**Jour. Math. Comp. Anal.** 134 (2001) 113-126.

# AN INTEGRAL WITH THREE PARAMETERS. PART 2.

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Abstract. In this paper we use the exact expression for the integral

$$I(a,b;r) := \int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2}$$

to give unifying proofs of several results in Classical Analyisis.

## 1. Introduction

In this paper we present some consequences of the closed-form evaluation

$$I(a,b;r) := \int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2} = \frac{B(r-\frac{1}{2},\frac{1}{2})}{2^{\frac{1}{2}+r}\,(a+1)^{r-\frac{1}{2}}}, \tag{1.1}$$

described in [2]. Here $B$ is the beta function. Some of the integrals given here are new in the sense that they cannot be computed by a symbolic language or found in a table of integrals. We have used Mathematica 4.0 and Maple V as sources for the former, and Gradshteyn and Ryzhik [12] for the latter. The classical results presented in Section 3 are admitedly well known. The value of our contribution lies in the fact that these results (and many more) follow directly from the *master formula* (1.6).

The main result of [2] is reproduced here in Theorem 1.1 below. Conditions on the parameters $a$, $b$ and $r$ that guarantee convergence of the integral $I(a,b;r)$ will always be assumed; these are the *only* conditions on the parameters: in particular, $a > -1$ and $r > \frac{1}{2}$ are sufficient, but $r$ *is not* required to be an integer.

**Theorem 1.1.** The four integrals

$$I_1 = \int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2} \tag{1.2}$$

$$I_2 = \int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{dx}{x^2} \tag{1.3}$$

$$I_3 = \int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r dx \tag{1.4}$$

$$I_4 = \frac{1}{2}\int_0^\infty \left( \frac{x^2}{x^4+2ax^2+1} \right)^r \cdot \frac{x^2+1}{x^2}\, dx \tag{1.5}$$

have the common value

$$I(a,b;r) = \frac{B(r-\frac{1}{2},\frac{1}{2})}{2^{\frac{1}{2}+r}\,(a+1)^{r-\frac{1}{2}}}. \tag{1.6}$$

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*Date*: January 16, 2002.

1991 *Mathematics Subject Classification.* Primary 33.

*Key words and phrases.* Eulerian functions, Integrals.

The results presented here are restricted to evaluation of integrals by strictly elementary methods. The techniques developed in [2] are an essential part of the algorithm for integration of rational functions given in [6, 7]. We expect that the examples discussed here will lead to a general technique for the evaluation of integrals of the form

$$I \quad := \quad \int_0^\infty R_1(x) \, \ln R_2(x) \, dx \tag{1.7}$$

for rational functions $R_1$, $R_2$.

## 2. Examples from a master formula

The master formula (1.6) was used in [2] to evaluate a large class of definite integrals. In this section we produce additional families of integrals which can be derived directly from it.

### 2.1. **Special evaluations.**

The master formula (1.6) gives, for specific values of the parameters, the values of certain definite integrals.

**Example 2.1.1**. The choice $a = 2$, $r = 4/3$ and $b = 2$ gives

$$\int_0^\infty \frac{x^{2/3} \, dx}{(x^4+4x^2+1)^{4/3}} \quad = \quad \frac{\sqrt{\pi}\, \Gamma\left(\frac{5}{6}\right)}{2\sqrt[6]{7776}\, \Gamma\left(\frac{4}{3}\right)}. \tag{2.1}$$

We could not (directly) evaluate (2.1) with a symbolic language.

**Example 2.1.2**. Letting $a = 1/2$ and $r = 4$ in (1.3) yields

$$\int_0^\infty \frac{x^6 \, dx}{(x^4+x^2+1)^4} \quad = \quad \frac{5\pi}{864\sqrt{3}}. \tag{2.2}$$

The evaluation of (2.2) by Mathematica 4.0 took 20.81 seconds.

**Example 2.1.3**. The fact that the integral in (1.2) is independent of the parameter $b$ may be used to evaluate additional integrals. For example, $a = 1/2$, $r = 4$ as above and now $b = 5$ yields

$$\int_0^\infty \frac{x^6 \,(x^2+1) \, dx}{(x^5+1)(x^4+x^2+1)^4} \quad = \quad \frac{5\pi}{864\sqrt{3}}. \tag{2.3}$$

Mathematica 4.0 evaluated (2.3) in 331.26 seconds, a large part of which was employed in simplifying the answer.

### 2.2. **New integrals by differentiation.**

Several interesting evaluations can be obtained from the special case $a = 1$ in (1.6) written as

$$\int_0^\infty \left( \frac{x}{x^2+1} \right)^{c+1} \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2} \quad = \quad 2^{-(c+1)} B\left(\tfrac{c}{2}, \tfrac{1}{2}\right) \tag{2.4}$$

where $c := 2r-1$.

