AN INTRODUCTION TO p-ADIC ANALYSIS

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The p-adic numbers were *invented* at the end of the 19^{th}-century in order to analyze some diophantine equations. My original interest in these issues came from learning a beautiful theorem that states *a quadratic form* $ax^2 + bxy + cy^2 = 0$ has solutions in $\mathbb{Z}$, if and only if it has solutions in $\mathbb{R}$ and in every p-adic field $\mathbb{Q}_p$. Essentially this means that $ax^2+bxy+cy^2 \equiv 0 \bmod p$ can be solved. The fact that this in not true for higher order equations is even more remarkable: Selmer proved that $3x^3 + 4y^3 + 5z^3 = 0$ has no integer solutions (aside from $x = y = z = 0$) but it has solutions modulo p for every prime p.

The course will begin by asking a simple question: suppose we have a norm defined on $\mathbb{Q}$, that is, $\|x\| \geq 0$, $\|xy\| = \|x\| \cdot \|y\|$ and some kind of triangle inequality. We would like to know what is the completion of $\mathbb{Q}$ under this norm. It turns out that you either have $\mathbb{R}$ or one of the p-adic fields.

The subject is a fascinating combination of Algebra, Number Theory, Analysis and Topology. The fact that convergence is defined differently leads to amusing identities, such as $\sum_{n=1}^{\infty} n \cdot n! = -1$.

The emphasis of the course will be in developing enough tools to understand a result of H. Cohn.

Theorem 1. *Let $g(n)$ be the number of tilings of a $2n \times 2n$ square with 1×2 dominos. It is known that $g(n) = 2^n \times f(n)^2$, where $f(n)$ is an odd positive integer. Then f is uniformly continuous in the 2-adic metric.*

My current interest in p-adic numbers comes from recent work with T. Amdeberham and D. Manna on the 2-adic valuation of Stirling numbers. These numbers, $S(n,k)$, count the number of ways to split a set of n elements into k non-empty subsets. They satisfy the recurrence $S(n,k) = S(n-1,k-1)+kS(n-1,k)$. We are interested in the exact power of 2 that divides $S(n,k)$, this is called the 2-adic valuation and is denoted by ν_2. This problem occured to us when we discovered that

$$\nu_2(S(2^n,k)) = s_2(k) - 1, \tag{1}$$

where $s_2(k)$ is the sum of the binary digits of k. It turns out that we knew about $s_2(k)$ from our previous. We expected that $\nu_2(S(n,k))$ could follow from there, and we have some partial results This was a naive thought. Lundell and Clarke have established, under a conjectured statement, the following:

Theorem 2. *Consider the function $f_{0,5}(x) = 5 + 10 \cdot 3^x + 5^x$. Let u_0 be the unique zero of $f_{0,5}(x)$ that satisfies $u_0 \in 2\mathbb{Z}_2$ and u_1 be the unique zero of $f_{0,5}(x)$ that satisfies $u_1 \in 1 + 2\mathbb{Z}_2$. Here $\mathbb{Z}_2$ is the ring of 2-adic integers. Then*

$$\nu_2(S(n,5)) = \begin{cases} -1 + \nu_2(n - n_0) & \text{if } n \text{ is even,} \\ -1 + \nu_2(n - n_1) & \text{if } n \text{ is odd.} \end{cases}$$

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There is much more to be done in this problem. The graphs of the 2-adic valuations of $S(n,k)$ describe the complexity of this problem. Why does 104 look so different than 195?. My hope is that the course will motivate students to solve this question.

