Wallis-Ramanujan-Schur-Feynman

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1 Wallis' infinite product for π

Among the earliest analytic expressions for π one finds two infinite products: the first one given by Vieta [21] in 1593

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

and the second by Wallis [22] in 1655

$$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots. \tag{1.1}$$

In this journal, T. Osler [15] has presented the remarkable formula

$$\frac{2}{\pi} = \prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}.$$

This equation becomes Wallis' product when $p = 0$ and Vieta's formula as $p \to \infty$. It is surprising that such a connection between the two products was not discovered earlier.

The collection [1] contains both original papers of Vieta and Wallis as well as other fundamental papers in the history of π. Indeed, there are many good historical sources on π. The text by P. Eymard and J. P. Lafon [6] is an excellent place to start.

Wallis' formula (1.1) is equivalent to

$$W_n := \prod_{k=1}^{n}\frac{(2k)\cdot(2k)}{(2k-1)\cdot(2k+1)} = \frac{2^{4n}}{\binom{2n}{n}\binom{2n+1}{n}(n+1)} \to \frac{\pi}{2} \tag{1.2}$$

as $n \to \infty$. This may be established using Stirling's approximation

$$m! \sim \sqrt{2\pi m}\left(\frac{m}{e}\right)^m.$$

Alternatively, there are many elementary proofs of (1.2) in the literature. Among them, [23] and [12] have recently appeared in this journal.

Section 3 presents a proof of (1.2) based on the evaluation of the rational integral

$$G_n := \frac{2}{\pi} \int_0^\infty \frac{dx}{(x^2+1)^n}. \tag{1.3}$$

This integral is discussed in the next section. The motivation to generalize (1.3) has produced interesting links to symmetric functions from Combinatorics and to one-loop Feynman diagrams from Particle Physics. The goal of this work is to present these connections.

2 A rational integral and its trigonometric version

The method of partial fractions reduces the integration of a rational function to an algebraic problem: the factorization of its denominator. The integral (1.3) correspondes to the presence of purely imaginary poles. See [3] for a treatment of these ideas.

A recurrence for G_n is obtained by writing $1 = (x^2+1) - x^2$ for the numerator of (1.3) and integrating by parts. The result is

$$G_{n+1} = \frac{2n-1}{2n} G_n. \tag{2.1}$$

Since $G_1 = 1$ it follows that

$$G_{n+1} = \frac{1}{2^{2n}} \binom{2n}{n}. \tag{2.2}$$

The choice of a new variable is one of the fundamental tools in the evaluation of definite integrals. The new variable, if carefully chosen, usually simplifies the problem or opens up unsuspected possibilities. Trigonometric changes of variables are considered *elementary* because these functions appear early in the scientific training. Unfortunately, this hides the fact that this change of variables introduces a *transcendental function* with a multivalued inverse. One has to proceed with care.

The change of variables $x = \tan\theta$ in the definition (1.3) of G_n gives

$$G_{n+1} = \frac{2}{\pi} \int_0^{\pi/2} (\cos\theta)^{2n} \, d\theta.$$

In this context, the recurrence (2.1) is obtained by writing

$$(\cos\theta)^{2n} = (\cos\theta)^{2n-2} - \frac{\sin\theta}{2n-1} \frac{d}{d\theta} (\cos\theta)^{2n-1}$$

and then integrating by parts. Yet another recurrence for G_n is obtained by a double-angle substitution in

$$G_{n+1} = \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^n d\theta,$$

and a binomial expansion (observe that the odd powers of cosine integrate to zero). It follows that

$$G_{n+1} = 2^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} G_{k+1}.$$

Thus, proving (2.2) is equivalent to the finite sum identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} 2^{-2k} \binom{n}{2k} \binom{2k}{k} = 2^{-n} \binom{2n}{n}. \tag{2.3}$$

There are many possible ways to prove this identity. For instance, it is a perfect candidate for the truly 21st century WZ-method [16] that provides automatic proofs; or, as pointed out by M. Hirschhorn in [10], it is a disguised form of the Chu-Vandermonde identity

$$\sum_{k \geq 0} \binom{x}{k} \binom{y}{k} = \binom{x+y}{x} \tag{2.4}$$

(which was discovered first in 1303 by Zhu Shijie). Namely, upon employing Legendre's duplication formula for the gamma function

$$\Gamma(\tfrac{1}{2})\Gamma(2z+1) = 2^{2z}\Gamma(z+1)\Gamma(z+\tfrac{1}{2})$$

the identity (2.3) rewrites as

$$\sum_{k \geq 0} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}-\frac{1}{2}}{k} = \binom{n-\frac{1}{2}}{\frac{n}{2}-\frac{1}{2}}.$$

