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Journal of Number Theory 116 (2006) 200–229
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

The evaluation of Tornheim double sums, Part 1

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Received 8 June 2004; revised 16 March 2005

Available online 5 July 2005

Communicated by D. Goss

Abstract

We provide an explicit formula for the Tornheim double series in terms of integrals involving the Hurwitz zeta function. We also study the limit when the parameters of the Tornheim sum become natural numbers, and show that in that case it can be expressed in terms of definite integrals of triple products of Bernoulli polynomials and the Bernoulli function $A_k(q) := k\zeta'(1-k, q)$.

MSC: primary 33

Keywords: Hurwitz zeta function

1. Introduction

The function

$$T(a,b,c) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^a\, m^b\, (n+m)^c}, \quad a,b,c \in \mathbb{C}, \tag{1.1}$$

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doi:10.1016/j.jnt.2005.04.008

was introduced by Tornheim in [25]. We provide here an analytic expression for $T(a,b,c)$ in terms of the integrals

$$I(a,b,c) = \int_0^1 \zeta(1-a,q)\zeta(1-b,q)\zeta(1-c,q)\,dq \tag{1.2}$$

and

$$J(a,b,c) = \int_0^1 \zeta(1-a,q)\zeta(1-b,q)\zeta(1-c,1-q)\,dq. \tag{1.3}$$

Here $\zeta(z,q)$ is the Hurwitz zeta function,

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}, \tag{1.4}$$

defined for $z \in \mathbb{C}$ and $q \neq 0, -1, -2, \ldots$. The series (1.4) converges for $\operatorname{Re} z > 1$ and $\zeta(z,q)$ admits a meromorphic extension to the complex plane with a single pole at $z = 1$ as its only singularity.

In the case where the parameters a, b, c in (1.1) are positive integers, the Tornheim sum can be expressed in terms of the Riemann zeta function

$$\zeta(z) = \zeta(z,1) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{1.5}$$

its derivatives, and integrals related to the families (1.2) and (1.3), as given in Theorem 1.1 below.

We use the notation

$$\bar{\zeta}(z,q) := \zeta(1-z,q), \tag{1.6}$$

defined for $z \neq 0$ and $q \neq 0, -1, -2, \ldots$.

The results presented here are a continuation of [11,12] where we have provided many explicit evaluations of definite integrals containing $\zeta(z,q)$ in the integrand. For instance, if $\operatorname{Re} a > 0$, then

$$\int_0^1 \bar{\zeta}(a,q)\,dq = 0, \tag{1.7}$$

and, for $\operatorname{Re} a > 1, \operatorname{Re} b > 1$, we have

$$\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(b,q)\,dq = \frac{2\Gamma(a)\Gamma(b)}{(2\pi)^{a+b}}\,\zeta(a+b)\cos\left(\frac{\pi}{2}(a-b)\right) \tag{1.8}$$

and

$$\int_0^1 \bar{\zeta}(a,q)\bar{\zeta}(b,1-q)\,dq = \frac{2\Gamma(a)\Gamma(b)}{(2\pi)^{a+b}}\,\zeta(a+b)\cos\left(\frac{\pi}{2}(a+b)\right). \tag{1.9}$$

Lerch's evaluation [30],

$$\frac{d}{dz}\,\zeta(z,q)\bigg|_{z=0} = \ln\,\Gamma(q) - \ln\,\sqrt{2\pi}, \tag{1.10}$$

yields integrals involving the loggamma function. For instance,

$$L_1 = \int_0^1 \ln\,\Gamma(q)\,dq = \ln\,\sqrt{2\pi} \tag{1.11}$$

and

$$L_2 = \int_0^1 \ln^2\,\Gamma(q)\,dq = \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma L_1}{3} + \frac{4}{3}L_1^2 - \frac{A\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2} \tag{1.12}$$

with

$$A = \gamma + \ln\,2\pi. \tag{1.13}$$

We expect that the methods developed here will provide analytic expressions for the constant

$$L_3 = \int_0^1 \ln^3\,\Gamma(q)\,dq. \tag{1.14}$$

The series $T(a,b,c)$, for $a,b,c \in \mathbb{R} - \mathbb{N}$, is given in Theorem 2.4 in terms of integrals (1.2) and (1.3). The evaluation of the Tornheim series for integer values of the parameters are expressed in terms of some definite integrals:

Theorem 1.1. *The Tornheim sums $T(n_1,n_2,n_3)$ can be expressed as a finite expression of the Riemann zeta function, its derivatives and the integrals*

$$K_{m,n} = \int_0^1 \psi^{(-m)}(q)B_n(q)\ln\Gamma(q)\,dq,$$

$$K^*_{m,n} = \int_0^1 \psi^{(-m)}(1-q)B_n(q)\ln\Gamma(q)\,dq,$$

$$Z_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(q) \ln\Gamma(q)\,dq,$$
$$Z^*_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(1-q) \ln\Gamma(q)\,dq.$$

Here $B_n(q)$ is the Bernoulli polynomial, $\Gamma(q)$ is the classical gamma function and $\psi^{(-m)}(q) = \psi(-m, q)$, where

$$\psi(z,q) = e^{-\gamma z} \frac{\partial}{\partial z}\left[e^{\gamma z} \frac{\zeta(z+1,q)}{\Gamma(-z)}\right] \tag{1.15}$$

is the generalization of the polygamma function introduced by the authors in [13], with z an arbitrary complex variable. Some properties of $\psi(z,q)$ are given in Appendix B. The closed form evaluation of the integrals in Theorem 1.1 will be discussed in a future paper. The proof of Theorem 1.1 is given in Sections 3 and 4.

The series (1.1) converges for $\operatorname{Re} a$, $\operatorname{Re} b$, $\operatorname{Re} c > 1$. Matsumoto [19] showed that it can be continued as a meromorphic function to $\mathbb{C}^3$, with all its singularities located on the subsets of $\mathbb{C}^3$ defined by one of the equations

$$a + c = 1 - l, b + c = 1 - l, a + b + c = 2 \quad \text{with } l \in \mathbb{N}_0.$$

The literature contains many techniques to evaluate some particular cases of $T(a, b, c)$. For instance, the case $c = 0$ is evaluated simply as

$$T(a, b, 0) = \zeta(a)\,\zeta(b).$$

The elementary identity

$$T(a, b-1, c+1) + T(a-1, b, c+1) = T(a, b, c) \tag{1.16}$$

and the symmetry rule

$$T(a, b, c) = T(b, a, c) \tag{1.17}$$

has been used by Huard et al. [17] to give the explicit expression

$$T(a,b,c) = \sum_{i=1}^{a} \binom{a+b-i-1}{a-i} T(i, 0, N-i) + \sum_{i=1}^{b} \binom{a+b-i-1}{b-i} T(i, 0, N-i)$$

in the case that both a and b are positive integers. Here $N = a+b+c$. If N is an odd positive integer greater than 1, then the sum $T(i, 0, N-i)$ is evaluated as

$$T(i,0,N-i) = (-1)^i \sum_{j=0}^{\lfloor (N-i-1)/2 \rfloor} \binom{N-2j-1}{i-1} \zeta(2j)\zeta(N-2j)$$
$$+(-1)^i \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{N-2j-1}{N-i-1} \zeta(2j)\zeta(N-2j) + \zeta(0)\zeta(N).$$

The evaluation of $T(i, 0, N-i)$ in the case N even remains open. The techniques introduced in this paper have allowed us to evaluate the sum $T(a, 0, c)$ in terms of integrals similar to the ones discussed here. Details will appear in [14].

The multiple zeta value, also called Euler sums, are defined by

$$\zeta(i_1, i_2, \ldots, i_k) = \sum \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{1.18}$$

where the sum extends over $n_1 > n_2 > \cdots > n_k > 0$. The sum $T(a, 0; c)$ is $\zeta(c, a)$. A general introduction to these sums is provided in Chapter 3 of Borwein et al. [6].