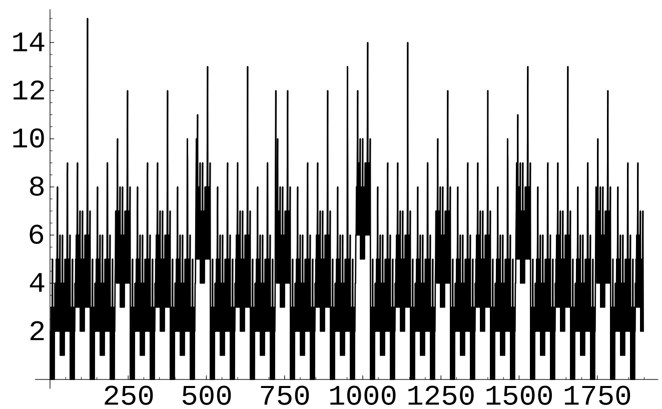

FIGURE 1. The data for $S(n,104)$.

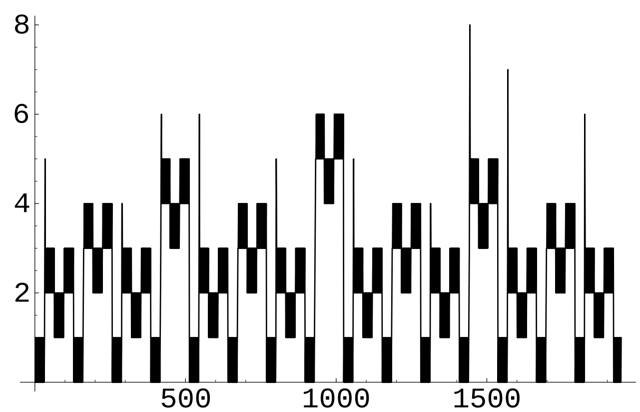

FIGURE 2. The data for $S(n,195)$.

The course will be run as a seminar. There will be some introductory talks given by me. The rest of the semester will consist of presentation by the students.

Books for background.

(1) F. Gouvea. p-adic numbers. Springer-Verlag, 1997.

Papers that will be presented by students.

(1) T. Amdeberham, D. Manna, V. Moll. The 2-adic valuation of a sequence arising from a rational integral. Submitted to INTEGERS, 2006
(2) T. Amdeberham, D. Manna, V. Moll. The 2-adic valuation of Stirling numbers. Preprint
(3) F. Clarke. Hensel's lemma and the divisibility by primes of Stirling-like numbers. Journal of Number Theory, 52, 69-84, 1995.
(4) H. Cohn. 2-adic behavior of domino tilings. Electronic Journal of Combinatorics, 6, 1-14,1999.
(5) D. M. Davis. Divisibility by 2 of Stirling-like numbers. Proc. American Mathematical Society 110, 597-600, 1990.
(6) I. Gessell-T. Lengyel. On the order of Stirling numbers and alternating binomial coefficients sums. Fibonnacci Quarterly, 39, 444-454, 2001.
(7) F. T. Howard. Congruences for the Stirling numbers and associated Stirling numbers. Acta Arithmetica 55, 29-41, 1990.
(8) H. Kwong. Minimum periods of $S(n,k)$ modulo M. Fibonnacci Quarterly 27, 217-221, 1989.
(9) H. Kwong. Periodicities of a class of infinite integer sequences modulo M. Journal of Number Theory 31, 64-79, 1989.
(10) T. Lengyel. On the divisibility by 2 of the Stirling numbers of the second kind. Fibonnacci Quarterly, 32, 194-201, 1994.
(11) T. Lengyel. Characterizing the 2-adic order of the logarithm. Fibonnacci Quarterly, 32, 397-401, 1994.
(12) T. Lengyel. The order of the Fibonnacci and Lucas numbers. Fibonnacci Quarterly, 33, 234-239, 1995.
(13) A. Lundell. A divisiblity property for Stirling numbers. Journal of Number Theory, 10, 35-54, 1978.
(14) A. Nijenhuis-H. Wilf. Periodicities of partition functions and Stirling numbers modulo p. Journal of Number Theory, 25, 308-312, 1987.
(15) S. De Wannemacker. On the 2-adic orders of Stirling numbers of the second kind. INTEGERS, 5(1), A-21, 2005.

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