This is a special case of (2.4). Another, particularly nice and direct, proof of (2.3), as kindly pointed out by one of the referees, is obtained from looking at the constant coefficient of

$$\left(\frac{x}{2} + \frac{x^{-1}}{2} + 1\right)^n = 2^{-n} \left(x^{1/2} + x^{-1/2}\right)^{2n}.$$

Remark 2.1. The idea of double-angle reduction lies at the heart of the *rational Landen transformations.* These are polynomial maps on the coefficient of the integral of a rational function that preserve its value. See [13] for a survey on Landen transformations and open questions.

3 A squeezing method

In this section we employ the explicit expression for G_n, given in (2.2), to establish Wallis' formula (1.1). This approach is also contained in Stewart's calculus text book [19] in the form of several guided exercises (45, 46, and 68 of Section 7.1). The proof is based on analyzing the integrals

$$I_n := \int_0^{\pi/2} (\sin x)^n \, dx.$$

The formula

$$I_{2n} = \int_0^{\pi/2} (\sin x)^{2n} \, dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

follows from (2.2) by symmetry. Its companion integral

$$I_{2n+1} = \int_0^{\pi/2} (\sin x)^{2n+1} \, dx = \frac{(2n)!!}{(2n+1)!!}$$

is of the same flavor. Here $n!! = n(n-2)(n-4)\cdots\{1 \text{ or } 2\}$ denotes the double factorial. The ratio of these two integrals gives

$$W_n I_{2n}/I_{2n+1} = \frac{\pi}{2}$$

where W_n is defined by (1.2). The convergence of W_n to $\pi/2$ now follows from the inequalities $1 \leq I_{2n}/I_{2n+1} \leq 1 + 1/(2n)$. This in turn is equivalent to

$$2n \int_0^{\pi/2} (\sin x)^{2n} \, dx \leq (2n+1) \int_0^{\pi/2} (\sin x)^{2n+1} \, dx.$$

The proof that $I_{2n}/I_{2n+1} \leq 1 + 1/(2n)$ follows directly from the bound $I_{2n} \leq I_{2n-1}$ and the recurrence $(2n+1)I_{2n+1} = 2nI_{2n-1}$. Alternatively, observe that the function

$$f(s) = s \int_0^{\pi/2} (\sin x)^s \, dx$$

is increasing. This may be seen from the change of variables $t = \sin x$ and a series expansion of the new integrand yielding

$$f'(s) = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \binom{2k}{k} \frac{2k+1}{(2k+s+1)^2} > 0. \tag{3.1}$$

Remark 3.1. Comparing the series (3.1) at $s = 0$ with the limit

$$f'(0) = \lim_{s \to 0} \frac{f(s)}{s} = \lim_{s \to 0} \int_0^{\pi/2} \sin^s x \, dx = \frac{\pi}{2}$$

immediately proves

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{2^{-2k}}{2k+1} = \frac{\pi}{2}.$$

This value may also be obtained by letting $x = \frac{1}{2}$ in the series

$$\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k}}{2k+1} = \frac{\text{arcsin} 2x}{2x}.$$

The reader will find in [11] a host of other interesting series that involve the central binomial coefficients.

4 An example of Ramanujan and a generalization

A natural generalization of Wallis' integral (1.3) is given by

$$G_n(\boldsymbol{q}) = \frac{2}{\pi} \int_0^\infty \prod_{k=1}^n \frac{1}{x^2 + q_k^2}\, dx, \tag{4.1}$$

where $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ with $q_k \in \mathbb{C}$. This notation will be employed throughout. Similarly, $\boldsymbol{q}^\alpha$ is used to denote $(q_1^\alpha, q_2^\alpha, \ldots, q_n^\alpha)$. As the value of the integral (4.1) is independent under a change of sign of the parameters q_k, it is assumed that $\operatorname{Re} q_k > 0$. Note that the integral $G_n(\boldsymbol{q})$ is a symmetric function of $\boldsymbol{q}$ that reduces to G_n in the special case $q_1 = \ldots = q_n = 1$.