The identities of Tornheim [25] for $T(n_1, n_2, n_3)$ are based on an elementary identity for series: let f be monotone decreasing and $f(x) \to c$ as $x \to \infty$ and define

$$\varphi(n, m; f) = \frac{f(m)}{n(n+m)} + \frac{f(n)}{m(n+m)} - \frac{f(n+m)}{nm}.$$

Then

$$\sum_{n,m=1}^{\infty} \varphi(n, m; f) = 2 \sum_{r=1}^{\infty} \frac{f(r) - c}{r^2}.$$

The special case $f(x) = 1/x$ yields

$$T(1,1,1) = 2\zeta(3) \tag{1.19}$$

and $f(x) = 1/x^{a-2}$ produces the relation

$$2T(a-2, 1, 1) - T(1, 1, a-2) = 2\zeta(a).$$

Among the many evaluation presented in [25] we mention

$$T(1,1,a-2) = (a-1)\zeta(a) - \sum_{i=2}^{a-2} \zeta(i)\zeta(a-i),$$
$$T(a-2,1,1) = \tfrac{1}{2}T(1,1,a-2) + \zeta(a)$$

and

$$T(1,0,a-1) = \tfrac{1}{2}T(1,1,a-2).$$

Subbarao and Sitaramachandrarao [23] give

$$T(2n,2n,2n) = \tfrac{4}{3}\sum_{i=0}^{n} \binom{4n-2i-1}{2n-1} \zeta(2i)\zeta(6n-2i), \tag{1.20}$$

and this is complemented by Huard et al. [17] with

$$T(2n+1,2n+1,2n+1) = -4\sum_{i=0}^{n} \binom{4n-2i+1}{2n} \zeta(2i)\zeta(6n-2i+3). \tag{1.21}$$

Boyadzhiev [7,8] has given elementary proofs of an expression for $T(a,b,c)$ in terms of the function

$$S(r,p) = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^p}.$$

Here $H_n^{(r)} = 1^{-r} + 2^{-r} + \cdots + n^{-r}$ is the generalized harmonic number. In [8] the author establishes recurrences for the sums $S(r,p)$ that permit to express them as products of zeta values in the case $N = r + p$ odd.

Tornheim double sums and other related ones appear as special cases of the zeta function $\zeta_{\mathfrak{g}}(s)$ of a semi-simple Lie algebra defined as

$$\zeta_{\mathfrak{g}}(s) = \sum_{\rho} \dim(\rho)^{-s},$$

where the sum is over all the finite-dimensional representations of $\mathfrak{g}$. Zagier [32] states that the special case $\mathfrak{g} = \mathfrak{sl}(3)$ yields (1.20). The nomenclature for $T(a,b,c)$ is not standard: Zagier [32] and also Crandall and Buhler [10] call $T(a,b,c)$ the *Witten zeta* *function*. Tsumura [27] has evaluated some special cases of the sum

$$W(p,q,r,s) = \sum_{m,n=1}^{\infty} \frac{1}{m^p n^q (m+n)^r (m+2n)^s}$$

under the parity restriction $p+q+r+s$ is odd. This is the Witten sum corresponding to $SO(5)$.

These sums also have appeared in connection with knots and Feynman diagrams, see [18] for details.

The Bernoulli function

$$A_k(q) = k\zeta'(1-k,q), \quad k \in \mathbb{N} \tag{1.22}$$

introduced in [13], plays an important role in the evaluations presented here. Adamchik [2] proved the identity

$$\zeta'(1-k,q) = \zeta'(1-k) + \sum_{j=0}^{k-1} (-1)^{k-1-j} j! Q_{j,k-1}(q) \ln \Gamma_{j+1}(q), \tag{1.23}$$

where

$$Q_{k,n}(q) = \sum_{j=k}^{n} (1-q)^{n-j} \binom{n}{j} \left\{ {j \atop k} \right\} \tag{1.24}$$

is the Stirling polynomial and the generalized gamma function $\Gamma_n(q)$ is defined inductively via

$$\begin{aligned} \Gamma_{n+1}(q+1) &= \frac{\Gamma_{n+1}(q)}{\Gamma_n(q)}, \\ \Gamma_1(q) &= \Gamma(q), \\ \Gamma_n(1) &= 1. \end{aligned} \tag{1.25}$$

Notation:

$\Gamma(q)$ is the gamma function
$\zeta(z)$ is the Riemann zeta function defined in (1.5)
$\zeta(z,q)$ is the Hurwitz zeta function defined in (1.4)

$\bar{\zeta}(z,q)$	is a shorthand for $\zeta(1-z,q)$
$\zeta_{\pm}(z,q)$	denotes the combination $\zeta(z,q)\pm\zeta(z,1-q)$
$\psi(z,q)$	is the generalized polygamma function, defined in (1.15)
$\psi^{(-n)}(q)$	is the balanced negapolygamma function, defined in (B.5)
$A_n(q)$	is the Bernoulli function, defined in (3.6)
$B_n(q)$	is the Bernoulli polynomial of degree n, defined in (A.1) and (A.2)
B_n	is the n-th Bernoulli number
H_n	is the n-th harmonic number, $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$
h_n	is a shorthand for H_{n-1}
γ	is Euler's constant
A	equals $\gamma+\ln 2\pi$
$A_{\pm}$	equals $A^2\pm\frac{\pi^2}{4}$

2. The main identity

We now provide an analytic expression for the Tornheim double series $T(a,b,c)$ in terms of the integrals (1.2) and (1.3). The analysis of its behavior as the parameters become integers is described in Section 3. The proof employs the Fourier representation for $\bar{\zeta}(z,q)$:

$$\bar{\zeta}(z,q) = \frac{2\Gamma(z)}{(2\pi)^z}\left[\cos\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi qn)}{n^z} + \sin\left(\frac{\pi z}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi qn)}{n^z}\right], \tag{2.1}$$

valid for $\operatorname{Re} z > 1$ and $0 < q \leqslant 1$, given in [4].

From (2.1) we obtain

$$2f_c(z)\sum_{n=1}^{\infty}\frac{\cos 2\pi qn}{n^z} = \bar{\zeta}(z,q)+\bar{\zeta}(z,1-q) \tag{2.2}$$

and

$$2f_s(z)\sum_{n=1}^{\infty}\frac{\sin 2\pi qn}{n^z} = \bar{\zeta}(z,q)-\bar{\zeta}(z,1-q), \tag{2.3}$$

where

$$f_c(z) = \frac{2\Gamma(z)}{(2\pi)^z}\cos\left(\frac{\pi z}{2}\right) \quad \text{and} \quad f_s(z) = \frac{2\Gamma(z)}{(2\pi)^z}\sin\left(\frac{\pi z}{2}\right). \tag{2.4}$$

For a function $h(a,b,c)$ we denote

$$h^{\text{sym}}(a,b,c) = h(a,b,c)+h(b,c,a)+h(c,a,b) \tag{2.5}$$

and

$$h^{n\mathrm{sym}}(a,b,c) = -h(a,b,c) + h(b,c,a) + h(c,a,b). \tag{2.6}$$

Proposition 2.1. *Let $a, b, c \in \mathbb{R}$. Then $T(a,b,c)$ satisfies the relations*

$$f_c(a) f_c(b) f_c(c) T^{\mathrm{sym}}(a,b,c) = I(a,b,c) + J^{\mathrm{sym}}(a,b,c) \tag{2.7}$$

and

$$f_s(a) f_s(b) f_c(c) T^{n\mathrm{sym}}(a,b,c) = I(a,b,c) - J^{n\mathrm{sym}}(a,b,c), \tag{2.8}$$

where f_c and f_s are defined in (2.4).

Proof. Multiply three series of cosine type in (2.2) to obtain that

$$8 f_c(a) f_c(b) f_c(c) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{n^a\, m^b\, r^c} \int_0^1 \cos(2\pi q n)\, \cos(2\pi q m)\, \cos(2\pi q r)\, dq$$

equals

$$\int_0^1 [\bar{\zeta}(a,q) + \bar{\zeta}(a,1-q)][\bar{\zeta}(b,q) + \bar{\zeta}(b,1-q)][\bar{\zeta}(c,q) + \bar{\zeta}(c,1-q)]\, dq.$$

The identities

$$\begin{aligned} 4\cos(u)\cos(v)\cos(w) &= \cos(u+v+w) + \cos(u+v-w) \\ &\quad + \cos(u-v+w) + \cos(u-v-w) \end{aligned}$$

and

$$\int_0^1 \cos(2\pi q j)\, dq = \begin{cases} 0 & \text{if } j \neq 0, \\ 1 & \text{if } j = 0, \end{cases}$$

reduce the left-hand side to $2 f_c(a) f_c(b) f_c(c) T^{\mathrm{sym}}(a,b,c)$. To complete the proof of the first identity, we expand the products of zeta functions to write the integral as a sum of eight different integrals, which can be reduced to the right-hand side of (2.7), by selectively performing the the change of variable $q \to 1-q$ in half of them.