**Example 2.2.1.** Differentiating (2.4) with respect to $b$ yields a 2-parameter family of vanishing integrals:

$$H(b,c) := \int_0^\infty \frac{x^{b+c-1}\,\ln x}{(x^2+1)^c\,(x^b+1)^2}dx \quad = \quad 0. \tag{2.5}$$

The vanishing of $H(b,c)$ can be established directly by the change of variable $x \mapsto 1/x$.

**Note**. Observe that the 3-parameter integral

$$H_1(a,b,c) \quad := \quad \int_0^\infty \frac{x^a\,\ln x}{(x^2+1)^c\,(x^b+1)^2}dx \tag{2.6}$$

is not identically zero. For example, Mathematica 4.0 yields

$$\int_0^\infty \frac{x^2\,\ln x}{(x^2+1)(x+1)^2}\,dx \quad = \quad \frac{\pi^2}{16}$$

and

$$\int_0^\infty \frac{x\,\ln x}{(x^2+1)\,(x^3+1)^2}\,dx \quad = \quad \frac{2\pi}{81}(\pi - 3\sqrt{3}).$$

**Example 2.2.2.** Differentiation of (2.5) with respect to the parameter $b$ yields

$$\int_0^\infty \frac{x^{b+c-1}\,(x^b-1)\,(\ln x)^2}{(x^2+1)^c\,(x^b+1)^3}\,dx \quad = \quad 0. \tag{2.7}$$

The identity (2.7) also follows directly from the change of variable $x \mapsto 1/x$.

**Note**. More interesting examples can be obtained by forcing the parameter $c$ in (2.4) to be a function of $b$ before differentiating. For instance $c := b-1$ yields

$$2\int_0^\infty \frac{\ln^2 x}{(x+1)^3}\,dx \quad = \quad \int_0^\infty \frac{\ln x\,\ln(x^2+1)}{(x+1)^2}\,dx$$

at $b=1$. Mathematica 4.0 evaluates both sides as $\pi^2/3$. For $b=2$ we have

$$2\int_0^\infty \frac{x^2\,\ln^2 x}{(x^2+1)^4}\,dx \quad = \quad \int_0^\infty \frac{x^2\,\ln x\,\ln(x^2+1)}{(x^2+1)^3}\,dx.$$

Mathematica 4.0 gives the correct result $\pi(\pi^2-8)/64$ for the right-hand side, but (incorrectly) gives 0 for the integral on the left[1].

**Example 2.2.3.** Differentiaton of (2.5) with respect to $c$ yields the identity

$$\int_0^\infty \frac{x^{b+c-1}\,\ln^2 x}{(x^2+1)^c\,(x^b+1)^2}\,dx \quad = \quad \int_0^\infty \frac{x^{b+c-1}\,\ln x\,\ln(x^2+1)}{(x^2+1)^c\,(x^b+1)^2}\,dx. \tag{2.8}$$

For $b=c=1$ we have

$$\int_0^\infty \frac{x\,\ln^2 x}{(x^2+1)(x+1)^2}\,dx \quad = \quad \int_0^\infty \frac{x\,\ln x\,\ln(x^2+1)}{(x^2+1)(x+1)^2}\,dx. \tag{2.9}$$

[1]V. Adamchik has informed us that this problem has been corrected for the next edition of Mathematica.

Mathematica 4.0 evaluates both integrals as $\pi^2(3\pi-8)/48$, but it takes 1.47 seconds to evaluate the left-hand side and 82.61 seconds to evaluate the right-hand side.

**Example 2.2.4**. The case $a=1$ of (1.6) yields

$$\int_0^\infty \left(\frac{x}{x^2+1}\right)^{2r} \cdot \frac{x^2+1}{x^b+1} \cdot \frac{dx}{x^2} \quad = \quad 2^{-2r} B\left(r-\tfrac{1}{2}, \tfrac{1}{2}\right). \tag{2.10}$$

Differentiating (2.10) $m$ times with respect to $r$ gives

$$\int_0^\infty f^{2r-1}(x) \ln^m\left(f(x)\right) \frac{dx}{x(x^b+1)} \quad = \quad \frac{\sqrt{\pi}}{2^{m+1}} \left(\frac{d}{dr}\right)^m \frac{\Gamma(r-\frac{1}{2})}{\Gamma(r)\, 2^{2r-1}}, \tag{2.11}$$

where $f(x) := x/(x^2+1)$.