The special case $n = 4$ appears as Entry 13, Chapter 13, of B. Berndt's volume 2 of Ramanujan's Notebooks [2], in the form[1]:

Example 4.1. *Let q_1, q_2, q_3 and q_4 be positive real numbers. Then*

$$\frac{2}{\pi} \int_0^\infty \frac{dx}{(x^2+q_1^2)(x^2+q_2^2)(x^2+q_3^2)(x^2+q_4^2)} = \frac{(q_1+q_2+q_3+q_4)^3 - (q_1^3+q_2^3+q_3^3+q_4^3)}{3q_1q_2q_3q_4(q_1+q_2)(q_2+q_3)(q_1+q_3)(q_1+q_4)(q_2+q_4)(q_3+q_4)}.$$

Using partial fractions the following general formula for $G_n(\boldsymbol{q})$ is obtained. In the next section a representation in terms of Schur functions is presented.

Lemma 4.2. *Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ be distinct and $\operatorname{Re} q_k > 0$. Then*

$$G_n(\boldsymbol{q}) = \sum_{k=1}^n \frac{1}{q_k} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{q_j^2 - q_k^2}. \tag{4.2}$$

Proof. Observe first that if $b_1, b_2, \ldots, b_n$ are distinct then

$$\prod_{k=1}^n \frac{1}{y + b_k} = \sum_{k=1}^n \frac{1}{y + b_k} \prod_{\substack{j=1 \\ j \neq k}}^n \frac{1}{b_j - b_k}. \tag{4.3}$$

Replacing y by x^2 and b_k by q_k^2 and using the elementary integral

$$\frac{2}{\pi} \int_0^\infty \frac{dx}{x^2 + q^2} = \frac{1}{q}$$

produces the desired evaluation of $G_n(\boldsymbol{q})$. □

[1] A minor correction from [2].

Remark 4.3. $G_n(\boldsymbol{q})$, as defined by (4.1), is a symmetric function in the q_i's which remains finite if two of these parameters coincide. Therefore, the factors $q_j - q_k$ in the denominator of the right hand side of (4.2) cancel out. This may be checked directly by combining the summands corresponding to j and k. Alternatively, note that the right hand side of (4.2) is symmetric while the critical factors $q_j - q_k$ in the denominator combine to the antisymmetric Vandermonde determinant. Accordingly, they have to cancel.

Example 4.4. The identities

$$\frac{2}{\pi}\int_0^\infty \prod_{j=1}^{n+1} \frac{1}{x^2+j^2}\,dx = \frac{1}{(2n+1)n!(n+1)!},$$

$$\frac{2}{\pi}\int_0^\infty \prod_{j=1}^{n+1} \frac{1}{x^2+(2j-1)^2}\,dx = \frac{1}{2^{2n}(2n+1)(n!)^2},$$

$$\frac{2}{\pi}\int_0^\infty \prod_{j=1}^{n} \frac{1}{x^2+1/j^2}\,dx = \frac{2A(2n-1,n-1)}{\binom{2n}{n}}$$

may be deduced inductively from Lemma 4.2. Here, $A(n,k)$ are the Eulerian numbers which count the number of permutations of n objects with exactly k descents. Recall that a permutation σ of the n letters $1, 2, \ldots, n$, here written as $\sigma(1)\,\sigma(2)\,\ldots\,\sigma(n)$, has a descent at position k if $\sigma(k) > \sigma(k+1)$. For instance, $A(3,1) = 4$ because there are 4 permutations of $1, 2, 3$, namely 1 3 2, 2 1 3, 2 3 1 and 3 1 2, which have exactly one descent.

The question of an explicit formula for the numerators appearing on the right-hand side of (4.2) is discussed in the next section.

5 Representation in terms of Schur functions

The expression for $G_n(\boldsymbol{q})$ developed in this section is given in terms of *Schur functions*. The reader is referred to [4] for a motivated introduction to these functions in the context of *alternating sign matrices* and to [17] for their role in the representation theory of the symmetric group. Among the many equivalent definitions for Schur functions, we now recall their definition in terms of quotients of alternants. This way, we are able to associate a Schur function not only to a partition but more generally to arbitrary vectors.

Here, a vector $\mu = (\mu_1, \mu_2, \ldots)$ means a finite sequence of real numbers. μ is further called a partition if $\mu_1 \geq \mu_2 \geq \ldots$ and all the parts μ_j are positive integers. Write $\mathbf{1}^n$ for the partition with n ones, and denote with $\lambda(n)$ the partition

$$\lambda(n) = (n-1, n-2, \ldots, 1).$$

Vectors and partitions may be added componentwise. In case they are of different length, the shorter one is padded with zeroes. For instance, one has

$\lambda(n+1) = \lambda(n) + \mathbf{1}^n$. Likewise, vectors and partitions may be multiplied by scalars. In particular, $a \cdot \mathbf{1}^n$ is the partition with n a's.