The second identity is obtained by considering the only other non-vanishing triple product integral, namely, that of $\sin(2\pi qn)\sin(2\pi qm)\cos(2\pi qr)$. $\square$

The case of Proposition 2.1 in which the parameters are integers will be our main interest in this paper. When the argument z of the function $\bar{\zeta}(z,q)$ is a positive integer n, this function reduces to a Bernoulli polynomial,

$$\bar{\zeta}(n,q) = -\frac{1}{n} B_n(q). \tag{2.9}$$

In this case, due to the reflection property of the Bernoulli polynomials,

$$B_k(1-q) = (-1)^k B_k(q), \tag{2.10}$$

the function $\bar{\zeta}(n,1-q)$ reduces simply to $\bar{\zeta}(n,q)$ up to a sign:

$$\bar{\zeta}(n,1-q) = (-1)^n \bar{\zeta}(n,q), \tag{2.11}$$

so that the J-type integrals reduce to I-type integrals.

Unfortunately, since the functions $f_c(n)$ and $f_s(n)$ vanish for n odd and n even, respectively, the identities of Proposition 2.1 for integer parameters $(a,b,c) = (n_1,n_2,n_3)$ are trivial except only in two cases:

(1) n_1, n_2, n_3 are all even,
(2) n_1, n_2 are both odd, and n_3 is even.

The first case is of special interest. It appears in [23] as *the reciprocity relation for a class of Tornheim series*.

Corollary 2.2. *Let n_1, n_2, $n_3 \in \mathbb{N}$ be even. Then*

$$T^{\text{sym}}(n_1,n_2,n_3) = (-1)^{(n_1+n_2+n_3)/2} \frac{(2\pi)^{n_1+n_2+n_3}}{2(n_1-1)!(n_2-1)!(n_3-1)!} I(n_1,n_2,n_3). \tag{2.12}$$

Corollary 2.3. *Let $n \in \mathbb{N}$. Then*

$$T(2n,2n,2n) = \tfrac{1}{3}(-1)^n (2\pi)^{6n} \sum_{k=0}^{n} \binom{4n-2k-1}{2n-1} \frac{B_{2k}B_{6n-2k}}{(2k)!(6n-2k)!}. \tag{2.13}$$

Proof. Corollary 2.2 and (2.9) yield

$$T(2n,2n,2n) = (-1)^{n+1} \frac{(2\pi)^{6n}}{6(2n)!^3} \int_0^1 B_{2n}(q)^3\, dq.$$

The value of the integral is given by Carlitz [9]. It can also be obtained directly from the formula for $B_n(q)^3$ given in Appendix A. □

Formula (2.13) agrees with formula (1.20) on account of the relation

$$\zeta(2k) = \frac{(-1)^{k+1}\,(2\pi)^{2k}\,B_{2k}}{2(2k)!}, \tag{2.14}$$

valid for $k \in \mathbb{N}_0$.

We now present an analytic expression for the Tornheim double series, valid for non-integer values of the parameters.

Theorem 2.4. *Let $a, b, c \in \mathbb{R}$ and define*

$$\lambda(z) = \frac{\Gamma(1-z)}{(2\pi)^{1-z}} = \frac{\pi}{(2\pi)^{1-z}\,\Gamma(z)\,\sin \pi z}. \tag{2.15}$$

For $a, b, c \notin \mathbb{N}$ we have

$$\begin{aligned} T(a,b,c) = {} & 4\lambda(a)\lambda(b)\lambda(c)\sin(\pi c/2)\left[\cos\left(\tfrac{\pi}{2}(a-b)\right)[J(c,a,b)+J(c,b,a)]\right. \\ & \left. -\cos\left(\tfrac{\pi}{2}(a+b)\right)[I(a,b,c)+J(a,b,c)]\right]. \end{aligned} \tag{2.16}$$

Proof. The difference of the two expressions stated in Theorem 2.1 yield

$$\begin{aligned} 2f_c(c)T(a,b,c) = {} & \left(\frac{1}{f_c(a)f_c(b)} - \frac{1}{f_s(a)f_s(b)}\right)[I(a,b,c)+J(a,b,c)] \\ & + \left(\frac{1}{f_c(a)f_c(b)} + \frac{1}{f_s(a)f_s(b)}\right)[J(c,a,b)+J(c,b,a)] \end{aligned} \tag{2.17}$$

and the result follows directly from here. The values of $a, b, c \in \mathbb{N}$ are excluded due to the singularity of $\lambda(z)$ for $z \in \mathbb{N}$. □

3. The limiting case

The goal of this section is to analyze the result of Theorem 2.4 as the parameters a, b, c approach positive integer values. The notation $a = n_1 + \varepsilon_1, b = n_2 + \varepsilon_2, c = n_3 + \varepsilon_3$ with $n_j \in \mathbb{N}$ and $\varepsilon_j \to 0$ is used.

We start by writing

$$T(a,b,c) =: \frac{(2\pi)^{a+b+c}}{16\Gamma(a)\Gamma(b)\Gamma(c)}\tilde{T}(a,b,c) \tag{3.1}$$

with, according to (2.4) and (2.17),

$$\tilde{T}(a,b,c) =: \frac{1}{c_c}\left[\frac{I(a,b,c)+J(a,b,c)+J(c,a,b)+J(c,b,a)}{c_a c_b} + \frac{-I(a,b,c)-J(a,b,c)+J(c,a,b)+J(c,b,a)}{s_a s_b}\right], \tag{3.2}$$

where

$$c_a = \cos(\pi a/2) \quad \text{and} \quad s_a = \sin(\pi a/2)$$

and c_b, s_b, c_c, s_c are similarly defined.

Our first task will be to obtain the limit of $c_c\tilde{T}(a,b,c)$ as both a and b approach positive integer values. The functions c_a and s_a have the property that, as the real number a approaches an integer, one of them tends to zero while the other tends to $+1$ or -1, depending on the parity. Explicitly, for n an integer and ε an infinitesimal quantity,

$$c_{n+\varepsilon} = \begin{cases} (-1)^{n/2} + o(\varepsilon), & n \text{ even}, \\ (-1)^{\frac{n+1}{2}}\frac{\pi}{2}\varepsilon + o(\varepsilon), & n \text{ odd}, \end{cases} \tag{3.3}$$

$$s_{n+\varepsilon} = \begin{cases} (-1)^{n/2}\frac{\pi}{2}\varepsilon + o(\varepsilon), & n \text{ even}, \\ (-1)^{\frac{n-1}{2}} + o(\varepsilon), & n \text{ odd}, \end{cases} \tag{3.4}$$

Hence, in order to compute the limit we are seeking, we need to expand the numerators inside square brackets in (3.2) up to order $\varepsilon_1\varepsilon_2$. This is accomplished by replacing both $\bar{\zeta}(a,q)$ and $\bar{\zeta}(b,q)$ by their Taylor series expansions around an integer value of its first argument,

$$\begin{aligned}\bar{\zeta}(n+\varepsilon,q) &= \bar{\zeta}(n,q) + \varepsilon\bar{\zeta}'(n,q) + o(\varepsilon) \\ &= -\frac{1}{n}\left[B_n(q) + \varepsilon A_n(q)\right] + o(\varepsilon),\end{aligned} \tag{3.5}$$

according to (2.9) and the definition of the Bernoulli function

$$A_k(q) =: k\zeta'(1-k,q), \tag{3.6}$$

studied in [12,13]. For instance, using the shorthand notation

$$\langle f(q)\rangle =: \int_0^1 f(q)\,dq \tag{3.7}$$

we have

$$J(c,a,b)|_{\substack{a=n_1+\varepsilon_1\\ b=n_2+\varepsilon_2}} = \frac{1}{n_1n_2}[\langle B_{n_1}(q)B_{n_2}(1-q)\bar{\zeta}(c,q)\rangle$$
$$+\varepsilon_1\langle A_{n_1}(q)B_{n_2}(1-q)\bar{\zeta}(c,q)\rangle$$
$$+\varepsilon_2\langle B_{n_1}(q)A_{n_2}(1-q)\bar{\zeta}(c,q)\rangle$$
$$+\varepsilon_1\varepsilon_2\langle A_{n_1}(q)A_{n_2}(1-q)\bar{\zeta}(c,q)\rangle] + o(\varepsilon_1\varepsilon_2). \tag{3.8}$$

