A FAMILY OF PALINDROMIC POLYNOMIALS

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1. Introduction

The relation between the roots of a polynomial $P(z)$ and its coefficients was one of the driving forces in the development of Algebra. The desire to obtain a closed-form expression for the roots of $P(z) = 0$ ended with a negative solution at the hands of N. Abel and E. Galois at the beginning of the 19^{th} century: *a generic polynomial equation of degree 5 or more cannot be solved by radicals.* The reader should be aware of the existence of analytic expressions for the roots of a polynomial. Naturally these involve non-algebraic functions: for the quintic equation this is done using *elliptic functions*, as explained in the beautiful text [3], and for higher degree the formulas involve the so-called *theta functions*.

In spite of this set-back, the study of roots of polynomials has continued throughout the centuries. Classical results include Newton's statement that if P has only negative real roots then the coefficients a_j of P form a *logconcave sequence*, that is, $a_j^2 - a_{j-1}a_{j+1} \geq 0$. For starters, the reader is refered to [4]. Logconcave sequences arise in many combinatorial contexts, the simplest of which are the binomial coefficients $\left\{ \binom{n}{k} : 0 \leq k \leq n \right\}$. Of course, there are artificial ways to construct logconcave polynomials. Here is a simple-minded example

$$A(z) = \sum_{r=0}^{m} r(m-r)z^r. \tag{1.1}$$

The reader will see this polynomial appearing later in this paper, in a natural way.

It is well-known that if a polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ has only *unimodular* roots (i.e. $|z| = 1$) then it is either *palindromic* (i.e. $a_k = a_{n-k}$) or *anti-palindromic* (i.e. $a_k = -a_{n-k}$). The converse is not true however; for example $P(z) = z^2 - 3z + 1$. In light of this, the polynomials $F_{\lambda,m}(z)$ defined below, being anti-palindromic, are perfect candidates for a study on unimodular zeros.

The main task in this work aims at locating the roots of the polynomial

$$\begin{aligned} F_{\lambda,m}(z) &= (z^m+1)(z-1) - \lambda(z^m-1)(z+1) \\ &= (1-\lambda)z^{m+1} - (1+\lambda)z^m + (1+\lambda)z - (1-\lambda), \end{aligned} \tag{1.2}$$

as a function of the parameter $\lambda \in \mathbb{R}$. Observe that, for $\lambda = 0$, this polynomial reduces to $(z^m+1)(z-1)$ and all its roots are on the unit circle $|z| = 1$.

The critical case $\lambda = 1/m$, where the equation $F_{\lambda,m}(z) = 0$ turns out to be

$$(m-1)z^{m+1} - (m+1)z^m + (m+1)z - (m-1) = 0, \tag{1.3}$$

Date: August 3, 2011.
2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Polynomials, zeros, unimodular zeros, logconcavity.

appeared in the context of the so-called *interior transmission problem*. This is a nonselfadjoint boundary-value problem which has attracted a lot of attention in the recent years. The problem relates the refractive index of a medium to a sequence of complex numbers called the transmission eigenvalues. In the spherically symmetric case, there is a natural special subset of those eigenvalues which coincides with the set of zeros of an entire function $D(\lambda)$. If the refractive index of the spherical medium (of radius b) is identically equal to an integer $m \geq 2$, then $D(\lambda)$ becomes $D(\lambda) = g(m\sqrt{\lambda})/2m\sqrt{\lambda}$, where

$$\begin{aligned} g(\tau) &= 2m\sin(\tau)\cos(m\tau) - 2\cos(\tau)\sin(m\tau) \\ &= (m-1)\sin((m+1)\tau) - (m+1)\sin((m-1)\tau). \end{aligned}$$

The reader finds further details in [1].

Setting $z = e^{i\tau}$, one obtains the polynomial of present concern:

$$4i(1/m)z^{m+1}g(\tau) = F_{1/m,m}(z).$$

The fact that the zeros of $F_{1/m,m}(z)$ lie on the unit circle implies immediately that the zeros of $g(\tau)$ are real, and hence those of $D(\lambda)$, i.e. the special transmission eigenvalues, are real too (which is, somehow, surprising). Distribution of the roots of $F_{\lambda,m}$ as a function of the parameter λ is presented in the next theorem. The proof is discussed in the upcoming sections.