Fix n and consider $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. Let $\mu = (\mu_1, \mu_2, \ldots)$ be a vector of length at most n. The corresponding alternant a_μ is defined as the determinant

$$a_\mu(\boldsymbol{q}) = \left|q_i^{\mu_j}\right|_{1 \le i,j \le n}.$$

Again, μ is padded with zeroes if necessary. Note that the alternant $a_{\lambda(n)}$ is the classical Vandermonde determinant

$$a_{\lambda(n)}(\boldsymbol{q}) = \left|q_i^{n-j}\right|_{1 \le i,j \le n} = \prod_{1 \le i < j \le n} (q_i - q_j).$$

The Schur function s_μ associated with the vector μ can now be defined as

$$s_\mu(\boldsymbol{q}) = \frac{a_{\mu+\lambda(n)}(\boldsymbol{q})}{a_{\lambda(n)}(\boldsymbol{q})}.$$

If μ is a partition with integer entries this is a symmetric polynomial. Indeed, as μ ranges over the partitions of m of length at most n, the Schur functions $s_\mu(\boldsymbol{q})$ form a basis of the homogeneous symmetric polynomials in $\boldsymbol{q}$ of degree m.

The Schur functions include as special cases the *elementary symmetric functions* e_k and the *complete homogeneous symmetric functions* h_k. Namely, $e_k(\boldsymbol{q}) = s_{\mathbf{1}^k}(\boldsymbol{q})$ and $h_k(\boldsymbol{q}) = s_{(k)}(\boldsymbol{q})$.

The next result expresses the integral $G_n(\boldsymbol{q})$ defined in (4.1) as a quotient of Schur functions.

Theorem 5.1. *Let* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *and* $\operatorname{Re} q_k > 0$. *Then*

$$G_n(\boldsymbol{q}) = \frac{s_{\lambda(n-1)}(\boldsymbol{q})}{s_{\lambda(n+1)}(\boldsymbol{q})} = \frac{s_{\lambda(n-1)}(\boldsymbol{q})}{e_n(\boldsymbol{q}) s_{\lambda(n)}(\boldsymbol{q})}. \tag{5.1}$$

Proof. From the previous definition of Schur functions, the right hand side of (5.1) becomes

$$\frac{s_{\lambda(n-1)}(\boldsymbol{q})}{e_n(\boldsymbol{q})\, s_{\lambda(n)}(\boldsymbol{q})} = \frac{a_{\lambda(n-1)+\lambda(n)}(\boldsymbol{q})}{e_n(\boldsymbol{q}) a_{2\lambda(n)}(\boldsymbol{q})}.$$

Observe that $a_{2\lambda(n)}(\boldsymbol{q}) = |q_i^{2n-2j}|_{i,j} = a_{\lambda(n)}(\boldsymbol{q}^2)$ is simply the Vandermonde determinant with q_i replaced by q_i^2. Next, expand the determinant $a_{\lambda(n-1)+\lambda(n)}$ by the last column (which consists of 1's only) to find

$$a_{\lambda(n-1)+\lambda(n)}(\boldsymbol{q}) = e_n(\boldsymbol{q}) \sum_{k=1}^{n} \frac{(-1)^{n-k}}{q_k} a_{\lambda(n-1)}(q_1^2, q_2^2, \ldots, q_{k-1}^2, q_{k+1}^2, \ldots, q_n^2).$$

Therefore

$$\frac{a_{\lambda(n-1)+\lambda(n)}(\boldsymbol{q})}{e_n(\boldsymbol{q})\, a_{2\lambda(n)}(\boldsymbol{q})} = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{q_k} \prod_{\substack{i<j \\ i,j \ne k}} (q_i^2 - q_j^2) \Big/ \prod_{i<j} (q_i^2 - q_j^2). \tag{5.2}$$

Observe that the only terms that do not cancel in the quotient above are those for which $i = k$ or $j = k$. The change of sign required to transform the factors $q_k^2 - q_j^2$ to $q_j^2 - q_k^2$ eliminates the factor $(-1)^{n-k}$. The expression on the right hand side of (5.2) is precisely the value (4.2) of the integral $G_n(\boldsymbol{q})$ produced by partial fractions.