Using the reflection property (2.10) of the Bernoulli polynomials and the invariance of the integration (3.7) under the change of variable $q \to 1-q$, we find the following result for the numerators inside square brackets in (3.2) (the upper sign corresponds to the numerator of the first term and the lower sign corresponds to the numerator of the second term):

$$\pm I(a,b,c) \pm J(a,b,c) + J(c,a,b) + J(c,b,a)$$
$$= \frac{1}{n_1n_2}\{\pm[\pm1+(-1)^{n_1}][\pm1+(-1)^{n_2}]\langle B_{n_1}(q)B_{n_2}(q)\bar{\zeta}(c,q)\rangle$$
$$+\varepsilon_1[\pm1+(-1)^{n_2}]\langle A_{n_1}(q)B_{n_2}(q)\bar{\zeta}_+(c,q)\rangle$$
$$+\varepsilon_2[\pm1+(-1)^{n_1}]\langle B_{n_1}(q)A_{n_2}(q)\bar{\zeta}_+(c,q)\rangle$$
$$+\varepsilon_1\varepsilon_2[\pm\langle A_{n_1}(q)A_{n_2}(q)\bar{\zeta}_+(c,q)\rangle + \langle A_{n_1}(q)A_{n_2}(1-q)\bar{\zeta}_+(c,q)\rangle]\}$$
$$+o(\varepsilon_1\varepsilon_2), \tag{3.9}$$

where

$$\bar{\zeta}_+(z,q) := \bar{\zeta}(z,q) + \bar{\zeta}(z,1-q). \tag{3.10}$$

We now examine the behavior of the Tornheim sum as $\varepsilon_1, \varepsilon_2 \to 0$. The limiting value is obtained from (3.3), (3.4) and (3.9). Observe that $T(a,b,c) = T(b,a,c)$ so only three cases are presented.

Theorem 3.1. *Suppose n_1, $n_2 \in \mathbb{N}$ and $c \in \mathbb{R}\backslash\mathbb{N}$. Then the Tornheim double series $T(n_1,n_2,c)$ are given by*

$$(-1)^{\frac{n_1+n_2}{2}}\frac{(2\pi)^{n_1+n_2}}{4n_1!n_2!}\frac{(2\pi)^c}{\Gamma(c)\cos(\pi c/2)}\left[\int_0^1 B_{n_1}(q)B_{n_2}(q)\bar{\zeta}(c,q)\,dq - \frac{1}{\pi^2}\int_0^1 \right.$$
$$\left.\times A_{n_1}(q)A_{n_2}(q)\bar{\zeta}_+(c,q)\,dq + \frac{1}{\pi^2}\int_0^1 A_{n_1}(q)A_{n_2}(1-q)\bar{\zeta}_+(c,q)\,dq\right] \tag{3.11}$$

for n_1, n_2 even;

$$(-1)^{\frac{n_1+n_2+1}{2}} \frac{(2\pi)^{n_1+n_2}}{4n_1!n_2!} \frac{(2\pi)^c}{\Gamma(c)\cos(\pi c/2)} \left[\frac{1}{\pi} \int_0^1 B_{n_1}(q) A_{n_2}(q) \bar{\zeta}_+(c,q)\,dq \right.$$
$$\left. + \frac{1}{\pi} \int_0^1 A_{n_1}(q) B_{n_2}(q) \bar{\zeta}_+(c,q)\,dq \right] \tag{3.12}$$

for n_1 even and n_2 odd, *and*

$$(-1)^{\frac{n_1+n_2}{2}} \frac{(2\pi)^{n_1+n_2}}{4n_1!n_2!} \frac{(2\pi)^c}{\Gamma(c)\cos(\pi c/2)} \left[\int_0^1 B_{n_1}(q) B_{n_2}(q) \bar{\zeta}(c,q)\,dq - \frac{1}{\pi^2} \int_0^1 \right.$$
$$\left. \times A_{n_1}(q) A_{n_2}(q) \bar{\zeta}_+(c,q)\,dq - \frac{1}{\pi^2} \int_0^1 A_{n_1}(q) A_{n_2}(1-q) \bar{\zeta}_+(c,q)\,dq \right] \tag{3.13}$$

for n_1, n_2 odd.

The final step in the process is to let $c = n_3 + \varepsilon_3$ and let $\varepsilon_3 \to 0$. For n even we simply have

$$\frac{\bar{\zeta}_+(n,q)}{\cos(\pi n/2)} = -\frac{2}{n}(-1)^{n/2} B_n(q),$$

whereas for n odd,

$$\lim_{c \to n} \frac{\bar{\zeta}_+(c,q)}{\cos(\pi c/2)} = -\frac{2}{n}(-1)^{\frac{n+1}{2}} \frac{1}{\pi}[A_n(q) + A_n(1-q)].$$

The value of $T(n_1, n_2, n_3)$ is thus expressed in terms of integrals of triple products of the Bernoulli polynomials $B_k(q) = -k\zeta(1-k,q)$ and the function $A_k(q) = k\zeta'(1-k,q)$.

Define the following families of integrals:

$$R_1(n_1, n_2, n_3) = \int_0^1 B_{n_1}(q) B_{n_2}(q) B_{n_3}(q)\,dq, \tag{3.14}$$

$$R_2(n_1, n_2, n_3) = \frac{1}{\pi} \int_0^1 B_{n_1}(q) B_{n_2}(q) A_{n_3}(q)\,dq, \tag{3.15}$$

$$R_3(n_1, n_2, n_3) = \frac{1}{\pi^2} \int_0^1 A_{n_1}(q) A_{n_2}(q) B_{n_3}(q)\,dq, \tag{3.16}$$

$$R_4(n_1,n_2,n_3) = \frac{1}{\pi^2}\int_0^1 A_{n_1}(q)A_{n_2}(1-q)B_{n_3}(q)\,dq, \tag{3.17}$$

$$R_5(n_1,n_2,n_3) = \frac{1}{\pi^3}\int_0^1 A_{n_1}(q)A_{n_2}(q)A_{n_3}(q)\,dq, \tag{3.18}$$

$$R_6(n_1,n_2,n_3) = \frac{1}{\pi^3}\int_0^1 A_{n_1}(q)A_{n_2}(q)A_{n_3}(1-q)\,dq. \tag{3.19}$$

These integrals are all symmetric under interchange of their first two arguments (n_1 and n_2), except for R_4 which is antisymmetric if n_3 is odd.

Define

$$p(n) = \begin{cases} (-1)^{n/2}, & n \text{ even},\\ (-1)^{\frac{n+1}{2}}, & n \text{ odd}. \end{cases} \tag{3.20}$$

Theorem 3.2. *Let $\alpha = n_1+n_2+n_3$. Then the Tornheim double series $T(n_1,n_2,n_3)$ is given by*

$$T(n_1,n_2,n_3) = p(\alpha)\frac{(2\pi)^\alpha}{2n_1!\,n_2!\,n_3!}T_R(n_1,n_2,n_3) \tag{3.21}$$

where $T_R(n_1,n_2,n_3)$ can be expressed in terms of the functions $R_j : 1\leqslant j\leqslant 6$ as follows:

Case 1: *n_1, n_2 and n_3 are even*:

$$T_R(n_1,n_2,n_3) = -\tfrac{1}{2}R_1(n_1,n_2,n_3) + R_3(n_1,n_2,n_3) - R_4(n_1,n_2,n_3). \tag{3.22}$$

Case 2: *n_1 and n_2 are even*; *n_3 is odd*:

$$\begin{aligned} T_R(n_1,n_2,n_3) = {} & -R_2(n_1,n_2,n_3) + R_5(n_1,n_2,n_3) \\ & + R_6(n_1,n_2,n_3) - R_6(n_3,n_1,n_2) - R_6(n_3,n_2,n_1). \end{aligned} \tag{3.23}$$

