Duality of Abelian Groups

A project for the mathematics graduate program in the lagniappe semester at Tulane University, Summer 2006

1. Background

¿From basic linear algebra one knows that every finite dimensional vector space V has a *dual* $\widehat{V}$, the vector space of all linear maps from V to the ground field.
If $V = \mathbb{R}^n$, then an element $\omega \in \hat{V}$ acts on a vector $x = (x_1, \dots, x_n)$ in the form $\omega(x) = a_1x_1 + \cdots + a_nx_n$ for a suitable n-tuple $(a_1, \dots, a_n)$. Thus $\hat{V}$ is again isomorphic to $\mathbb{R}^n$. Solving a system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

now amounts to the following: Let W be the vetor space spanned by

$$\omega_1 = (a_{11}, \dots, a_{1n}), \dots, \omega_m = (a_{m1}, \dots, a_{mn}),$$

in $\hat{V}$ we must find the space of solutions $W^\perp$ of all vectors $x \in V$ such that $\omega(x) = 0$ for all $\omega \in W$. (A method called the Gauss algorithm yields one way of solving this problem.)

This dual space $\hat{V}$ captures the entire structure of V since $\hat{\hat{V}}$, the *double dual* (that is, the dual of the dual) is naturally isomorphic to V itself. For infinite dimensional real vector spaces duals can be constructed in exactly the same fashion, but the double dual is vastly greater than the original space—unless one uses a bit of topology or functional analysis and equips $\widehat{V}$ with the so-called topology of pointwise convergence and allows in $\widehat{\widehat{V}}$ only the *continuous* functionals $\widehat{V} \to \mathbb{R}$. Then once again, $V \cong \widehat{\widehat{V}}$, and the whole duality remains valid as in the finite dimensional situation.
This set-up works for an abelian group and indeed for an abelian topological group A if one takes as dual $\widehat{A}$ the abelian group of all continuous homomorphisms $A \to \mathbb{T}$ where $\mathbb{T}$ is the additive topological group $\mathbb{R}/\mathbb{Z}$ of all real numbers modulo 1, that is, he group which is isomorphic to the group of all complex numbers of absolute value. These homomorphisms are called *characters*, $\widehat{A}$ is also called the *character group* of A, while $\mathbb{Z}$ is called the *circle group*. If the group A is sufficiently tame, say, locally compact or a vector space of the type discussed above, then $A \cong \widehat{\widehat{A}}$, but this is not always the case. There is recent research interest in a class of abelian topological groups A which one can describe as closed subgroups of groups of the form $R^I \times \mathbb{T}^J \times \mathbb{Z}^K$ for the additive group $\mathbb{R}$ of real numbers, the circle group $\mathbb{T}$, the discrete additive group $\mathbb{Z}$ of integers, and suitable sets I, J, and K. Let us call such groups A *pro-groups* (because the pros are interested in them). It is not fully understood yet how pro-groups behave under duality.

Location of Duality of Abelian Groups in the Universe of Mathematics
—Pure Mathematics
—Interdisciplinary in the triangle of Algebra, Topology, and Analysis
—Topological Algebra and Abstract Harmonic Analysis
—Relevant Traditions at Tulane: The algebra of abelian groups, topological groups and semigroups, functional analysis.

The Project.

(i) We shall study the basics of the duality of topological abelian groups. The seminar supervisor will give some introductory lectures explaining the basic ideas.
(ii) The participants in the seminar will read selected textbook literature and original articles and report to the seminar on well defined assignments.
(iii) The instructor will be available for individual consultations during the study period of the lagniappe semester.
(iv) The research projects include the computations of character groups of familiar abelian groups and, as original research projects, not so familiar groups in the domain of pro-groups.
(v) We are using literature that is either available in the library or that will be provided by the instructor.

Prerequisites.

We need a good knowledge of linear algebra, some topology such as might be provided in basic analysis courses or in basic point set topology courses. Motivated undergraduates and any graduate student having passed first year courses should be amply qualified to follow the seminar. One of the purposes of the seminar, however, is to fill in prequisites, should they be missing, either in the plenary sessions or in individual advising.

References.

[1] Hewitt, E., and K. A. Ross, Abstract Harmonic Analysis I, Springer-Verlag Berlin etc., 1963.

[A classical source book with a strong analysis flavor]

[2] Hofmann, K. H., and S. A. Morris, The Structure of Compact Groups, de Gruyter Berlin, 1998, xvii+835pp.

[A source book from which relevant selections will be taken, notably from Appendix 1 for abelian groups, Chapters 1, 7, 8 for duality. A second edition of this book will appear in 2006.]

[3] Hofmann, K. H., S. A. Morris, and D. Poguntke, *The exponential function of locally connected compact abelian groups*, Forum Mathematicum **16** (2003), 1–16

[4] Hofmann, K. H., and S. A. Morris, *The Structure of abelian pro-Lie groups*, Mathematische Zeitschrift **248** (2004), 867–891

[Original research projects are to be selected from the articles [3,4] under the supervision of the instructor.]

[5] Hofmann, K. H., and S. A. Morris, The Lie Theory of Connected Pro-Lie Groups, EMS Publishing House, Zürich, 2006, xii+671 pp., to appear.

[Parts of this book will be provided by the instructor.]

Instructor
Karl H. Hofmann