It remains to show that $e_n(\boldsymbol{q}) s_{\lambda(n)}(\boldsymbol{q}) = s_{\lambda(n+1)}(\boldsymbol{q})$. This amounts to the identity

$$q_1 q_2 \cdots q_n \left| q_i^{2n-2j} \right|_{i,j} = \left| q_i^{2n-2j+1} \right|_{i,j}$$

which follows directly by inserting the factors q_i one at a time per row. □

The next example illustrates Theorem 5.1 with the principal specialization of the parameters $\boldsymbol{q}$.

Example 5.2. The special case $q_k = q^k$ produces the evaluation

$$\frac{2}{\pi} \int_0^\infty \prod_{k=1}^{n} \frac{1}{x^2 + q^{2k}} = \frac{1}{q^{n^2}} \prod_{j=1}^{n-1} \frac{1 - q^{2j-1}}{1 - q^{2j}}. \tag{5.3}$$

This can be obtained inductively from Lemma 4.2 but may also be derived from Theorem 5.1 in combination with the evaluation (6.1) of the principal specialization of Schur functions as in Theorem 7.21.2 of [18].

Taking the limit $q \to 1$ in (5.3) reproduces formula (2.2) for G_n. In other words, (5.3) is a q-analog [7] of (2.2). Similarly,

$$\frac{\pi_q}{1+q} = q^{1/4} \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}} \frac{1 - q^{2n}}{1 - q^{2n+1}}$$

is a useful q-analog of Wallis' formula (1.2) which naturally appears in [8] where Gosper studies q-analogs of trigonometric functions (in fact, Gosper arrives at the above expression as a definition for π_q while q-generalizing the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$).

The proof of Theorem 5.1 extends to the following more general result.

Lemma 5.3.

$$\sum_{k=1}^{n} \frac{1}{q_k^{\alpha-\beta}} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{q_j^\alpha - q_k^\alpha} = \frac{s_\lambda(\boldsymbol{q})}{s_\mu(\boldsymbol{q})}$$

where

$$\begin{aligned} \lambda &= (\alpha - 1) \cdot \lambda(n) - \beta \cdot \mathbf{1}^{n-1}, \\ \mu &= (\alpha - 1) \cdot \lambda(n+1) - (\beta - 1) \cdot \mathbf{1}^n. \end{aligned}$$

As a consequence, one obtains the following integral evaluation which generalizes the evaluation of $G_n(\boldsymbol{q})$ given in Theorem 5.1.

Theorem 5.4. *Let* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *and* $\operatorname{Re} q_k > 0$. *Further, let* $\alpha > 0$ *and* $0 < \beta < \alpha n$ *be given such that* β *is not an integer multiple of* α. *Then*

$$G_{n,\alpha,\beta}(\boldsymbol{q}) := \frac{\sin(\pi\beta/\alpha)}{\pi/\alpha} \int_0^\infty \frac{x^{\beta-1}}{\prod_{k=1}^n (x^\alpha + q_k^\alpha)} \, dx = \frac{s_\lambda(\boldsymbol{q})}{s_\mu(\boldsymbol{q})}$$

where λ *and* μ *are as in Lemma 5.3.*

Proof. Upon writing $\beta = b\alpha + \beta_1$ for $b < n$ a positive integer and $0 < \beta_1 < \alpha$, the assertion follows from the partial fraction decomposition

$$\frac{x^{b\alpha}}{\prod_{k=1}^n (x^\alpha + q_k^\alpha)} = (-1)^b \sum_{k=1}^n \frac{q_k^{b\alpha}}{x^\alpha + q_k^\alpha} \prod_{j \neq k} \frac{1}{q_j^\alpha - q_k^\alpha},$$

the integral evaluation

$$\int_0^\infty \frac{x^{\beta_1 - 1} dx}{x^\alpha + q^\alpha} = \frac{1}{q^{\alpha - \beta_1}} \frac{\pi/\alpha}{\sin(\pi\beta_1/\alpha)},$$

and Lemma 5.3. □

6 Schur functions in terms of SSYT

The Schur function $s_\lambda(\boldsymbol{q})$ associated to a partition λ also admits a representation in terms of *semi-standard Young tableaux* (SSYT). The reader will find information about this topic in [4]. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the Young diagram of shape λ is an array of boxes, arranged in left-justified rows, consisting of λ_1 boxes in the first row, λ_2 in the second row ending with λ_n boxes in the nth row. A SSYT of shape λ is a filling of the boxes of the Young diagram of shape λ with positive integers. These integers are restricted to be weakly increasing across rows (repetitions are allowed) and strictly increasing down columns. From this point of view, the Schur function $s_\lambda(\boldsymbol{q})$ is defined as

$$s_\lambda(\boldsymbol{q}) = \sum_T \boldsymbol{q}^T$$

where the sum is over all SSYT of shape λ with entries from $\{1, 2, \ldots, n\}$. The symbol $\boldsymbol{q}^T$ is a monomial in the variables q_j, where the exponent of q_j is the number of appearances of j in T. For example, the array shown in Figure 1 is a tableau T for the partition $(6, 4, 3, 3)$. The corresponding monomial $\boldsymbol{q}^T$ is given by $q_1 q_2^3 q_3 q_4^3 q_5^4 q_6^2 q_7 q_8$.

