

## A DOZEN INTEGRALS: RUSSELL-STYLE

TEWODROS AMDEBERHAN AND VICTOR H. MOLL

On June 15, 1876, the Proceedings of the Royal Society of London published the paper [1] by Mr. W. H. L. Russell entitled *On certain integrals*. The paper starts with *The following are certain integrals which will, I hope, be found interesting*. The rest of the paper is a list of 12 definite integrals starting with

$$\int_0^{\infty} dz e^{-(r+1)z+xe^{-z}} = e^x (1 - r/x + r(r-1)/x^2 - \dots) / x.$$

In honor of Russell notable achievements in the evaluation of integrals, we hope the reader will find the next list interesting:

$$(1) \quad \int_0^{\infty} x \left( \frac{\gamma \sinh \gamma x}{\cosh^2 \gamma x} e^{-x^2/\pi^2} + \frac{\sqrt{\pi} \sinh x}{\cosh^2 x} e^{-\gamma^2 x^2} \right) dx = \int_0^{\infty} \frac{e^{-x^2/\pi^2} dx}{\cosh \gamma x}$$

$$(2) \quad \int_0^{\infty} x \left( \frac{1}{\pi} e^{-x^2/\pi^2} + \frac{1}{\sqrt{\pi}} e^{-x^2} \right) \frac{\sinh x dx}{\cosh^2 x} = \int_0^{\infty} \frac{e^{-x^2} dx}{\cosh \pi x}$$

$$(3) \quad \int_0^{\infty} x \left( e^{-x^2/\pi} + 2e^{-16x^2/\pi} \right) \frac{\sinh 2x dx}{\cosh^2 2x} = \int_0^{\infty} \frac{e^{-4x^2/\pi} dx}{\cosh 4x}$$

$$(4) \quad \int_0^{\infty} \left( \frac{\sinh x}{\cosh^2 x} + \frac{\pi^{3/2} \sinh \pi x}{\cosh^2 \pi x} \right) x e^{-x^2} dx = \int_0^{\infty} \frac{e^{-x^2} dx}{\cosh x}$$

$$(5) \quad \int_0^{\infty} x e^{-x^2/\pi} \frac{\sinh x dx}{\cosh^2 x} = \int_0^{\infty} \frac{e^{-4x^2/\pi} dx}{\cosh 2x}$$

$$(6) \quad \int_0^1 \frac{x^{-\ln x} dx}{1+x^2} = \int_0^{\infty} \frac{e^{-4x^2/\pi} dx}{\cosh 2\sqrt{\pi}x}$$

$$(7) \quad \int_0^{\infty} \frac{x^2 e^{-x^2} dx}{\cosh \sqrt{\pi}x} = \frac{1}{4} \int_0^{\infty} \frac{e^{-x^2} dx}{\cosh \sqrt{\pi}x}$$

$$(8) \quad \int_0^{\infty} \left( e^{-x^2/\pi^2} + \pi^{5/2} e^{-x^2} \right) \frac{x^2 dx}{\cosh x} = \frac{\pi^2}{2} \int_0^{\infty} \frac{e^{-x^2/\pi^2} dx}{\cosh x}$$

$$(9) \quad \int_0^{\infty} \left( \sqrt{\pi} e^{-x^2/3} + 9\sqrt{3}\pi^{-2} e^{-3x^2/\pi^2} \right) \frac{x^2 dx}{\cosh x} = \frac{3\pi\sqrt{3}}{2} \int_0^{\infty} \frac{e^{-3x^2} dx}{\cosh \pi x}$$

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$$(10) \quad \int_0^\infty \left( \pi^5 e^{-\pi^3 x^2/G} + G^{5/2} e^{-Gx^2/\pi} \right) \frac{x^2 dx}{\cosh \pi x} = \frac{\pi G^{3/2}}{2} \int_0^\infty \frac{e^{-Gx^2/\pi} dx}{\cosh \pi x}$$

$$(11) \quad \int_0^\infty x(3 - 4\pi x^2) \frac{e^{-\pi x(x+1)} dx}{\sinh \pi x} = \frac{1}{2\pi}$$

$$(12) \quad \int_0^\infty \frac{\sin^2 x}{\cosh x + \cos x} \frac{dx}{x^2} + \frac{2}{\pi} \int_0^{e^{-\pi/2}} \frac{\tan^{-1} x}{x} dx = \frac{\pi}{4}$$

Here,

$$(13) \quad \gamma := \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n$$

is Euler's constant and

$$(14) \quad G := \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2},$$

is Catalan's constant. We note the fact that both of these constants have so far resisted all attempts at proofs of irrationality.

Every formula can be checked by observing that if  $f(y) = 1/\cosh(y)$  and  $M$  is the transformation

$$M(f)(y) = \int_0^\infty e^{-x^2} f(xy) dx,$$

then we have the elementary relation

$$\frac{df}{dx} = -xf(y)\sqrt{1-f^2(y)}$$

and also

$$yM(f)(y) = \sqrt{\pi} M \left( f \left( \frac{\pi}{y} \right) \right).$$

Applying these to the left hand side integrands produces the right-hand side. In the last few calculations, the role of the functions  $1/\sinh y$  and  $e^{-x^2}$  has to be interchanged.

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#### REFERENCES

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DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139  
*E-mail address:* [tewodros@math.mit.edu](mailto:tewodros@math.mit.edu)

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118  
*E-mail address:* [vhm@math.tulane.edu](mailto:vhm@math.tulane.edu)