Theorem 1.1. *The polynomial $F_{\lambda,m}(z)$ always has $z=1$ as a root. The remaining roots solve the equation $H_{\lambda,m}(z)=0$, where $H_{\lambda,m}(z) = F_{\lambda,m}(z)/(z-1)$. Moreover, $H_{\lambda,m}(z)$ has*
(a) at least $m-2$ unimodular roots with (possibly) two exceptions;
(b) the exceptions are reciprocal real roots;
(c) all roots are simple except for a double root at $z=1$;
(d) if $\lambda \leq 1/m$ then all roots are unimodular;
(e) in the special case $\lambda = 1$, the polynomial $H_{\lambda,m}$ is of degree m having $z=0$ as a simple root, the rest being unimodular.

2. Some basic properties

In this section we collect some basic properties of the polynomial $F_{\lambda,m}(z)$. The first lemma establishes part (e) of Theorem 1.1.

Lemma 2.1. *The value $z=0$ is a root of $F_{\lambda,m}(z)=0$ only for $\lambda=1$. In this case, the other $m-1$ roots are roots of unity; hence unimodular.*

Proof. This follows directly from $F_{\lambda,m}(0) = \lambda - 1$ and $F_{1,m}(z) = -2z(z^{m-1}-1)$. □

The polynomial $F_{\lambda,m}(z)$ has real coefficients, therefore the complex roots occur as conjugates pairs. The next result shows that the roots can be partitioned into groups each consisiting of four elements.

Lemma 2.2. *The polynomial $F_{\lambda,m}(z)$ satisfies*

$$F_{\lambda,m}(1/z) = -\frac{1}{z^{m+1}}F_{\lambda,m}(z). \tag{2.1}$$

Thus each root $r \in \mathbb{C}$ is part of a quartet $\{r, \bar{r}, 1/r, 1/\bar{r}\}$.

Proof. The relation (2.1) is straightforward. Naturally, the value $\lambda = 1$ which leads to the root $z = 0$ has to be excluded. The roots in the quartet can coalesce, as in the case of $r = 1$ where they are all equal. □

Lemma 2.3. *The value $z = 1$ is always a root of $F_{\lambda,m}$. This root is simple for $\lambda \neq 1/m$. Also, $F_{\lambda,m}(z) = (z-1)H_{\lambda,m}(z)$ factorizes as*

$$H_{\lambda,m}(z) = (1-\lambda)z^m - 2\lambda \sum_{j=1}^{m-1} z^j + (1-\lambda). \tag{2.2}$$

Proof. Clearly $F_{\lambda,m}(1) = 0$. Similarly $F'_{\lambda,m}(1) = 2(1-\lambda m)$ gives the simplicity statement. The expression for the quotient $F_{\lambda,m}(z)/(z-1)$ is obtained by writing $z^k = (z^k - 1) + 1$ in (1.2). □

Lemma 2.4. *The value $z = -1$ is a root of $F_{\lambda,m} = 0$ for m odd. This root is simple for m odd and $\lambda \neq m$.*

Proof. Put $z = -1$ in Lemma 2.2. The value $F'_{\lambda,m}(-1) = 2(\lambda - m)$ gives the simplicity statement. □

3. The roots of $F_{\lambda,m}(z) = 0$

This section is devoted to a proof of Theorem 1.1. The discussion is divided into cases according to the value of λ.

Case 1: $\lambda \leq 0$.

Proposition 3.1. *If $\lambda \leq 0$, then $F_{\lambda,m}(z)$ possesses only unimodular roots.*

Proof. The equation $F_{\lambda,m}(z) = 0$ implies $z^m = \dfrac{(1-\lambda)-(1+\lambda)z}{(1-\lambda)z-(1+\lambda)}$. Denote the moduli by $A = |(1-\lambda)-(1+\lambda)z|$, $B = |(1-\lambda)z-(1+\lambda)|$ with $z = x+iy$. Then, the claim holds for $\lambda = 0$ since the equation becomes $z^m = -1$. For $\lambda < 0$ a direct calculation gives $A^2 = B^2 - 4\lambda(1-|z|^2)$. This yields

$$|z|^{2m} = 1 - \frac{4\lambda}{B^2}(1-|z|^2). \tag{3.1}$$

This equation is solvable if and only if $|z| = 1$, hence the assertion follows. □

Case 2: $0 < \lambda < 1/m$.

The analysis is based on the fact that non-unimodular roots bifurcate from a value λ for which $F_{\lambda,m}$ has multiple unimodular roots. It is shown that this does not occur for $\lambda < 1/m$.