The number $N(\mu)$ of SSYT of shape μ can be obtained by letting $q \to 1$ in the formula

$$s_\mu(1, q, q^2, \ldots, q^{n-1}) = \prod_{1 \leq i < j \leq n} \frac{q^{\mu_i + n - i} - q^{\mu_j + n - j}}{q^{j-1} - q^{i-1}} \tag{6.1}$$

1	2	2	4	5	5
2	3	4	5		
4	6	6			
5	7	8			

Figure 1: A tableau T for the partition $(6, 4, 3, 3)$

(see page 375 of [18]). This yields

$$N(\mu) = \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j + j - i}{j - i}. \tag{6.2}$$

The evaluation (2.2) of Wallis' integral (1.3) may be recovered from here as

$$G_{n+1} = \frac{s_{\lambda(n)}(\mathbf{1}^{n+1})}{s_{\lambda(n+2)}(\mathbf{1}^{n+1})} = \frac{N(\lambda(n))}{N(\lambda(n+2))} = \frac{1}{2^{2n}} \binom{2n}{n}.$$

7 A counting problem

The k-central binomial coefficients $c(n, k)$, defined by the generating function

$$(1 - k^2 x)^{-1/k} = \sum_{n \geq 0} c(n, k) x^n,$$

are given by

$$c(n, k) = \frac{k^n}{n!} \prod_{m=1}^{n-1} (1 + km).$$

For $k = 2$ these coefficients reduce to the central binomial coefficients $\binom{2n}{n}$. The numbers $c(n, k)$ are integers in general and their divisibility properties have been studied in [20]. In particular, the authors establish that the k-central binomial coefficients are always divisible by k and characterize their p-adic valuations.

The next result attempts an interpretation of what the numbers $-c(n, -k)$ count.

Corollary 7.1. *Let λ and μ be the partitions given by*

$$\begin{aligned} \lambda &= (k-1) \cdot \lambda(n) - \mathbf{1}^{n-1}, \\ \mu &= (k-1) \cdot \lambda(n+1). \end{aligned}$$

Then the integer $-c(n, -k)$ enumerates the ratio between the total number of SSYT of shapes λ and μ times the factor k^{2n-1}/n.

Proof. By Theorem 5.4 and (6.2),

$$G_{n,k,1}(\mathbf{1}^n) = \frac{s_\lambda(\mathbf{1}^n)}{s_\mu(\mathbf{1}^n)} = \prod_{m=1}^{n-1} \frac{km - 1}{km}.$$

The claim follows. ∎

Remark 7.2. R. Stanley pointed out some interesting Schur function quotient results. See exercises 7.30 and 7.32 in [18].

8 An integral from Gradshteyn and Ryzhik

It is now demonstrated how the previous results may be used to prove an integral evaluation found as entry 3.112 in [9]. The main tool is the (dual) Jacobi-Trudi identity which expresses a Schur function in terms of elementary symmetric functions. Namely, if λ is a partition such that its conjugate λ' (the unique partition whose Young diagram, see Section 6, is obtained from the one of λ by interchanging rows and columns) has length at most m then

$$s_\lambda = \left| e_{\lambda'_i - i + j} \right|_{1 \le i,j \le m}.$$

This identity may be found for instance in [18, Corollary 7.16.2].

Theorem 8.1. *Let f_n and g_n be polynomials of the form*

$$\begin{aligned} g_n(x) &= b_0 x^{2n-2} + b_1 x^{2n-4} + \ldots + b_{n-1}, \\ f_n(x) &= a_0 x^n + a_1 x^{n-1} + \ldots + a_n \end{aligned}$$

and assume that all roots of f_n lie in the upper half-plane. Then

$$\int_{-\infty}^{\infty} \frac{g_n(x)dx}{f_n(x) f_n(-x)} = \frac{\pi i}{a_0} \frac{M_n}{\Delta_n}$$

where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}, \qquad M_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & & 0 \\ 0 & a_1 & a_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & a_n \end{vmatrix}.$$