In the special case $r=1$ the right-hand side of (2.11) is a linear combination of products of the constants $\pi$, $\ln 2$ and the values of the zeta function at odd integers. Assigning the weights $w(\pi) = w(\ln 2) = 1$ and $w(\zeta(j)) = j$ and defining the weight of a monomial in additive fashion, we see that the value of this integral is a *homogeneous* polynomial of weight $m$. For example

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \ln\left(\frac{x}{x^2+1}\right) \frac{dx}{(x^2+1)^2} &= -\frac{\ln 2}{2} \\
\frac{1}{\pi}\int_0^\infty \ln^2\left(\frac{x}{x^2+1}\right) \frac{dx}{(x^2+1)^2} &= \frac{1}{48}(48\ln^2 2 + \pi^2) \\
\frac{1}{\pi}\int_0^\infty \ln^3\left(\frac{x}{x^2+1}\right) \frac{dx}{(x^2+1)^2} &= -\frac{1}{8}(16\ln^3 2 + 3\zeta(3) + \pi^2 \ln 2\,) \\
\frac{1}{\pi}\int_0^\infty \ln^4\left(\frac{x}{x^2+1}\right) \frac{dx}{(x^2+1)^2} &= \frac{19}{960}\pi^4 + \frac{1}{2}\pi^2\ln^2 2 + 4\ln^4 2 + 3\zeta(3)\ln 2.
\end{aligned}$$

Mathematica 4.0 evaluates only the first two cases.

### 2.3. **A family of polynomials.**

In this subsection we produce by differentiation a family of vanishing integrals.

**Theorem 2.1.** There exists a family of polynomials $Q_n(t) := a_0^n + a_1^n t + \cdots + a_n^n t^n$ such that

$$\int_0^\infty \frac{x^{b+c-1}\, \ln^{n+1} x}{(x^b+1)^{n+2}\,(x^2+1)^c} \times Q_n(-x^b)\, dx = 0. \tag{2.12}$$

The polynomials $Q_n$ are symmetric (i.e. $Q_n(t^{-1}) = t^{-n}Q_n(t)$), of degree $n$, have positive integer coefficients, and satisfy the differential-difference equation

$$Q_{n+1}(t) \quad = \quad t(1-t)\,\frac{d}{dt}Q_n(t) + (1+(n+1)t)\,Q_n(t). \tag{2.13}$$

Moreover, the coefficients $a_j^n$ are unimodal with mode $\lfloor \frac{n}{2} \rfloor$.

*Proof.* Differentiate the integrand in (2.5) $n$ times to produce

$$\left(\frac{d}{db}\right)^n \frac{x^{b+c-1}\,\ln x}{(x^2+1)^c\,(x^b+1)^2} \quad = \quad \frac{x^{b+c-1}\,\ln^{n+1} x}{(x^b+1)^{n+2}\,(x^2+1)^c} \times Q_n(-x^b) \tag{2.14}$$

for some function $Q_n(t)$. Equation (2.13) then follows by induction, and we thus see that $Q_n(t)$ is a polynomial (using the initial condition $Q_0(t) = 1$). Equation (2.13) also yields the recursion

$$\begin{aligned} a_0^{n+1} &= a_0^n \\ a_j^{n+1} &= (j+1)a_j^n + (n+2-j)a_{j-1}^n \quad \text{for } 1 \leq j \leq n \\ a_{n+1}^{n+1} &= a_n^n, \end{aligned}$$

from which we conclude the positivity and the symmetry of the coefficients of $Q_n$.

Finally, we show unimodality by induction. For $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ we have

$$\begin{aligned} a_j^{n+1} - a_{j-1}^{n+1} &= (j+1)a_j^n + (n+2-2j)a_{j-1}^n - (n+3-j)a_{j-2}^n \\ &\geq (n+3-j)\left(a_{j-1}^n - a_{j-2}^n\right), \end{aligned}$$

and the second part follows by symmetry. □

The first values of $Q_n(t)$ are

$$\begin{aligned} Q_0(t) &= 1 \\ Q_1(t) &= t+1 \\ Q_2(t) &= t^2+4t+1 \\ Q_3(t) &= t^3+11t^2+11t+1 \\ Q_4(t) &= t^4+26t^3+66t^2+26t+1 \\ Q_5(t) &= t^5+57t^4+302t^3+302t^2+57t+1. \end{aligned} \tag{2.15}$$

**Note**. Unimodal polynomials are linked to integration formulae. For example, we have shown in [3, 4, 5] that $P_m(a)$ defined via

$$P_m(a) \quad := \quad \frac{1}{\pi} 2^{m+3/2}(a+1)^{m+1/2} \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \tag{2.16}$$

is a unimodal polynomial and is given explicitly by

$$P_m(a) \quad = \quad 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{2.17}$$