Lemma 2.2 shows that if r is a root of $F_{\lambda,m}(z)$, then so is $1/r$. Assume that, as λ increases from $\lambda = 0$ to some $\lambda^* > 0$, one of the roots of $F_{\lambda^*,m}(z)$ has modulus strictly bigger than 1. It follows that there is a second root with modulus less than 1. The continuity of the number of roots as a function of λ shows that these non-unimodular roots must bifurcate from a multiple root.

Proposition 3.2. *For $0 < \lambda < 1/m$, the polynomial $F_{\lambda,m}(z)$ has no multiple unimodular roots.*

Proof. A multiple root $z = r \in \mathbb{C}$ satisfies both

$$F_{\lambda,m}(r) = (1-\lambda)r^{m+1} - (1+\lambda)r^m + (1+\lambda)r - (1-\lambda) = 0, \text{ and} \tag{3.2}$$

$$F'_{\lambda,m}(r) = (m+1)(1-\lambda)r^m - m(1+\lambda)r^{m-1} + (1+\lambda) = 0. \tag{3.3}$$

Multiply (3.2) by $m+1$ and (3.3) by r and subtract to obtain

$$r^m = mr - \frac{(m+1)(1-\lambda)}{1+\lambda}. \tag{3.4}$$

Replace in (3.2) to obtain

$$m(1-\lambda^2)r^2 - 2(m - 2\lambda + m\lambda^2)r + m(1-\lambda^2) = 0. \tag{3.5}$$

This equation (in the variable r) has discriminant is $\Delta = -16\lambda(m-\lambda)(1-m\lambda)$. If $\lambda < 0$, then $\Delta > 0$ and the unimodular root is real. Therefore $r = \pm 1$ and these are easy to rule out using (3.5). In the range $0 < \lambda < 1/m$, square (3.4) to get

$$(1+\lambda)^2 r^{2m} = m^2(1+\lambda)^2 r^2 - 2m(m+1)(1-\lambda^2)r + (m+1)^2(1-\lambda)^2. \tag{3.6}$$

From (3.5), it follows that

$$r^2 = \frac{2(m - 2\lambda + m\lambda^2)}{m(1-\lambda^2)} r - 1, \tag{3.7}$$

and replacing in (3.6) gives

$$\begin{aligned} (1+\lambda)^2(1-\lambda)r^{2m} &= 2m(1+\lambda)(2m\lambda - \lambda^2 - 1)r \\ &\quad + (1-\lambda)(1-\lambda-2m\lambda)(1-\lambda+2m). \end{aligned} \tag{3.8}$$

From (3.4) it follows that $r = \frac{1}{m} r^m + \frac{(m+1)(1-\lambda)}{m(1+\lambda)}$, and replacing in (3.8) gives $(1-\lambda^2)r^{2m} - 2(2m\lambda - \lambda^2 - 1)r^m + (1-\lambda^2) = 0$. This is now solved to arrive at

$$r^m - 1 = \frac{2(m\lambda - 1) \pm 2iE}{1-\lambda^2}. \tag{3.9}$$

The solution of (3.5) implies

$$r - 1 = \frac{2\lambda(m\lambda - 1) \pm 2iE}{m(1-\lambda^2)}, \tag{3.10}$$

where $E = \sqrt{\lambda(m-\lambda)(1-m\lambda)} > 0$. Then

$$\left| \frac{r^m - 1}{r - 1} \right| = m \left| \frac{(m\lambda - 1) \pm iE}{\lambda(m\lambda - 1) \pm iE} \right| = m \sqrt{\frac{(m\lambda-1)^2 + E^2}{\lambda^2(m\lambda-1)^2 E^2}} > m, \tag{3.11}$$

using $\lambda < 1$. On the other hand $|r| = 1$, so that

$$\left| \frac{r^m - 1}{r - 1} \right| = \left| r^{m-1} + \cdots + 1 \right| \le m. \tag{3.12}$$

This contradiction completes the proof. □

Corollary 3.3. *For $0 \le \lambda < 1/m$, all roots of $F_{\lambda,m}(z)$ are unimodular.*

Case 3: $\lambda = 1/m$.

The equation $F_{1/m,m}(z) = 0$ is written as

$$\frac{1}{m} F_{1/m,m}(z) = (m-1)\left(z^{m+1} - 1\right) - (m+1)z\left(z^{m-1} - 1\right) = 0. \tag{3.13}$$

The next result came as a pleasent surprise.