Proof. Write $f_n(x) = a_0 \prod_{j=1}^n (x - iq_j)$. By assumption, $\operatorname{Re} q_j > 0$. Further,

$$f_n(x) f_n(-x) = (-1)^n a_0^2 \prod_{j=1}^n (x^2 + q_j^2).$$

Let $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. It follows from Theorem 5.4 that

$$\int_{-\infty}^{\infty} \frac{x^{2\beta} dx}{f_n(x) f_n(-x)} = \frac{(-1)^{n+\beta}\pi}{a_0^2} \frac{s_{\lambda(n-1)-2\beta\cdot\mathbf{1}^{n-1}}(\boldsymbol{q})}{s_{\lambda(n+1)-2\beta\cdot\mathbf{1}^n}(\boldsymbol{q})} = \frac{(-1)^n\pi}{a_0^2} \frac{s_{\lambda'}(\boldsymbol{q})}{s_{\lambda(n+1)}(\boldsymbol{q})}$$

where $\lambda = \lambda(n-1) + 2\cdot\mathbf{1}^\beta$. The latter equality is obtained by writing the quotient of (generalized) Schur functions as a quotient of alternants, multiplying the k-th row with $q_k^{2\beta}$ each, and reordering the columns of the determinant in the numerator. The right-hand side now is a quotient of Schur functions to which the Jacobi-Trudi identity may be applied.

$$s_{\lambda(n+1)}(\boldsymbol{q}) = |e_{n+1-2k+j}(\boldsymbol{q})|_{1\le k,j\le n} = |e_{2k-j}(\boldsymbol{q})|_{1\le k,j\le n} \,.$$

Note that $e_k(\boldsymbol{q}) = i^k a_k$. Hence, $s_{\lambda(n+1)}(\boldsymbol{q}) = i^{n(n+1)/2}\Delta_n$. The term $s_{\lambda'}(\boldsymbol{q})$ is dealt with analogously. The claim follows by expanding the determinant M_n with respect to the first row. □

9 A sum related to Feynman diagrams

Particle scattering in quantum field theory is usually described in terms of Feynman diagrams. A Feynman diagram is a graphical representation of a particular term arsing in the expansion of the relevant quantum mechanical scattering amplitude as a power series in the coupling constants that parametrize the strengths of the interactions.

From the mathematical point of view, a Feynman diagram is a graph to which a certain function is associated. If the graph has circuits (*loops*, in the physics terminology) then this function is defined in terms of a number of integrals over the 4-dimensional momentum space $(k_0, \boldsymbol{k})$, where k_0 is the *energy* integration variable and $\boldsymbol{k}$ is a 3-dimensional momentum variable.

Feynman diagrams also appear in calculations of the thermodynamic properties of a system described by quantum fields. In this context, the integral over the energy-component of a Feynman loop diagram is replaced by a summation over discrete energy values. These Matsubara sums were introduced in [14]. A general method to compute these sums in terms of an associated integral was presented in [5].

These techniques, applied to the expression (4.2) for the integral $G_n(\boldsymbol{q})$ give the value of the sum associated with the one-loop Feynman diagram consisting of n vertices and vanishing external momenta, $N_i = 0$, as depicted in Figure 2.

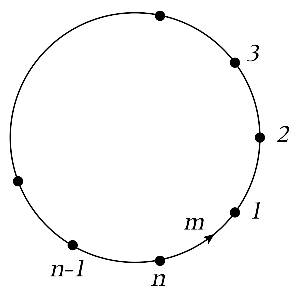

Figure 2: The one-loop Feynman diagram with n vertices and vanishing external momenta. m is the summation variable associated to each of the internal lines.

The Matsubara sum associated to the diagram in Figure 2 is

$$M_n(\boldsymbol{q}) := \sum_{m=-\infty}^{\infty} \prod_{k=1}^{n} \frac{1}{m^2 + q_k^2} \tag{9.1}$$

where the variables q_k are related to the kinematic energies carried by the (virtual) particles in the Feynman diagram. This sum was denoted by S_G in [5]; the notation has been changed here to avoid confusion.