## 2.4. **New integrals by integration.**

Integrating (2.11) with respect to $b$ from $b_1$ to $b_2$ and using

$$\int_{b_1}^{b_2} \frac{db}{x^b+1} \quad = \quad \frac{1}{\ln x} \ln\left(\frac{x^{b_2}\left(x^{b_1}+1\right)}{x^{b_1}\left(x^{b_2}+1\right)}\right)$$

we obtain

$$\begin{aligned} \int_0^\infty \left(\frac{x}{x^2+1}\right)^{2r-1} \left[\ln\left(\frac{x}{x^2+1}\right)\right]^m \ln\left(\frac{x^{b_2}\left(x^{b_1}+1\right)}{x^{b_1}\left(x^{b_2}+1\right)}\right) \frac{dx}{x \ln x} \\ = (b_2-b_1)\frac{\sqrt{\pi}}{2^{m+1}}\left(\frac{d}{dr}\right)^m \frac{\Gamma(r-\frac{1}{2})}{\Gamma(r)\, 2^{2r-1}}. \end{aligned} \tag{2.18}$$

**Example 2.4.1**. The case $m = 0$, $b_2 = 1$ and $b_1 = 2p+1$ with $p \in \mathbb{N}$ yields

$$\int_0^\infty \left( \frac{x}{x^2+1} \right)^{2r-1} \times \left\{ \frac{\ln(x^{2p} - x^{2p-1} + \cdots - x + 1)}{\ln x} - 2p \right\} \frac{dx}{x} = -\frac{p}{2^{2r-1}} B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right).$$

Using

$$\int_0^\infty \frac{x^{2r-2}\, dx}{(x^2+1)^{2r-1}} = \tfrac{1}{2} B\left(r - \tfrac{1}{2}, r - \tfrac{1}{2}\right)$$

and

$$\tfrac{1}{2} B\left(r - \tfrac{1}{2}, r - \tfrac{1}{2}\right) - 2^{-2r+1} B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\pi}{2^{4r-3}} \binom{2r-2}{r-1}$$

we get[2]

$$\int_0^\infty \left( \frac{x}{x^2+1} \right)^{2r+1} \times \left\{ \frac{\ln(x^{2p} - x^{2p-1} + \cdots - x + 1)}{\ln x} \right\} \frac{dx}{x} = \frac{p\pi}{2^{4r+1}} \binom{2r}{r}. \tag{2.19}$$

The case $r = 0$ and $p = 1$ gives

$$\int_0^\infty \frac{\ln(x^2 - x + 1)}{(x^2+1)\, \ln x}\, dx = \frac{\pi}{2}. \tag{2.20}$$

We were unable to evaluate (2.20) symbolically.

**Example 2.4.2**. We now sum (2.19) from $r = 0$ to $r = \infty$. The series resulting from the right-hand side can be summed by Mathematica as

$$\frac{p\pi}{2} \sum_{r=0}^{\infty} 2^{-4r} \binom{2r}{r} = \frac{p\,\pi}{\sqrt{3}}.$$

A more traditional evaluation is discussed in Theorem 3.6. Summing the integrals on the left-hand side we obtain

$$\int_0^\infty \frac{x^2+1}{x^4+x^2+1} \times \left\{ \frac{\ln(x^{2p} - x^{2p-1} + \cdots - x + 1)}{\ln x} \right\} dx = \frac{p\pi}{\sqrt{3}}. \tag{2.21}$$

**Example 2.4.3**. For $p = 1$ we get

$$\int_0^\infty \frac{x^2+1}{x^4+x^2+1} \times \frac{\ln(x^2 - x + 1)}{\ln x}\, dx = \frac{\pi}{\sqrt{3}}.$$

**Example 2.4.4**. Differentiating (2.19) with respect to the parameter $p$ yields

$$\int_0^\infty \frac{x^{2(r+p)}\, dx}{(x^2+1)^{2r+1}(x^{2p}+1)} = \frac{\pi}{2^{4r+1}} \binom{2r}{r}.$$

**Example 2.4.5**. Differentiating (2.21) with respect to the parameter $r$ yields

$$\int_0^\infty \frac{x^{2r}}{(x^2+1)^{2r+1}\, \ln x} \cdot \ln\left( \frac{x}{x^2+1} \right) \cdot \ln\left( \frac{x^{2p}+1}{x+1} \right) dx = \frac{d}{dr} \left( \frac{p\pi}{2^{4r+2}} \frac{\Gamma(2r+1)}{\Gamma^2(r+1)} \right).$$

[2]Observe the change in the exponent from the usual $2r-1$ to $2r+1$.