Theorem 3.4. *The value $z = 1$ is a triple root of $F_{1/m,m}(z) = 0$. The quotient polynomial $F_{1/m,m}(z)/(z-1)^3$ is given by the logconcave polynomial $A(x)$ defined in* (1.1)*; that is,*

$$F_m(z) = (z-1)^3 A(x) = (z-1)^3 \sum_{r=1}^{m} r(m-r)z^r.$$

Proof. It has already been established that $z = 1$ is a root of any $F_{\lambda,m}(z)$. A direct computation shows that $F'_{1/m,m}(1) = 0$, $F''_{1/m,m}(1) = 0$ and $F'''_{1/m,m}(1) = m^2 - 1$. Therefore $z = 1$ is a triple root.

Observe that $F_{1/m,m}(z) = m(m-1)\left(z^{m+1}-1\right) - m(m+1)z\left(z^{m-1}-1\right)$ means

$$\frac{F_{1/m,m}(z)}{m(z-1)} = (m-1)\sum_{j=0}^{m} z^j - (m+1)\sum_{j=0}^{m-2} z^{j+1}. \tag{3.14}$$

It follows that $F_{1/m,m}(z) = m(z-1)\sum_{j=0}^{m} \alpha_j z^j$ with

$$\alpha_j = \begin{cases} m-1 & \text{if } j = 0 \text{ or } j = m \\ -2 & \text{if } 1 \leq j \leq m-1. \end{cases}$$

The relation $\alpha_0 + \alpha_1 + \cdots + \alpha_m = 0$ yields

$$\begin{aligned} F_{1/m,m}(z) &= m(z-1)\sum_{j=0}^{m} \alpha_j\left(z^j - 1\right) = m(z-1)^2 \sum_{j=1}^{m} \alpha_j \sum_{k=0}^{j-1} z^k \\ &= m(z-1)^2 \sum_{k=0}^{m-1} \left(\sum_{j=k}^{m-1} \alpha_{j+1} \right) z^k. \end{aligned}$$

Thus $\sum_{j=k+1}^{m} \alpha_j = -\sum_{j=0}^{k} \alpha_j = -(m-1-2k)$ gives $-\frac{F_{1/m,m}(z)}{m(z-1)^2} = \sum_{k=0}^{m-1} (m-1-2k)z^k$.

The value $\sum_{k=0}^{m-1}(m-1-2k) = 0$ shows that

$$F_{1/m,m}(z) = -m(z-1)^2 \sum_{k=0}^{m-1}(m-1-2k)(z^k-1) = -m(z-1)^3 \sum_{r=0}^{m-2} \left(\sum_{k=r}^{m-2} (m-2k-3) \right) z^r.$$

Evaluating the internal sum implies the final result. □

Case 4: $1/m < \lambda$.

A new technique is introduced for this range of λ, which can be used to furnish alternative proofs of the results for $\lambda \leq 1/m$ presented earlier.

Theorem 3.5. *The polynomial $H_{\lambda,m}(z)$ has at least $m-2$ unimodular roots.*

Recall $H_{\lambda,m}(z) = F_{\lambda,m}(z)/(z-1)$. Then

$$H_{\lambda,m}(z) = (1-\lambda)z^m - 2\lambda(z + \cdots + z^{m-1}) + (1-\lambda). \tag{3.15}$$

This time, the roots are grouped according to the parity of m. Details are given when m is even, the case m odd is left to the reader.

Assume m is even, say $m = 2n$. Denote $u^{(k)} = z^k + 1/z^k$. Then

$$z^{-n} H_{\lambda,2n}(z) = -2\lambda + (1-\lambda)u^{(n)} - 2\lambda \sum_{k=1}^{n-1} u^{(k)}. \tag{3.16}$$

The key observation is the content of the next statement.

Lemma 3.6. *A complex number $z = e^{i\theta}$ is a root of $H_{\lambda,2n}(z) = 0$ if and only if the equation*

$$(1+\lambda)\sin\left(\frac{2n-1}{2}\theta\right) = (1-\lambda)\sin\left(\frac{2n+1}{2}\theta\right) \tag{3.17}$$

has a real solution θ.

