations of $A_{l,m}$ for p odd present phenomena different from those explained for the case $p = 2$. Figure 4 shows the plot of $\nu_{17}(A_{1,m})$ where we observe linear growth. Experimental data suggest that, for any odd prime p, one has

$$\nu_p(A_{l,m}) \sim \frac{m}{p-1}. \tag{8.13}$$

Figure 5 depicts the error term $\nu_{17}(A_{1,m}) - m/16$. The structure of the error remains to be explored.

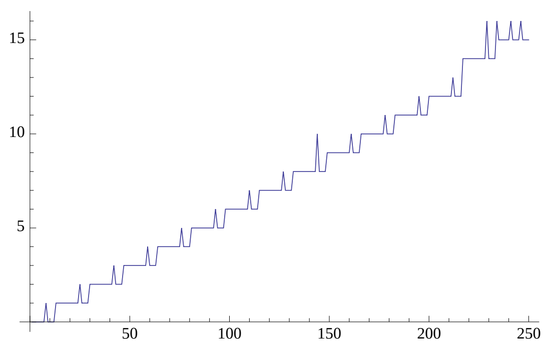

FIGURE 4. The valuation $\nu_{17}(A_{1,m})$

9. UNIMODALITY AND LOG-CONCAVITY

A finite sequence of real numbers $\{a_0, a_1, \cdots, a_m\}$ is said to be *unimodal* if there exists an index $0 \leq j \leq m$ such that $a_0 \leq a_1 \leq \cdots \leq a_j$ and $a_j \geq a_{j+1} \geq \cdots \geq a_m$.

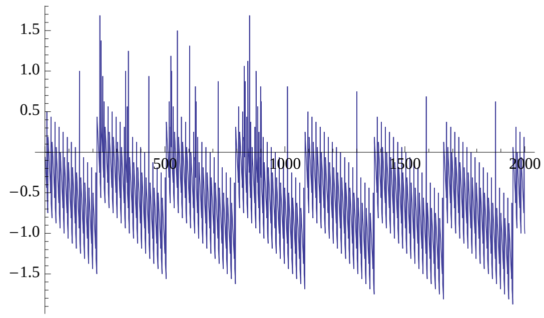

FIGURE 5. The error term $\nu_{17}(A_{1,m}) - m/16$

A polynomial is said to be unimodal if its sequence of coefficients is unimodal. The sequence $\{a_0, a_1, \cdots, a_m\}$ with $a_j \geq 0$ is said to be *logarithmically concave* (or *log-concave* for short) if $a_{j+1}a_{j-1} \leq a_j^2$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is log-concave then it is unimodal [36].

Unimodal polynomials arise often in combinatorics, geometry, and algebra, and have been the subject of considerable research in recent years. The reader is referred to [29] and [14] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal.

For $m \in \mathbb{N}$, the sequence $\{d_{l,m} : 0 \leq l \leq m\}$ is unimodal. This is a consequence of the following criterion established in [6].

Theorem 9.1. *Let a_k be a nondecreasing sequence of positive numbers and let $A(x) = \sum_{k=0}^{m} a_k x^k$. Then $A(x+1)$ is unimodal.*

We applied this theorem to the polynomial

$$A(x) := 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} x^k \tag{9.1}$$

that satisfies $P_m(x) = A(x+1)$. The criterion was extended in [1] to include the shifts $A(x+j)$ and in [32] for arbitrary shifts. The original proof of the unimodality of $P_m(a)$ can be found in [7].

In [26] we conjectured the log-concavity of $\{d_{l,m} : 0 \leq l \leq m\}$. This turned out a more difficult question. Here we describe some of our failed attempts.

1) A result of Brenti [14] states that if $A(x)$ is log-concave then so is $A(x+1)$. Unfortunately this does not apply in our case since (9.1) is not log-concave. Indeed,

$$\begin{aligned} 2^{4m-2k}\left(a_k^2 - a_{k-1}a_{k+1}\right) &= \binom{2m}{m-k}^2 \binom{m+k}{m}^2 \times \\ &\times \left(1 - \frac{k(m-k)(2m-2k+1)(m+k+1)}{(k+1)(m+k)(2m-2k-1)(m-k+1)}\right) \end{aligned}$$

and this last factor could be negative—for example, for $m = 5$ and $j = 4$. The number of negative terms in this sequence is small, so perhaps there is a way out of this.