For $r = 1$ and $p = 1$ we obtain

$$\int_0^\infty \frac{x^2}{(x^2+1)^3\ \ln x} \cdot \ln\left(\frac{x}{x^2+1}\right) \cdot \ln\left(\frac{x^2+1}{x+1}\right)\, dx \quad = \quad -\frac{\pi}{32}(4\ln 2 - 1).$$

None of the last three examples could be done symbolically.

## 3. Some results from Classical Analysis

In this section we derive several classical results from (1.6). We start with Wallis' formula, continue with Legendre's duplication formula for the gamma function, and conclude with the generating function for the central binomial coefficients.

**Theorem 3.1. Wallis' formula**. Let $r > -1$. Then

$$\int_0^{\pi/2} \sin^r \varphi \, d\varphi \quad = \quad \tfrac{1}{2} B\left(\tfrac{r+1}{2}, \tfrac{1}{2}\right). \tag{3.1}$$

*Proof.* The formulae for $I_2$ and $I_3$ in the special case $a = 1$ yield

$$\int_0^\infty x^{-2}\,(x + 1/x)^{-2r} dx \quad = \quad 2^{-2r} B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right) \tag{3.2}$$

and

$$\int_0^\infty (x + 1/x)^{-2r} dx \quad = \quad 2^{-2r} B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right). \tag{3.3}$$

The sum of (3.2) and (3.3) gives

$$\int_0^\infty (1 + x^{-2})\,(x + 1/x)^{-2r} dx \quad = \quad 2^{1-2r} B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right), \tag{3.4}$$

and the change of variable $x \mapsto \tan\varphi$ then converts (3.4) into (3.1).

□

**Corollary 3.2.** Let $b, c \in \mathbb{R}^+$, $a > -\sqrt{bc}$ and $r > 1/2$. Then

$$\int_0^\infty \left(\frac{x^2}{bx^4 + 2ax^2 + c}\right)^r dx \quad = \quad \frac{B\left(r - \frac{1}{2}, \frac{1}{2}\right)}{2^{r+1/2}\sqrt{b}\,(a + \sqrt{bc})^{r-1/2}}. \tag{3.5}$$

*Proof.* Let $x \mapsto (c/b)^{1/4} u$ and use (1.6).

□

**Corollary 3.3.** Let $b, c \in \mathbb{R}^+$ and $a > -\sqrt{bc}$. Then

$$\int_0^\infty \frac{x^2\, dx}{bx^4 + 2ax^2 + c} \quad = \quad \frac{\pi}{2\sqrt{2b(a + \sqrt{bc})}} \tag{3.6}$$

and

$$\int_0^\infty \frac{dx}{bx^4 + 2ax^2 + c} \quad = \quad \frac{\pi}{2\sqrt{2c(a + \sqrt{bc})}}. \tag{3.7}$$

In particular,

$$N_{0,4}(a;0) := \int_0^\infty \frac{dx}{x^4 + 2ax^2 + 1} \quad = \quad \frac{\pi}{2\sqrt{2(a+1)}}. \tag{3.8}$$

*Proof.* Let $r = 1$ in Corollary 3.2 to produce (3.6). The change of variables $x \mapsto 1/x$ yields (3.7). □

**Example 3.1.1**. Letting $a = 2$, $b = 5$ and $c = 9$ in (3.7) gives

$$\int_0^\infty \frac{dx}{5x^4 + 4x^2 + 9} = \frac{\pi}{2\sqrt{18(2 + 3\sqrt{5})}}. \tag{3.9}$$

**Note**. The integral $N_{0,4}(a;0)$ is part of the family

$$N_{r,4}(a;m) := \int_0^\infty \frac{x^{2r}\,dx}{(x^4 + 2ax^2 + 1)^{m+1}}, \; r \in \mathbb{N} \cup \{0\}, \tag{3.10}$$

which is evaluated in [7] as

$$\frac{\pi}{2^{3m+3/2}\,(1+a)^{m+1/2}} \times \sum_{j=0}^{m-r} 2^j (1+a)^j \binom{2m-2j}{m-j} \binom{m-r+j}{2j} \binom{2j}{j} \binom{m}{j}^{-1}$$

provided $0 \leq r \leq m$. If $m + 1 \leq r \leq 2m + 1$, the change of variable $x \mapsto 1/x$ yields $N_{r,4}(a;m) = N_{2m+1-r,4}(a;m)$. For example, $a = 4,\ m = 11$ and $r = 5$ give

$$\int_0^\infty \frac{x^{10}\,dx}{(x^4 + 8x^2 + 1)^{12}} = \frac{195240969\,\pi}{104857600000000000\sqrt{10}}.$$

**Theorem 3.4. Legendre's duplication formula for $\Gamma$.** For $r \in \mathbb{R}^+$ we have

$$\Gamma(2r) = 2^{2r-1}\frac{\Gamma(r + \frac{1}{2})\Gamma(r)}{\sqrt{\pi}}, \tag{3.11}$$

which yields

$$\Gamma(n + \tfrac{1}{2}) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi} \tag{3.12}$$

for $n \in \mathbb{N}$.