Case 3: *n_1 is even*, *n_2 is odd*, *and n_3 is even*:

$$T_R(n_1,n_2,n_3) = -R_2(n_3,n_1,n_2) - R_2(n_3,n_2,n_1). \tag{3.24}$$

Case 4: *n_1 is even*; *n_2 and n_3 are odd*:

$$\begin{aligned} T_R(n_1,n_2,n_3) = {} & R_3(n_3,n_1,n_2) + R_3(n_3,n_2,n_1) \\ & + R_4(n_1,n_3,n_2) + R_4(n_2,n_3,n_1). \end{aligned} \tag{3.25}$$

Case 5: n_1 *and* n_2 *are odd*; n_3 *is even*:

$$T_R(n_1,n_2,n_3) = -\tfrac{1}{2} R_1(n_1,n_2,n_3) + R_3(n_1,n_2,n_3) + R_4(n_1,n_2,n_3). \tag{3.26}$$

Case 6: n_1, n_2 *and* n_3 *are odd*:

$$\begin{aligned} T_R(n_1,n_2,n_3) &= -R_2(n_1,n_2,n_3) + R_5(n_1,n_2,n_3) \\ &\quad + R_6(n_1,n_2,n_3) + R_6(n_3,n_1,n_2) + R_6(n_3,n_2,n_1). \end{aligned} \tag{3.27}$$

The closed form evaluation of the Tornheim sums $T(n_1,n_2,n_3)$ has thus been reduced to that of the integrals R_j. A partial evaluation of these integrals is presented in the next section, in terms of new family of integrals, closely related to R_j.

4. A new family of integrals

The evaluation of the integrals R_j is most conveniently organized in terms of a new family of integrals Q_j, defined in terms of the balanced generalized polygamma function, introduced in [13], by

$$Q_1(n_1,n_2,n_3) = \int_0^1 B_{n_1}(q)B_{n_2}(q)B_{n_3}(q)\,dq, \tag{4.1}$$

$$Q_2(n_1,n_2,n_3) = \int_0^1 B_{n_1}(q)B_{n_2}(q)\psi^{(-n_3)}(q)\,dq, \tag{4.2}$$

$$Q_3(n_1,n_2,n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(q)B_{n_3}(q)\,dq, \tag{4.3}$$

$$Q_4(n_1,n_2,n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(1-q)B_{n_3}(q)\,dq, \tag{4.4}$$

$$Q_5(n_1,n_2,n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(q)\psi^{(-n_3)}(q)\,dq, \tag{4.5}$$

$$Q_6(n_1,n_2,n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(q)\psi^{(-n_3)}(1-q)\,dq. \tag{4.6}$$

These integrals satisfy the same symmetry properties as their R-analogs. In addition, the Q-integrals satisfy homogeneous recursion relations, which allow their evaluation in terms of a few basic ones. The relation among the families R_j and Q_j using the identity

$$A_m(q) = m!\psi^{(-m)}(q) + h_m B_m(q) \tag{4.7}$$

is given in Appendix C.

In this section we present recurrences for the integrals Q_j. The initial conditions require a variety of definite integrals listed below:

- The integrals

$$N_{m,n} = \int_0^1 \psi^{(-m)}(q) B_n(q)\, dq, \tag{4.8}$$

$$M_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(q)\, dq, \tag{4.9}$$

  and

$$M^*_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(1-q)\, dq, \tag{4.10}$$

  which will be evaluated in Section 5.
- The families of integrals

$$K_{m,n} = \int_0^1 \psi^{(-m)}(q) B_n(q)\, \ln\Gamma(q)\, dq, \tag{4.11}$$

$$K^*_{m,n} = \int_0^1 \psi^{(-m)}(1-q) B_n(q)\, \ln\Gamma(q)\, dq, \tag{4.12}$$

$$Z_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(q) \ln\Gamma(q)\, dq, \tag{4.13}$$

  and

$$Z^*_{m,n} = \int_0^1 \psi^{(-m)}(q)\psi^{(-n)}(1-q) \ln\Gamma(q)\, dq. \tag{4.14}$$

  The closed form evaluation of these functions is left as an open question.

The integral Q_1. The explicit value of $Q_1(n_1, n_2, n_3)$ was given by Carlitz [9]:

$$Q_1(n_1, n_2, n_3) = (-1)^{n_3+1} n_3! \sum_{k=0}^{\lfloor (n_1+n_2-1)/2 \rfloor} \left[n_1 \binom{n_2}{2k} + n_2 \binom{n_1}{2k} \right] \times \frac{(n_1+n_2-2k-1)!}{(n_1+n_2+n_3-2k)!} B_{2k} B_{n_1+n_2+n_3-2k}. \tag{4.15}$$

The integral Q_2. This integral is obtained directly from the formula given in [13],

$$\int_0^1 \zeta(z+1, q)\psi(z', q)\, dq = 2(2\pi)^{z+z'} \Gamma(-z) \left\{ \frac{\pi}{2} \zeta(-z-z') \sin \frac{\pi}{2}(z-z') + [(\gamma + \ln 2\pi) \times \zeta(-z-z') - \zeta'(-z-z')] \cos \frac{\pi}{2}(z-z') \right\}, \tag{4.16}$$

valid for $\operatorname{Re} z, \operatorname{Re} z' < 0$ and $\operatorname{Re}(z+z') < -1$. The evaluation at $z = -m$ and $z' = -n$, with $m, n \in \mathbb{N}$ gives

$$\int_0^1 B_m(q)\psi^{(-n)}(q)\, dq = \frac{2m!}{(2\pi)^{m+n}} \left\{ \frac{\pi}{2} \zeta(m+n) \sin \frac{\pi}{2}(m-n) - [(\gamma + \ln 2\pi) \times \zeta(m+n) - \zeta'(m+n)] \cos \frac{\pi}{2}(m-n) \right\}. \tag{4.17}$$

The evaluation of $Q_2(n_1, n_2, n_3)$ follows from (4.17) and the representation (A.3) for the product of two Bernoulli polynomials:

Theorem 4.1. *Let* $n_1, n_2, n_3 \in \mathbb{N}$ *and let* $\alpha = n_1 + n_2 + n_3$. *Then*

$$Q_2(n_1, n_2, n_3) = 2(-1)^{n_3} \sum_{k=0}^{k(n_1,n_2)} \left[n_1 \binom{n_2}{2k} + n_2 \binom{n_1}{2k} \right] \frac{(n_1+n_2-2k-1)!}{(2\pi)^{\alpha-2k}} \times (-1)^k B_{2k} \left\{ \frac{\pi}{2} \sin \frac{\pi\alpha}{2} \zeta(\alpha-2k) - \cos \frac{\pi\alpha}{2} [(\gamma + \ln 2\pi) \times \zeta(\alpha-2k) - \zeta'(\alpha-2k)] \right\}, \tag{4.18}$$

where $k(n_1, n_2) = \operatorname{Max}\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$. *The constant term in* (A.3) *gives no contribution on account that the function* $\psi^{(-n)}(q)$ *is balanced for* $n \in \mathbb{N}$. *The integrals* Q_2 *are only needed in the case that* $\alpha = n_1 + n_2 + n_3$ *is odd, equal to* $2N + 1$, *say. In* *this case*, $\sin(\pi\alpha/2) = (-1)^N$ *and* $\cos(\pi\alpha/2) = 0$, *so that*

$$Q_2(n_1, n_2, n_3) = \pi(-1)^{N+n_3} \times \sum_{k=0}^{k(n_1,n_2)} \left[n_1 \binom{n_2}{2k} + n_2 \binom{n_1}{2k} \right] \times \frac{(n_1+n_2-2k-1)!}{(2\pi)^{\alpha-2k}} (-1)^k B_{2k} \zeta(\alpha - 2k). \tag{4.19}$$

Proof. The details are elementary. □

The integral Q_3. We now produce a recurrence for

$$Q_3(n_1, n_2, n_3) = \int_0^1 \psi^{(-n_1)}(q) \psi^{(-n_2)}(q) B_{n_3}(q)\, dq.$$

The basic tools are the relations

$$\frac{d}{dq} \psi^{(-m)}(q) = \psi^{(-m+1)}(q), \tag{4.20}$$

and

$$\frac{d}{dq} B_m(q) = m B_{m-1}(q), \tag{4.21}$$

valid for $m \in \mathbb{N}_0$, and the fact that both the negapolygamma functions and the Bernoulli polynomials are balanced for the range of indices we wish to consider, i.e., $\psi^{(-m)}(1) = \psi^{(-m)}(0)$, for $m \geqslant 2$, and $B_m(1) = B_m(0)$, for all m.