2) The coefficients $d_{l,m}$ satisfy many recurrences. For example,

$$d_{j+1}(m) = \frac{2m+1}{j+1} d_j(m) - \frac{(m+j)(m+1-j)}{j(j+1)} d_{j-1}(m). \tag{9.2}$$

This can be found by a direct application of WZ method. Therefore, $d_{l,m}$ is log-concave provided

$$j(2m+1)d_{j-1}(m)d_j(m) \leq (m+j)(m+1-j)d_{j-1}(m)^2 + j(j+1)d_j(m)^2. \tag{9.3}$$

We have conjectured that the smallest value of the expression

$$(m+j)(m+1-j)d_{j-1}(m)^2 + j(j+1)d_j(m)^2 - j(2m+1)d_{j-1}(m)d_j(m) \tag{9.4}$$

is $2^{2m}m(m+1)\binom{2m}{m}^2$ and it occurs at $j = m$. This would imply the log-concavity of $\{d_{l,m} : 0 \leq l \leq m\}$. Unfortunately, it has not yet been proven.

Actually we have conjectured that the $d_{l,m}$ satisfy a stronger version of log-concavity. Given a sequence $\{a_j\}$ of positive numbers, define a map

$$\mathfrak{L}\left(\{a_j\}\right) := \{b_j\}$$

by $b_j := a_j^2 - a_{j-1}a_{j+1}$. Thus $\{a_j\}$ is log-concave if $\{b_j\}$ has positive coefficients. The nonnegative sequence $\{a_j\}$ is called *infinitely log-concave* if any number of applications of $\mathfrak{L}$ produces a nonnegative sequence.

Conjecture 9.2. *For each fixed $m \in \mathbb{N}$, the sequence $\{d_{l,m} : 0 \leq l \leq m\}$ is infinitely log-concave.*

The log-concavity of $\{d_{l,m} : 0 \leq l \leq m\}$ has recently been established by M. Kauers and P. Paule [20] as an applications of their work on establishing inequalities by automatic means. The starting point is the triple sum expression in Section 2 written as

$$d_{l,m} = \sum_{j,s,k} \frac{(-1)^{k+j-l}}{2^{3(k+s)}} \binom{2m+1}{2s} \binom{m-s}{k} \binom{2(k+s)}{k+s} \binom{s}{j} \binom{k}{l-j}.$$

Using the RISC package Multisum [35] they derive the recurrence

$$2(m+1)d_{l,m+1} = 2(l+m)d_{l-1,m} + (2l+4m+3)d_{l,m}. \tag{9.5}$$

The positivity of $d_{l,m}$ follows directly from here. To establish the log-concavity of $d_{l,m}$ the new recurrence

$$4l(l+1)d_{l+1,m} = -2(2l-4m-3)(l+m+1)d_{l,m} + 4(l-m-1)(m+1)d_{l,m+1}$$

is derived automatically and the log-concavity of $d_{l,m}$ is reduced to establishing the inequality

$$d_{l,m}^2 \quad \geq \quad \frac{4(m+1)\left(4((l-m-1)(m+1) - (2l^2-4m^2-7m-3)d_{l,m+1}d_{l,m}\right)}{16m^3+16lm^2+40m^2+28lm+33m+9l+9}$$

The 2-log-concavity of $\{d_{l,m} : 0 \leq l \leq m\}$, that is $\mathfrak{L}^{(2)}(\{d_{l,m}\}) \geq \{0, 0, \ldots 0\}$ remains an open question. At the end of [20] the authors state that "...we have little hope that a proof of 2-logconcavity could be completed along these lines, not to mention that a human reader would have a hard time digesting it."

The general concept of infinite log-concavity has generated some interest. D. Uminsky and K. Yeats [31] have studied the action of $\mathfrak{L}$ on sequences of the form

$$\{\cdots, 0, 0, 1, x_0, x_1, \cdots, x_n, \cdots, x_1, x_0, 1, 0, 0, \cdots\} \tag{9.6}$$

and

$$\{\cdots, 0, 0, 1, x_0, x_1, \cdots, x_n, x_n, \cdots, x_1, x_0, 1, 0, 0, \cdots\} \tag{9.7}$$

and established the existence of a large unbounded region in the positive orthant of $\mathbb{R}^n$ that consists only of infinitely log-concave sequences $\{x_0, \ldots, x_n\}$. P. McNamara and B. Sagan [25] have considered sequences satisfying the condition $a_k^2 \geq ra_{k-1}a_{k+1}$. Clearly this implies log-concavity of $r \geq 1$. Their techniques apply to the rows of the Pascal triangle. Choosing appropriate r-factors and a computer verification procedure, they obtain the following.