*Proof.* Let $b = a^2$ and $c = 1$ in Corollary 3.2 to produce

$$\int_0^\infty \frac{x^{2r}\,dx}{(ax^2 + 1)^{2r}} = \frac{B\left(r - \frac{1}{2}, \frac{1}{2}\right)}{2^{2r}\,a^{r+1/2}}. \tag{3.13}$$

The change of variable $ax^2 \mapsto t$ and the relation

$$B(p,q) = \int_0^\infty \frac{z^{p-1}\,dz}{(z+1)^{p+q}}$$

then yield

$$2^{2r-1}\,B\left(r - \tfrac{1}{2}, r + \tfrac{1}{2}\right) = B\left(r - \tfrac{1}{2}, \tfrac{1}{2}\right), \tag{3.14}$$

from which (3.11) follows after using the functional equation

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{3.15}$$

and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The relation $\Gamma(n) = (n-1)!$ yields (3.12). □

**Note**. An alternate proof of Legendre's duplication formula is discussed in [16]. The Mellin transform

$$\mathfrak{M}(f(x))(s) \quad := \quad \int_0^\infty f(x)x^{s-1}dx$$

satisfies

$$\mathfrak{M}(f * g) \quad = \quad \mathfrak{M}(f) \cdot \mathfrak{M}(g) \tag{3.16}$$

where

$$(f * g)(x) \quad := \quad \int_0^\infty f(x/u)g(u)\frac{du}{u}.$$

The identity (3.16) applied to the functions $f(x) = e^{-x}$ and $g(x) = x^{-1/2}e^{-x}$ yields (3.11).

Still another proof of (3.11) is found in the text [9]. This clever proof is due to Serret. Here the integral

$$B(r,r) \quad = \quad \int_0^1 \left(x - x^2\right)^{r-1} \, dx = \int_0^1 \left(1/4 - (1/2 - x)^2\right)^{r-1} \, dx$$

is seen to be $2^{1-2p}B(1/2,p)$ by the change of variable $(1/2 - x)^2 \mapsto x/4$. The functional equation (3.15) then yields the result.

**Corollary 3.5.** Let $n \in \mathbb{N}$. Then

$$\int_0^{\pi/2} \sin^{2n+1} x \, dx \quad = \quad \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1} = \frac{2^{2n}\,(n!)^2}{(2n+1)!} \tag{3.17}$$

and

$$\int_0^{\pi/2} \sin^{2n} x \, dx \quad = \quad \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} = \frac{(2n)!\,\pi}{2^{2n+1}\,(n!)^2}. \tag{3.18}$$

*Proof.* Use Wallis' formula (3.1) and (3.12). □

**Theorem 3.6. Generating function for $\binom{2r}{r}$.** The generating function for the central binomial coefficients is given by

$$\sum_{r=0}^{\infty} \binom{2r}{r} t^r \quad = \quad \frac{1}{\sqrt{1-4t}}. \tag{3.19}$$

*Proof.* For $r \in \mathbb{N}$ the expression in (1.6) reduces to

$$\int_0^\infty \left( \frac{x^2}{x^4 + 2ax^2 + 1} \right)^r dx \quad = \quad 2\pi\sqrt{2(1+a)} \binom{2r-2}{r-1} [8(1+a)]^{-r}, \tag{3.20}$$

using (3.15) and (3.12) to simplify the right-hand side. Now sum (3.20) from $r = 1$ to $r = \infty$. The sum of the integrals on the left-hand side can be evaluated as

$$\int_0^\infty \frac{x^2\,dx}{x^4 + (2a-1)x^2 + 1} \quad = \quad \frac{\pi}{2\sqrt{2a+1}}$$

using (3.6). The proof is completed by letting $t = [8(1+a)]^{-1}$.

□

## 4. An integral of degree 8

We now employ (1.6) to evaluate the integral of a *symmetric* rational function of degree 8. We call a rational function symmetric if its denominator $Q$ satisfies $Q(1/x) = x^{deg(Q)}Q(x)$. These functions are the basis for an algorithm to integrate even rational functions developed in [7].