Theorem 4.2. *Let* $n_1, n_2, n_3 \in \mathbb{N}$ *with* $n_1, n_2 > 1$. *Then*

$$(n_3 + 1) Q_3(n_1, n_2, n_3) = -Q_3(n_1 - 1, n_2, n_3 + 1) - Q_3(n_1, n_2 - 1, n_3 + 1). \tag{4.22}$$

Proof. Start with

$$(n_3 + 1) Q_3(n_1, n_2, n_3) = \int_0^1 \psi^{(-n_1)}(q)\, \psi^{(-n_2)}(q) \frac{d}{dq} B_{n_3+1}(q)\, dq,$$

integrate by parts and observe that there is no contribution from the boundary. □

The recurrence shows that the value of $Q_3(n_1, n_2, n_3)$ can be obtained from the values of

$$Q_3(1, m, n) = K_{m,n} + \zeta'(0) N_{m,n}, \tag{4.23}$$

in view of $\psi^{(-1)}(q) = \ln\Gamma(q) + \zeta'(0)$, the symmetry of the integral Q_3 under interchange of its first two arguments, and the definitions (4.11) and (4.8) of the integrals $K_{m,n}$, $N_{m,n}$.

The integral Q_4. Similarly, for n_1, $n_2 > 1$, we have that

$$Q_4(n_1, n_2, n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(1-q)B_{n_3}(q)\,dq$$

satisfies the recurrence

$$(n_3+1)Q_4(n_1, n_2, n_3) = -Q_4(n_1-1, n_2, n_3+1) + Q_4(n_1, n_2-1, n_3+1), \tag{4.24}$$

so that it can be obtained from

$$Q_4(1, m, n) = K^*_{m,n} + \zeta'(0)(-1)^n N_{m,n}, \tag{4.25}$$

where $K^*_{m,n}$ is defined in (4.12).

The integrals Q_5 and Q_6. Similar arguments show that for $n_1, n_2 > 1$ we have

$$Q_5(n_1, n_2, n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(q)\psi^{(-n_3)}(q)\,dq,$$

$$Q_6(n_1, n_2, n_3) = \int_0^1 \psi^{(-n_1)}(q)\psi^{(-n_2)}(q)\psi^{(-n_3)}(1-q)\,dq,$$

satisfy the recurrences

$$Q_5(n_1, n_2, n_3) = -Q_5(n_1-1, n_2, n_3+1) - Q_5(n_1, n_2-1, n_3+1) \tag{4.26}$$

and

$$Q_6(n_1, n_2, n_3) = Q_6(n_1-1, n_2, n_3+1) + Q_6(n_1, n_2-1, n_3+1). \tag{4.27}$$

The initial conditions are

$$Q_5(1, m, n) = Z_{m,n} + \zeta'(0)M_{m,n}, \tag{4.28}$$

$$Q_6(1, m, n) = Z^*_{m,n} + \zeta'(0)M^*_{m,n}, \tag{4.29}$$

where $Z_{m,n}$, $Z^*_{m,n}$, $M_{m,n}$ and $M^*_{m,n}$ are defined in (4.13), (4.14), (4.9) and (4.10), respectively.

5. Evaluation of integrals

5.1. The product of a Bernoulli polynomial and a balanced negapolygamma

The explicit evaluation of the integral $N_{m,n}$ defined by (4.8) was obtained in (4.17) as a byproduct of the evaluation of Q_2:

$$N_{m,n} = \frac{2n!}{(2\pi)^{m+n}} \left\{ \frac{\pi}{2}\zeta(m+n)\sin\frac{\pi}{2}(n-m) - [(\gamma+\ln 2\pi)\zeta(m+n) - \zeta'(m+n)]\cos\frac{\pi}{2}(n-m) \right\}. \tag{5.1}$$

5.2. The product of two balanced negapolygammas

The integral of the product of two balanced negapolygamma functions has been given in [12]:

Let $k,\ k' \in \mathbb{N}$, $k_+ = k + k'$ and $k_- = k - k'$. Then

$$M_{k,k'} = \int_0^1 \psi^{(-k)}(q)\psi^{(-k')}(q)\,dq = \frac{2}{(2\pi)^{k_+}}\cos\left(\frac{\pi}{2}k_-\right)[A_+\zeta(k_+) - 2A\zeta'(k_+) + \zeta''(k_+)], \tag{5.2}$$

where $A = \gamma + \ln 2\pi$ and $A_\pm = A^2 \pm \pi^2/4$.

Similarly, one finds

$$M^*_{k,k'} = \int_0^1 \psi^{(-k)}(q)\psi^{(-k')}(1-q)\,dq = \frac{2}{(2\pi)^{k_+}}\left\{\cos\left(\frac{\pi}{2}k_+\right)[A_-\zeta(k_+) - 2A\zeta'(k_+) + \zeta''(k_+)] + \pi\sin\left(\frac{\pi}{2}k_+\right)[A\zeta(k_+) - \zeta'(k_+)]\right\}. \tag{5.3}$$

We have been unable to obtain closed form results for the integrals $K_{m,n}$ $K^*_{m,n}$, $Z_{m,n}$ and $Z^*_{m,n}$, which involve the kernel $\ln\Gamma(q)$ and one or two negapolygamma functions.

6. Some examples

In this section we describe some explicit evaluations of Tornheim sums. It is convenient to introduce the function

$$U_{m,n} = \int_0^1 \psi^{(-m)}(q)B_n(q)\ln\sin(\pi q)\,dq. \tag{6.1}$$

The identity

$$\ln\Gamma(q) + \ln\Gamma(1-q) = \ln\pi - \ln\sin(\pi q) \tag{6.2}$$

yields the relation

$$K_{m,n} + (-1)^n K_{m,n}^{*} = \ln\pi N_{m,n} - U_{m,n}. \tag{6.3}$$

Example 6.1. We consider the value of $T(1,1,2)$. This corresponds to Case 5 of Theorem 3.2. It yields

$$T(1,1,2) = 4\pi^4\left[-\frac{1}{2}R_1(1,1,2) + R_3(1,1,2) + R_4(1,1,2)\right], \tag{6.4}$$

and in terms of the Q_j-family,

$$T(1,1,2) = 4\pi^4\left[-\frac{1}{2}Q_1(1,1,2) + \frac{1}{\pi^2}Q_3(1,1,2) + \frac{1}{\pi^2}Q_4(1,1,2)\right]. \tag{6.5}$$

The Q_j-integrals are given by

$$Q_1(1,1,2) = \frac{1}{180},$$

$$Q_3(1,1,2) = K_{1,2} + \zeta'(0)N_{1,2},$$

$$Q_4(1,1,2) = K_{1,2}^{*} + \zeta'(0)N_{1,2}$$

and using the values $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ and $N_{1,2} = \zeta(3)/4\pi^2$ we obtain

$$T(1,1,2) = 4\pi^2\left(K_{1,2} + K_{1,2}^{*}\right) - \zeta(3)\ln(2\pi) - \frac{1}{90}\pi^4.$$

In terms of the U-function this can be written as

$$T(1,1,2) = -4\pi^2 U_{1,2} - \zeta(3)\ln 2 - \frac{1}{90}\pi^4. \tag{6.6}$$

The identities (1.16) and (1.17) yield $T(1,1,2) = 2T(0,1,3)$, and the method of Huard et al. [17] yields the values of $T(n_1,n_2,n_3)$ for $N = n_1+n_2+n_3$ odd and also for $N = 4$ and 6. For instance $T(0,1,3) = \frac{1}{4}\zeta(4)$. This yields an evaluation of an integral of type $U_{m,n}$: the value $T(1,1,2) = \zeta(4)/2$, the identity $\psi^{(-1)}(q) = \ln\Gamma(q) + \zeta'(0)$ and

$$\int_0^1 B_2(q)\,\ln(\sin \pi q)\,dq = -\frac{\zeta(3)}{2\pi^2}$$

given in Example 5.2 of [11] produce

$$\int_0^1 B_2(q)\ln\Gamma(q)\,\ln(\sin\pi q)\,dq = -\left(\frac{\pi^2}{240} + \frac{\ln(4\pi)\,\zeta(3)}{4\pi^2}\right). \tag{6.7}$$