Theorem 9.3. *For fixed* $n \leq 1450$*, the sequence* $\{\binom{n}{k} : 0 \leq k \leq n\}$ *is infinite log-concave.*

In particular, they looked for values of r for which the r-factor condition is preserved by the $\mathfrak{L}$-operator. The factor that works is $r = \frac{3+\sqrt{5}}{2}$ (the square of the golden mean). This technique can be used on a variety of finite sequences. See [25] for a complete discussion of the techinque.

McNamara and Sagan have also considered q-analogues of the binomial coefficients. In order to describe these extensions we introduce the basic notation and refer to the reader to [19] and [5] for more details on the world of q-analogues.

Let q be a variable and for $n \in \mathbb{N}$ define

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{9.8}$$

The *Gaussian-polynomial* or *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \tag{9.9}$$

where $[n]_q! := [1]_q[2]_q \cdots [n]_q$. The Gaussian polynomials have nonnegative coefficients. We will say that the sequence of polynomials $\{f_k(q)\}$ is *q-log-concave* if $\mathfrak{L}(f_k(q))$ is a sequence of polynomials with nonnegative coefficients. The extension of this definition to *infinite q-log-concavity* is made in the obvious way.

Observe that

$$\binom{n}{k} = \lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{9.10}$$

McNamara and Sagan have established the surprising result:

Theorem 9.4. *The sequence* $\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_{k \geq 0}$ *is not infinite q-logconcave.*

In fact they established that applying $\mathfrak{L}$ twice gives polyomials with some negative coefficients. As a compensation, they propose:

Conjecture 9.5. *The sequence* $\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_{n \geq k}$ *is infinite q-log-concave for all fixed* $k \geq 0$.

Another q-analog of the binomial coefficients that arisies in the study of quantum groups is defined by

$$\langle n \rangle := \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{1}{q^{n-1}}(1 + q^2 + q^4 + \cdots + q^{2n-2}). \tag{9.11}$$

From here we proceed as in the case of Gaussian polynomials and define

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle := \frac{\langle n \rangle!}{\langle k \rangle! \, \langle n-k \rangle!} \tag{9.12}$$

where $\langle n \rangle ! = \langle 1 \rangle \langle 2 \rangle \cdots \langle n \rangle$. For these coefficients McNamara and Sagan have proposed

Conjecture 9.6. *a) The row sequence* $\left\{ \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \right\}_{k \geq 0}$ *is infinitely q-log-concave for all* $n \geq 0$.
b) The column sequence $\left\{ \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \right\}_{n \geq k}$ *is infinitely q-log-concave for all fixed* $n \geq 0$.
c) For all integers $0 \leq u < v$, *the sequence* $\left\{ \left\langle \begin{smallmatrix} n+mu \\ mv \end{smallmatrix} \right\rangle \right\}_{m \geq 0}$ *is infinitely q-log-concave for all* $n \geq 0$.

This conjecture has been verified for all $n \leq 24$ with $v \leq 10$. When $u > v$, using

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \frac{1}{q^{nk-k^2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \tag{9.13}$$

one checks that the lowest degree of $\left\langle \begin{smallmatrix} n+u \\ v \end{smallmatrix} \right\rangle^2 - \left\langle \begin{smallmatrix} n+2u \\ 2v \end{smallmatrix} \right\rangle$ is -1, so the sequence is not even q-log-concave. Sagan and McNamara observe that when $u = v$, the quantum groups analoge has exactly the same behavior as the Gaussian polynomials.

Newton began the study of log-concave sequences by establishing the following result (paraphrased in Section 2.2 of [18]).

Theorem 9.7. *Let* $\{a_k\}$ *be a finite sequence of positive real numbers. Assume all the roots of the polynomial*

$$P[a_k; x] := a_0 + a_1 x + \cdots + a_n x^n \tag{9.14}$$

are real. Then the sequence $\{a_k\}$ *is log-concave.*

McNamara and Sagan [25] and, independently, R. Stanley have proposed the next conjecture.

Conjecture 9.8. *Let* $\{a_k\}$ *be a finite sequence of positive real numbers. If* $P[a_k; x]$ *has only real roots then the same is true for* $P[\mathfrak{L}(a_k); x]$.

This conjecture was also independently made by Fisk. See [25] for the complete details on the conjecture.

The polynomials $P_m(a)$ in (1.4) are the generating function for the sequence $\{d_{l,m}\}$ described here. It is an unfortunate fact that they do not have real roots [7] so these conjecture would not imply Conjecture 9.2. In spite of this, the asymptotic behavior of these zeros has remarkable properties. Dimitrov [17] has shown that, in the right scale, the zeros converge to a lemniscate.

The infinite-log-concavity of $\{d_{l,m}\}$ has resisted all our efforts. It remains to be established.

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