**Theorem 4.1.** Define

$$M_8(a_1,a_2;r) \;:=\; \int_0^\infty \left( \frac{x^4}{x^8+a_2x^6+2a_1x^4+a_2x^2+1}\right)^r dx, \tag{4.1}$$

where $r \in \mathbb{N}$ and $a_1 > \max\{-a_2-1, -\text{sign}(a_2+4)\times\left(a_2^2/8+1\right)\}$. Then

$$M_8(a_1,a_2;r) \;=\; c^{1/4-r}N_{0,4}\left(\frac{a_2+4}{2\sqrt{c}}; r-1\right), \tag{4.2}$$

where $c = 2(a_1+a_2+1)$.

*Proof.* The change of variable $x \mapsto 1/x$ yields a new form of the integral $M_8$:

$$M_8(a_1,a_2;r) \;=\; \int_0^\infty \left( \frac{x^4}{x^8+a_2x^6+2a_1x^4+a_2x^2+1}\right)^r \frac{dx}{x^2}. \tag{4.3}$$

Computing the average of these two forms and letting $x = \tan\theta$ and then $\psi = 2\theta$ produces

$$M_8(a_1,a_2;r) \;=\; 2^{-r+1}\int_0^\pi \frac{(1-C)^{2r-1}\,d\psi}{[\,(a_1-a_2+1)C^2+7(2-a_1-a_2)C+(17+3a_2+a_1)\,]^r},$$

where $C = \cos\psi$. The substitution $z = \text{cotg}\,\psi$ then gives

$$M_8(a_1,a_2;r) \;=\; 2^{-r+1}\int_0^\infty \frac{dz}{(8z^4+2(a_2+4)z^2+(a_1+a_2+1))^r}. \tag{4.4}$$

The change of variable $z \mapsto (8/(a_1+a_2+1))^{1/4}t$ and (3.10) yield (4.2). ☐

**Theorem 4.2.** The generating function for $\binom{4r+1}{2r}$ is

$$\sum_{r=0}^\infty \binom{4r+1}{2r} t^r \;=\; \frac{\sqrt{2}}{\sqrt{1-16t}\,\sqrt{1+\sqrt{1-16t}}}. \tag{4.5}$$

*Proof.* The result of Theorem 4.1 is summed form $r=1$ to $r=\infty$. The integrals on the left-hand side add up to

$$\int_0^\infty \frac{x^4\,dx}{x^8+a_2x^6+(2a_1-1)x^4+a_2x^2+1} \;=\; c^{-3/4}N_{0,4}\left(\frac{a_2+4}{2\sqrt{c}};0\right)$$

using Theorem 4.1 with $r=1$. We thus have

$$\sum_{r=0}^\infty c^{-r}N_{0,4}\left(\frac{a_2+4}{2\sqrt{c}};r\right) \;=\; \left(\frac{c}{c-1}\right)^{3/4} N_{0,4}\left(\frac{a_2+4}{2\sqrt{c-1}};0\right).$$

Now choose $a_2 = 2\sqrt{2(a_1-1)}$ so that the argument of $N_{0,4}$ in the series is 1 and employ

$$N_{0,4}(1;r) = \frac{1}{2}B\left(\tfrac{1}{2}, 2r+\tfrac{3}{2}\right) = \frac{\pi}{2^{4r+2}}\binom{4r+1}{2r} \tag{4.6}$$

to obtain (4.5). ☐

## 5. The Fresnel integrals

The evaluation of Fresnel integrals

$$F_0 = \int_0^\infty \cos x^2 \, dx \quad \text{and} \quad G_0 = \int_0^\infty \sin x^2 \, dx \tag{5.1}$$

by contour integration appears in Laurent's classical book [14]. According to Remmert [17] the expressions for $F_0$ and $G_0$ were known to Euler [10]. The calculation of (5.1) appears as an exercise in most texts in Complex Analysis, for instance in [1], p. 206 and in [15]. The evaluation of (5.1) by strictly real-variable methods appears in [9, 11, 13, 18, 19]. The proof presented here is a modification of Leonard's proof [13].

**Theorem 5.1.** The Fresnel integrals

$$F(t) := \int_0^\infty e^{-tx^2} \cos x^2 \, dx \quad \text{and} \quad G(t) := \int_0^\infty e^{-tx^2} \sin x^2 \, dx \tag{5.2}$$

are given by

$$F(t) = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{1+t^2}+t}{1+t^2}} \quad \text{and} \quad G(t) = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{1+t^2}-t}{1+t^2}}. \tag{5.3}$$

In particular

$$F_0 = \int_0^\infty \cos x^2 \, dx \quad \text{and} \quad G_0 = \int_0^\infty \sin x^2 \, dx \tag{5.4}$$

have the common value $\sqrt{\pi/8}$.