It follows that

$$U_{1,2} = -\frac{\pi^2}{240} - \frac{\ln 2\,\zeta(3)}{4\pi^2}. \tag{6.8}$$

Example 6.2. The explicit expression for $R_2(n_1,n_2,n_3)$ that can be obtained from (C.2) through (4.15) and (4.18) permits the evaluation of $T(n_1,n_2,n_3)$ in the case n_1, n_3 even and n_2 odd. For example

$$T(2,1,2) = \frac{\pi^2}{6}\,\zeta(3) - \frac{3}{2}\zeta(5),$$

$$T(2,3,2) = -\frac{\pi^2}{6}\zeta(5) + 2\zeta(7),$$

and

$$T(4,3,2) = \frac{\pi^4}{90}\,\zeta(5) + \frac{\pi^2}{6}\,\zeta(7) - \frac{5}{2}\zeta(9).$$

Example 6.3. Define $w = a+b+c$ to be the weight of the sum $T(a,b,c)$. The results of the procedure described above for sums of small weight are given below.
Weight 3:

$$T(1,1,1) = 4Z_{1,1} + 12Z^*_{1,1} - \zeta(3) + \ln 2\pi\left(\frac{A^2}{3} - \frac{\pi^2}{6} - \frac{4A\zeta'(2)}{\pi^2} + \frac{2\zeta''(2)}{\pi^2}\right),$$

Weight 4:

$$\begin{aligned} T(1,1,2) &= -4\pi^2 U_{1,2} - \frac{\pi^4}{90} - \zeta(3)\ln 2, \\ T(1,2,1) &= -4\pi^2(U_{1,2} + 2U_{2,1}), \end{aligned}$$

Weight 5:

$$\begin{aligned} T(1,1,3) &= -8\pi^2(K^*_{1,3} + 2Z_{1,3} + 2Z^*_{1,3} + 4Z^*_{3,1}) - \zeta(5) \\ &\quad + \ln 2\pi\left(\frac{\pi^2 A^2}{45} - \frac{\pi^2 A}{30} - \frac{\pi^4}{90} - \frac{4\zeta'(4)\,A}{\pi^2} + \frac{3\zeta'(4)}{\pi^2} + \frac{2\zeta''(4)}{\pi^2}\right), \end{aligned}$$

$$T(1,2,2) = \frac{\pi^2}{6}\zeta(3) - \frac{3}{2}\zeta(5),$$

$$\begin{aligned} T(1,3,1) &= -8\pi^2(K^*_{1,3} + 2Z_{3,1} + 2Z^*_{1,3} + 4Z^*_{3,1}) + \frac{\pi^2}{6}\zeta(3) - 2\zeta(5) \\ &\quad + \ln 2\pi\left(\frac{\pi^2 A^2}{45} - \frac{\pi^2 A}{30} - \frac{\pi^4}{90} - \frac{4\zeta'(4)A}{\pi^2} + \frac{3\zeta'(4)}{\pi^2} + \frac{2\zeta''(4)}{\pi^2}\right), \end{aligned}$$

$$\begin{aligned} T(2,2,1) &= 32\pi^2 Z_{2,2} + \frac{\pi^2}{3}\zeta(3) - 3\zeta(5) \\ &\quad + \ln 2\pi\left(-\frac{\pi^2 A^2}{45} - \frac{\pi^4}{180} + \frac{4\zeta'(4)A}{\pi^2} - \frac{2\zeta''(4)}{\pi^2}\right). \end{aligned}$$

We have produced some partial results in the evaluation of the integrals $K_{m,n}$, $K^*_{m,n}$ and $Z_{m,n}$, $Z^*_{m,n}$. These suggest that the value of the Tornheim sums can be expressed in terms of a small number of definite integrals. For instance, for $m \geqslant 3$ odd, we have the relation

$$\begin{aligned} K_{m,n} &= -mK_{m-1,n+1} + \ln\sqrt{2\pi}(N_{m,n} + mN_{m-1,n+1}) \\ &\quad - \int_0^1 B_m(q)\psi(q)\psi^{(-n-1)}(q)\,dq, \end{aligned}$$

which reduces the value of $K_{m,n}$ to that of $K_{1,m+n-1}$ plus the moments of the product $\psi(q)\psi^{(-j)}(q)$. Details will be presented elsewhere.

Acknowledgments

The first author would like to thank the Department of Mathematics at Tulane University for its hospitality, and the partial support of grant MECESUP FSM0204. The second author acknowledges the partial support of NSF-DMS 0070567.

Appendix A. The Bernoulli polynomials

The *Bernoulli polynomials* $B_n(q)$ defined by the generating function

$$\frac{xe^{xq}}{e^x-1} = \sum_{n=0}^{\infty} B_n(q)\frac{x^n}{n!}. \tag{A.1}$$

The *Bernoulli numbers* $B_n = B_n(0)$ satisfy

$$B_n(q) = \sum_{k=0}^{n} \binom{n}{k} B_k q^{n-k}. \tag{A.2}$$

For $n \geqslant 1$ we have the differential recursion $B_n'(q) = nB_{n-1}(q)$ and the symmetry rule $B_n(1-q) = (-1)^n B_n(q)$. In particular, $B_n(1) = B_n(0)$ for $n > 1$.

The Bernoulli polynomials $\{B_0(q), B_1(q), \ldots, B_n(q)\}$ form a basis for the space of polynomials of degree at most n. Thus the product $B_{n_1}(q)B_{n_2}(q)$ is a linear combination of $B_j(q)$ for $j = 0, \ldots, n_1+n_2$. It is a remarkable fact that this combination has the explicit form

$$\begin{aligned} B_{n_1}(q)B_{n_2}(q) = & \sum_{k=0}^{k(n_1,n_2)} \left[n_1\binom{n_2}{2k} + n_2\binom{n_1}{2k} \right] \frac{B_{2k}}{n_1+n_2-2k} B_{n_1+n_2-2k}(q) \\ & +(-1)^{n_1+1}\frac{n_1!\,n_2!}{(n_1+n_2)!} B_{n_1+n_2}, \end{aligned} \tag{A.3}$$

where $k(n_1,n_2) = \operatorname{Max}\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$. In terms of rescaled Bernoulli polynomials and numbers, defined by

$$\tilde{B}_n(q) = \frac{B_n(q)}{n!}, \qquad \tilde{B}_n = \tilde{B}_n(0) = \frac{B_n}{n!}, \tag{A.4}$$

relation (A.3) has the simpler form

$$\begin{aligned} \tilde{B}_{n_1}(q)\tilde{B}_{n_2}(q) = & \sum_{k=0}^{k(n_1,n_2)} \left[\binom{n_1+n_2-2k-1}{n_1-1} + \binom{n_1+n_2-2k-1}{n_2-1} \right] \\ & \times \tilde{B}_{2k}\tilde{B}_{n_1+n_2-2k}(q) + (-1)^{n_1+1}\tilde{B}_{n_1+n_2}. \end{aligned} \tag{A.5}$$

In theory, (A.3) yields expressions for a product of any number of Bernoulli polynomials. For example,

$$B_n^2(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k}B_{2n-2k}(q) + (-1)^{n+1}\frac{B_{2n}}{\binom{2n}{n}}, \tag{A.6}$$

or

$$\tilde{B}_n^2(q) = 2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \tilde{B}_{2k}\tilde{B}_{2n-2k}(q) + (-1)^{n+1}\tilde{B}_{2n}, \tag{A.7}$$

and

$$B_n^3(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n\binom{n}{2k}}{n-k} B_{2k} \sum_{j=0}^{n-k} \left[n\binom{2n-2k}{2j} + 2(n-k)\binom{n}{2j} \right] \frac{B_{2j}\, B_{3n-2k-2j}(q)}{3n-2k-2j}$$
$$+(-1)^{n+1}\left[\frac{B_{2n}}{\binom{2n}{n}} B_n(q) + 2n!^3 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \frac{B_{2k}B_{3n-2k}}{(2k)!(3n-2k)!} \right]. \tag{A.8}$$

or

$$\tilde{B}_n^3(q) = 2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \tilde{B}_{2k} \sum_{j=0}^{n-k} \left[\binom{3n-2k-2j-1}{n-1} \right.$$
$$\left. + \binom{3n-2k-2j-1}{2n-2k-1} \right] \tilde{B}_{2j}\, \tilde{B}_{3n-2k-2j}(q)$$
$$+(-1)^{n+1}\left[\tilde{B}_{2n}\tilde{B}_n(q) + 2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2n-2k-1}{n-1} \tilde{B}_{2k}\tilde{B}_{3n-2k} \right]. \tag{A.9}$$