Proof. Let $u = z + 1/z$ and replace it in (3.16) to obtain

$$\Phi(\theta) := \frac{1}{2} - \frac{1-\lambda}{2\lambda}\cos n\theta + \sum_{k=1}^{n-1}\cos k\theta = 0. \tag{3.18}$$

A use of the identity $\displaystyle\frac{1}{2} + \sum_{k=1}^{n-1}\cos k\theta = \frac{\sin((2n-1)\theta/2}{2\sin\theta/2}$ completes the proof. $\square$

The equation (3.17) may be expressed as

$$(1+\lambda)U_{2n-2}(w) = (1-\lambda)U_{2n}(w) \tag{3.19}$$

where $w = \cos\theta/2$ and U_n is the Chebyshev polynomial of the second kind. The interlacing of the roots of the polynomials $\{U_n\}_n$ is now employed to conclude that $H_{\lambda,m}(z) = 0$ has at least $m-2$ roots on the unit circle. Indeed, the zeros of the left-hand side of (3.17) are given by $\left\{0, \frac{2\pi}{2n-1}, \frac{4\pi}{2n-1}, \cdots, \frac{2(2n-1)\pi}{2n-1}\right\}$ and those of the right-hand side of (3.17) are $\left\{0, \frac{2\pi}{2n+1}, \frac{4\pi}{2n+1}, \cdots, \frac{2(2n+1)\pi}{2n+1}\right\}$. From these explicit values, and the interlacing of zeros of $\{U_n(w)\}_n$, it becomes certain that there are at least $2n-2$ crossings, which automatically renders as many roots of modulus 1 for $H_{\lambda,m}(z) = 0$. Observe that neither $\theta = 0$ nor $\theta = 2\pi$ contribute to the roots of $H_{\lambda,m}$, unless $\lambda = 1/m$.

The next result involves a head count of real roots. Note that for $\lambda \neq 1$, the polynomial $F_{\lambda,m}$ is of degree $m+1$, so the number of real roots is at most 3, with $z = 1$ always present.

Lemma 3.7. *Assume $1/m < \lambda < 1$. Aside from the roots at $z = 1$, the polynomial $F_{\lambda,m}(z)$ has exactly two other positive real zeros. It follows that $F_{\lambda,m}(z)$ has three real roots and $m-2$ unimodular roots.*

Proof. The first step is to verify that $F_{\lambda,m}(z)$ has at least three real zeros. This follows directly from the following facts: $F_{\lambda,m}(0) = \lambda - 1 < 0$, the slope at $z = 1$ is $F'_{\lambda,m}(0) = -2m(\lambda - 1/m) < 0$ and $F_{\lambda,m}(z) \sim (1-\lambda)z^{m+1}$ as $z \to \infty$.

Descartes' rule of signs [2] states that the number of positive roots of a polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by a multiple of 2. At present, the sequence of coefficients is $1-\lambda$, $-(1+\lambda)$, $1+\lambda$, $-(1-\lambda)$. In particular, for $\lambda < 1$ there are three sign changes. Thus, the number of positive roots is either 3 or 1. It follows that there are exactly three positive roots, as claimed. $\square$

Lemma 3.8. *Assume $\lambda > 1$ is real and m is an odd integer. Then if $1 < \lambda < m$, the polynomial $F_{\lambda,m}(z)$ has $z = -1$ as a root and two other negative real roots. The remaining $m-2$ roots are unimodular. For $\lambda \geq m$, the value $z = -1$ is a triple root and the remaining $m-2$ roots are unimodular.*

Proof. The result follows directly from the identity $\lambda F_{1/\lambda,m}(-z) = -F_{\lambda,m}(z)$. □

Lemma 3.9. *Assume $\lambda > 1$ is real and m is an even integer. Then $F_{\lambda,m}(z)$ has two distinct negative roots and $z = 1$ as the only positive root. The remaining $m-2$ roots are unimodular.*

Proof. The expression

$$F_{\lambda,2n}(-z) = (\lambda-1)z^{2n+1} - (\lambda+1)z^{2n} - (\lambda+1)z + (\lambda-1) \tag{3.20}$$

and the data $F_{\lambda,2n}(0) = \lambda - 1 > 0, F_{\lambda,2n}(-1) = -4$ and $F_{\lambda,2n}(z) \sim (\lambda-1)z^{2n+1}$ leads to the assertion. □

Lemma 3.10. *Assume $1/m < \lambda < 1$ is real and m is an even integer. Then $F_{\lambda,m}(z)$ has two distinct positive roots aside from $z = 1$. The remaining $m-2$ roots are unimodular.*

Proof. Observe that

$$G(z) = \frac{F_{\lambda,2n}(z)}{z-1} = (1-\lambda)z^{2n} - 2\lambda(z + z^2 + \cdots + z^{2n-1}) + (1-\lambda). \tag{3.21}$$

Then $G(0) = 1 - \lambda > 0$, $G(1) = 2(1 - m\lambda) < 0$ and $G(z) \sim (1-\lambda)z^{2n}$ yield the statement. □

Acknowledgements. The second author acknowledges the partial support of NSF-DMS 0713836.

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