*Proof.* The change of variable $x \mapsto \sqrt{x}$ converts (5.2) into

$$G(t) = \frac{1}{2} \int_0^\infty e^{-xt} \frac{\sin x}{\sqrt{x}} \, dx. \tag{5.5}$$

Observe that

$$\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

yields

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-xs}}{\sqrt{s}} ds, \tag{5.6}$$

so replacing (5.6) in (5.5) gives

$$G(t) = \frac{1}{2} \int_0^\infty e^{-xt} \sin x \left( \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-xs}}{\sqrt{s}} \, ds \right) dx.$$

Now reverse the order of integration and use

$$\int_0^\infty e^{-ax} \sin x \, dx = \frac{1}{1+a^2}$$

to obtain, with $s \mapsto u^2$,

$$G(t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{1+(t+s)^2} \frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{du}{1+(u^2+t)^2}.$$

Similarly,

$$F(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{(u^2+t)\,du}{1+(u^2+t)^2}. \tag{5.7}$$

The evaluation of $F$ and $G$ in [13] by partial fractions is elementary but long. In our proof the values of $G(t)$ and $F(t)$ are direct consequences of the quartic integrals evaluated in Section 3. From Corollary 3.3 we have

$$G(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{du}{u^4+2tu^2+(1+t^2)} = \frac{\sqrt{\pi}}{\sqrt{8(1+t^2)}} \times \sqrt{\sqrt{1+t^2}-t}$$

and

$$\begin{aligned} F(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{u^2\,du}{u^4+2tu^2+1+t^2} + \frac{t}{\sqrt{\pi}} \int_0^\infty \frac{du}{u^4+2tu^2+1+t^2}, \\ &= \frac{\sqrt{\pi}}{\sqrt{8(1+t^2)}} \times \sqrt{\sqrt{1+t^2}+t}. \end{aligned}$$

□

## 6. Conclusions

We have presented consequences of an exact evaluation of an integral with three parameters. Many classical results can be derived from it. In addition, we have been able to evaluate a large number of definite integrals which cannot be found in standard tables and cannot be evaluated by standard symbolic packages.

## References


[1] AHLFORS, L.: *Complex Analysis.* Third Ed., McGraw-Hill, 1979.

[2] BOROS, G. - MOLL, V.: *An integral with three parameters.* SIAM Review **40**, 1998, 972-980.

[3] BOROS, G. - MOLL, V.: *An integral hidden in Gradshteyn and Ryzhik.* Journal of Comp. Appl. Math. **106**, 1999, 361-368.

[4] BOROS, G. - MOLL, V.: *A criterion for unimodality.* Elec. Journal of Combinatorics, **6**, 1999, R10.

[5] BOROS, G. - MOLL, V.: *A sequence of unimodal polynomials.* Jour. Math. Anal. Appl. **237**, 1999, 272-287.

[6] BOROS, G. - MOLL, V.: *A rational Landen transformation. The case of degree six.* Contemporary Mathematics. *To appear.*

[7] BOROS, G. - MOLL, V.: *Landen transformations and the integration of rational functions.* Preprint.

[8] BROMWICH, T.J.: *An Introduction to the Theory of Infinite Series.* Second Edition, MacMillan & Co., New York, 1926.

[9] EDWARDS, J.: *A treatise of the Integral Calculus*, II. Chelsea Publ. Co., New York, 1922.

[10] EULER, L.: *De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum.* Opera Omnia, (1) **19**, 217-227.

[11] FLANDERS, H.: *On the Fresnel integrals.* Amer. Math. Monthly **89**, 1982, 264-266.

[12] GRADSHTEYN, I.S. - RYZHIK, I.M.: *Table of Integrals, Series and Products.* Fifth Edition, ed. Alan Jeffrey. Academic Press, 1994.

[13] LEONARD, L.E.: *More on Fresnel integrals.* Amer. Math. Monthly **95**, 1988, 431-433.

[14] LAURENT, H.: *Traite d'analyse*, Paris, 1888.

[15] OLDS, C.D.: *The Fresnel integrals.* Amer. Math. Monthly **75**, 1968, 285-286.

[16] RAO, S.K.: *A proof of Legendre's duplication formula.* Amer. Math. Monthly **62**, 1965, 620-621.

[17] REMMERT, R.: *Theory of Complex Functions.* Grad. Texts in Mathematics, **122**, Springer-Verlag, New York, 1990.

[18] YZEREN, J. van: *Moivre's and Fresnel's integrals by simple integration.* Amer. Math. Monthly **86**, 1979, 691-693.

[19] WEINSTOCK, R.: *Elementary evaluation of* $\int_0^\infty e^{-x^2} dx$, $\int_0^\infty \cos x^2 dx$, *and* $\int_0^\infty \sin x^2 dx$. Amer. Math. Monthly **97**, 1990, 39-42.

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