Integrating the relation (A.1) yields

$$\int_0^1 B_n(q)\, dq = 0, \quad \text{for } n \geqslant 1. \tag{A.10}$$

Apostol [3] gives a direct proof of

$$\int_0^1 B_{n_1}(q) B_{n_2}(q)\, dq = (-1)^{n_1+1} \frac{n_1!\, n_2!}{(n_1+n_2)!} B_{n_1+n_2}, \tag{A.11}$$

for $n_1,\ n_2 \in \mathbb{N}$.

The Bernoulli polynomials appear also as special values of the Hurwitz zeta function

$$\zeta(1-k, q) = -\frac{1}{k} B_k(q). \tag{A.12}$$

Appendix B. The generalized polygamma function $\psi(z, q)$

The *polygamma function* is defined by

$$\psi^{(m)}(q) = \frac{d^m}{dq^m}\psi(q), \quad m \in \mathbb{N}, \tag{B.1}$$

where

$$\psi(q) = \psi^{(0)}(q) = \frac{d}{dq}\ln\Gamma(q) \tag{B.2}$$

is the *digamma* function.

The function $\psi^{(m)}$ is analytic in the complex q-plane, except for poles (of order $m+1$) at all non-positive integers. Extensions of this function for m a negative integer have been defined by several authors [1,12,15]. These are the *negapolygamma* functions. For example, Gosper [15] defined

$$\begin{aligned}\psi_{-1}(q) &= \ln\Gamma(q),\\ \psi_{-k}(q) &= \int_0^q \psi_{-k+1}(t)\,dt, \quad k \geqslant 2,\end{aligned} \tag{B.3}$$

which were later reconsidered by Adamchik [1] in the form

$$\psi_{-k}(q) = \frac{1}{(k-2)!}\int_0^q (q-t)^{k-2}\ln\Gamma(t)\,dt, \quad k \geqslant 2. \tag{B.4}$$

These extensions can be expressed in terms of the derivative (with respect to its first argument) of the Hurwitz zeta function at the negative integers [1,15]. The definition (B.3) can be modified by introducing arbitrary constants of integration at every step. This yields different extensions differing by polynomials:

$$\psi_a^{(-m)}(q) - \psi_b^{(-m)}(q) = p_{m-1}(q),$$

satisfying

$$p_n(q) = \frac{d}{dq}p_{n+1}(q).$$

A new extension of $\psi^{(m)}(q)$ has been introduced in [12], in connection with integrals involving the polygamma and the loggamma functions. These are the *balanced negapolygamma* functions, defined for $m \in \mathbb{N}$ by

$$\psi^{(-m)}(q) := \frac{1}{m!}\left[A_m(q) - H_{m-1}B_m(q)\right]. \tag{B.5}$$

Here $H_r = 1 + 1/2 + \ldots + 1/r$ is the harmonic number ($H_0 = 0$), $B_m(q)$ is the mth Bernoulli polynomial, and

$$A_m(q) = m\,\zeta'(1 - m, q). \tag{B.6}$$

A function $f(q)$ is defined on $(0, 1)$ is called *balanced* if its integral over $(0, 1)$ vanishes and $f(0) = f(1)$. In [12] we have shown that

$$\frac{d}{dq}\psi^{(-m)}(q) = \psi^{(-m+1)}(q), \quad m \in \mathbb{N}. \tag{B.7}$$

The function $\psi(z, q)$ defined in (1.15) represents an extension of these polygamma families to $q \in \mathbb{C}$. Its main properties are presented in the next theorem. The details appear in [13].

Theorem B.1. *The generalized polygamma function $\psi(z, q)$ satisfies*:

- *For fixed $q \in \mathbb{C}$, the function $\psi(z, q)$ is an entire function of z.*
- *For $m \in \mathbb{Z}$: $\psi(m, q) = \psi^{(m)}(q)$.*
- *It satisfies*

$$\frac{\partial}{\partial q}\psi(z, q) = \psi(z + 1, q). \tag{B.8}$$

Appendix C. The relation between Q_j and R_j

Using the relation (4.7) we can express the integrals R_j, defined by (3.14)–(3.19) in terms of Q_j, defined by (4.1)–(4.6). Recall that, for $n > 1$ we have $h_n = 1 + 1/2 + \cdots + 1/(n-1)$ and $h_1 = 0$.

$$R_1(n_1, n_2, n_3) = Q_1(n_1, n_2, n_3), \tag{C.1}$$

$$\pi R_2(n_1, n_2, n_3) = n_3!\,Q_2(n_1, n_2, n_3) + h_{n_3} Q_1(n_1, n_2, n_3), \tag{C.2}$$

$$\begin{aligned}\pi^2 R_3(n_1, n_2, n_3) &= n_1!n_2!\,Q_3(n_1, n_2, n_3) + n_2!h_{n_1} Q_2(n_1, n_3, n_2) \\ &\quad + n_1!h_{n_2} Q_2(n_2, n_3, n_1) + h_{n_1} h_{n_2} Q_1(n_1, n_2, n_3),\end{aligned} \tag{C.3}$$

$$\begin{aligned}\pi^2 R_4(n_1, n_2, n_3) &= n_1!n_2!\,Q_4(n_1, n_2, n_3) + (-1)^{n_2} n_1!h_{n_2} Q_2(n_2, n_3, n_1) \\ &\quad + (-1)^{n_1+n_3} n_2!h_{n_1} Q_2(n_1, n_3, n_2) \\ &\quad + (-1)^{n_2} h_{n_1} h_{n_2} Q_1(n_1, n_2, n_3),\end{aligned} \tag{C.4}$$

$$\begin{aligned}\pi^3 R_5(n_1,n_2,n_3) = {} & n_1!n_2!n_3!Q_5(n_1,n_2,n_3) + n_1!n_2!h_{n_3}Q_3(n_1,n_2,n_3) \\ & + n_1!n_3!h_{n_2}Q_3(n_1,n_3,n_2) + n_2!n_3!h_{n_1}Q_3(n_2,n_3,n_1) \\ & + n_1!h_{n_2}h_{n_3}Q_2(n_2,n_3,n_1) + n_2!h_{n_1}h_{n_3}Q_2(n_1,n_3,n_2) \\ & + n_3!h_{n_1}h_{n_2}Q_2(n_1,n_2,n_3) + h_{n_1}h_{n_2}h_{n_3}Q_1(n_1,n_2,n_3), \quad \text{(C.5)}\end{aligned}$$

$$\begin{aligned}\pi^3 R_6(n_1,n_2,n_3) = {} & n_1!n_2!n_3!Q_6(n_1,n_2,n_3) + (-1)^{n_3}n_1!n_2!h_{n_3}Q_3(n_1,n_2,n_3) \\ & + n_1!n_3!h_{n_2}Q_4(n_1,n_3,n_2) + n_2!n_3!h_{n_1}Q_4(n_2,n_3,n_1) \\ & + (-1)^{n_3}n_1!h_{n_2}h_{n_3}Q_2(n_2,n_3,n_1) \\ & + (-1)^{n_3}n_2!h_{n_1}h_{n_3}Q_2(n_1,n_3,n_2) \\ & + (-1)^{n_1+n_2}n_3!h_{n_1}h_{n_2}Q_2(n_1,n_2,n_3) \\ & + (-1)^{n_3}h_{n_1}h_{n_2}h_{n_3}Q_1(n_1,n_2,n_3). \quad \text{(C.6)}\end{aligned}$$

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