Example 9.1. The first few Matsubara sums are

$$\begin{aligned}
M_1(q_1) &= \pi \frac{D_1}{q_1}, \\
M_2(q_1, q_2) &= \pi \frac{q_2 D_1 - q_1 D_2}{q_1 q_2 (q_2^2 - q_1^2)}, \\
M_3(q_1, q_2, q_3) &= \pi \frac{q_2 q_3 (q_2^2 - q_3^2) D_1 + q_3 q_1 (q_3^2 - q_1^2) D_2 + q_1 q_2 (q_1^2 - q_2^2) D_3}{q_1 q_2 q_3 (q_3^2 - q_2^2)(q_2^2 - q_1^2)(q_1^2 - q_3^2)}
\end{aligned}$$

with $D_j = \coth(\pi q_j)$.

Theorem 9.2. *The Matsubara sum $M_n(\boldsymbol{q})$ is given by*

$$M_n(\boldsymbol{q}) = \pi \sum_{k=1}^{n} \frac{\coth(\pi q_k)}{q_k} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{1}{q_j^2 - q_k^2}.$$

Proof. This follows from the partial fraction expansion

$$\prod_{k=1}^{n} \frac{1}{m^2 + q_k^2} = \sum_{k=1}^{n} \frac{1}{q_k^2 + m^2} \prod_{j \neq k} \frac{1}{q_j^2 - q_k^2}$$

which is a special case of (4.3), switching the order of summation, and employing the classical

$$\frac{\pi \coth(\pi z)}{z} = \sum_{m=-\infty}^{\infty} \frac{1}{z^2 + m^2}.$$

□

Proof 2. The method developed in [5] shows that

$$M_n(\boldsymbol{q}) = \pi \left[1 + \sum_{m=1}^{n} \mathrm{n_b}(q_m)(1 - R_m) \right] G_n(\boldsymbol{q}) \tag{9.2}$$

where $G_n(\boldsymbol{q})$ is the integral defined in (4.1),

$$\mathrm{n_b}(q) = \frac{1}{e^{2\pi q} - 1} = \frac{1}{2} \left(\coth \pi q - 1 \right),$$

and R_m is the reflection operator defined by

$$R_m f(q_1, \ldots, q_m, \ldots) = f(q_1, \ldots, -q_m, \ldots).$$

To use (9.2) combined with the evaluation (4.2) of $G_n(\boldsymbol{q})$ it is required to compute the action of each $1 - R_m$ on the summands of (4.2). Namely,

$$(1 - R_m)\frac{1}{q_k}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{q_j^2 - q_k^2} = \frac{2\delta_{km}}{q_k}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{q_j^2 - q_k^2}$$

where δ_{km} is the Kronecker delta. Therefore,

$$\mathrm{n_b}(q_m)(1 - R_m)G_n(\boldsymbol{q}) = \frac{2\,\mathrm{n_b}(q_m)}{q_m}\prod_{\substack{j=1\\ j\neq m}}^{n}\frac{1}{q_j^2 - q_m^2},$$

and the result follows from $2\,\mathrm{n_b}(q) = \coth(\pi q) - 1$. □

We close by giving an expansion of $M_n(\boldsymbol{q})$ in terms of symmetric functions. Starting with the classical expansion

$$\frac{\pi\coth q_k}{q_k} = \frac{1}{q_k^2} - 2\sum_{m=1}^{\infty}(-1)^m q_k^{2m-2}\zeta(2m),$$

where $\zeta(s)$ denotes the Riemann zeta function, it follows that

$$M_n(\boldsymbol{q}) = \sum_{k=1}^{n}\frac{1}{q_k^2}\prod_{j\neq k}\frac{1}{q_j^2 - q_k^2} - 2\sum_{m=1}^{\infty}(-1)^m\zeta(2m)\sum_{k=1}^{n}q_k^{2(m-1)}\prod_{j\neq k}\frac{1}{q_j^2 - q_k^2}.$$

Using the identity (h_j being the complete homogeneous symmetric function)

$$h_{m-n}(x_1, \ldots, x_n) = (-1)^{n-1}\sum_{k=1}^{n}x_k^{m-1}\prod_{j\neq k}\frac{1}{x_j - x_k},$$

which follows from Lemma 5.3 (or see page 450, Exercise 7.4 of [18]), this proves:

Corollary 9.3. *The Matsubara sum $M_n(\boldsymbol{q})$, defined in (9.1), is given by*

$$M_n(\boldsymbol{q}) = \frac{1}{e_n(\boldsymbol{q}^2)} + 2\sum_{m=0}^{\infty}(-1)^m\zeta(2m + 2n)h_m(\boldsymbol{q}^2).$$

10 Conclusions

The evaluation of definite integrals has the charming quality of taking the reader for a tour of many parts of mathematics. An innocent-looking generalization of one of the oldest formulas in analysis has been shown to connect the work of the four authors in the title.

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