MATH-1110 (DUPRÉ) FALL 2011 TEST 3 ANSWERS

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF EACH SHEET TURNED IN.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME ON EACH SHEET TURNED IN.

THIRD: WRITE YOUR FALL 2011 MATH-1110 SECTION NUMBER DIRECTLY UNDERNEATH YOU FIRST NAME ON EACH SHEET TURNED IN.

FOURTH: THERE ARE $N = 15$ **QUESTIONS AND EACH IS WORTH** $n = 20/3$ **POINTS. WRITE ALL YOUR ANSWERS NEATLY IN THE SPACE PROVIDED UNDER EACH QUESTION. NEATNESS COUNTS. IF I CANNOT READ IT WITHOUT STRAINING MY EYES YOU GET NO CREDIT.**

Suppose that X **is a random variable and that** $E(X) = 10$ **and the standard deviation of** X **is** $\sigma_X = 5$. **Let** T **denote the total of 16 observations of** X **and let** $\bar{X}$ **denote the average of 16 observations of** X.

1. What is $E(T)$?

In general, we have $E(T) = nE(X)$, so here $E(T) = 16 \cdot 10 = 160$.

2. What is $E(\bar{X})$?

In general we have $E(\bar{X}) = E(X)$, so here $E(\bar{X}) = 10$.

3. Assuming the 16 observations are all independent of each other, what is the standard deviation of T?

In general, for Independent observations, $\sigma_T = \sqrt{n} \cdot \sigma_X$, so here we have $\sigma_T = \sqrt{16} \cdot 5 = 20$.

4. Assuming the 16 observations are all independent of each other, what is the standard deviation of $\bar{X}$?

In general, for independent observations,

$$\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}},$$

so here we have

$$\sigma_{\bar{X}} = \frac{5}{\sqrt{20}} = \frac{5}{4} = 1.25.$$

5. Assuming the population size is $N = 20$ and sampling without replacement, what is the standard deviation of T?

Here we must take the standard deviation for T from the calculation for the case of independent observations and multiply by the correction factor for simple random sampling.

In general, this is

$$[\sigma_T]_{SRS} = [\sigma_T]_{IRS} \cdot \sqrt{\frac{N-n}{N-1}} \quad \text{and} \quad [\sigma_{\bar{X}}]_{SRS} = [\sigma_{\bar{X}}]_{IRS} \cdot \sqrt{\frac{N-n}{N-1}}$$

so here the result is

$$\sigma_T = 20 \cdot \sqrt{\frac{20-16}{20-1}} = 20 \cdot \sqrt{\frac{4}{19}} = \frac{20 \cdot 2}{\sqrt{19}} = \frac{40}{\sqrt{19}}, \quad \text{or about } 9.18.$$

6. If 60 percent of the voters in Duckburg support Proposition A which requires Scrooge to pay higher taxes, and if 20 are asked in a random sample, what is the probability that at most 10 will be found to support Proposition A?

A given duck either does or does not say yes to support for Proposition A, so the yes count here obeys the binomial distribution with $n = 20$ and success rate $p = 0.6$. To say the yes count total T is at most 10 is to say that $T \leq 10$, so we need $P(T \leq 10)$ which is the cumulative distribution function from Table 1 on page 684 of the textbook. We find in that table that

$$P(T \leq 10) = 0.245.$$

Suppose that the weight of Ducks in Duckburg is normally distributed with standard deviation 1.6 pounds. Suppose that an independent random sample of 16 ducks has a mean weight of 3.6 pounds.

7. What is the MARGIN OF ERROR in the 95 percrent confidence interval for true mean weight of ducks in Duckburg?

We know the distribution here is normal and we know the population standard deviation, so the MARGIN OF ERROR which I will denote by ME is given by

$$\text{ME} = z_C \cdot \frac{\sigma}{\sqrt{n}}, \quad \text{where } C = .95.$$

The value of z_C, for 95 percent confidence is the value which cuts off a tail of area $A = 0.025$, so from table 4 for t with infinite degrees of freedom, we see that $z_C = 1.960$, so the margin of error is

$$\text{ME} = (1.96)\frac{1.6}{\sqrt{16}} = (1.96)\frac{1.6}{4} = (1.96)(.4) = 0.784,$$

or

$$\text{ME} = 0.784.$$

8. What is the significance of this data as evidence that the true mean weight of ducks in Duckburg actually EXCEEDS 3 pounds?

This is a HYPOTHESIS TEST. We want to know the significance of the data for establishing $\mu > 3$, so as we seek to prove this, it must be the alternate hypothesis. The null hypothesis is $H_0: \ \mu \leq 3$, and this means we actually assume that $\mu_0 = 3$ under the null hypothesis. The alternate hypothesis is

$$H_{alt}: \ \mu > 3,$$

so the significance of the data is

$$\text{SIGNIFICANCE} = P(Z \geq z_{data}),$$

where z_{data} is the result of standardizing our value of $\bar{x}$ from the data using the assumed mean $\mu_0 = 3$ and the standard deviation $\sigma = 1.6$. Therefore

$$z_{data} = \frac{\bar{x} - \mu_0}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{3.6 - 3}{\frac{1.6}{\sqrt{16}}} = \frac{.6}{.4} = 1.5$$

Therefore the significance of the data is

$$\text{SIGNIFICANCE} = P(Z \geq 1.5) = P(Z \leq -1.5) = 0.0668,$$

from Table 3 in the textbook.

9. What is the significance of this data as evidence that the true mean weight of the ducks in Duckburg is LESS than 4 pounds?

This is just like the previous problem, a hypothesis test, but now the alternate hypothesis is $\mu < 4$, and the null hypothesis is $H_0: \ \mu \geq 4$, which means we assume the true mean to be $\mu_0 = 4$ under our null hypothesis. Our standardization of the data now becomes

$$z_{data} = \frac{\bar{x} - \mu_0}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{3.6 - 4}{\frac{1.6}{\sqrt{16}}} = \frac{-.4}{.4} = -1.$$

From the alternate hypothesis

$$H_{alt}: \ \mu < 4,$$

we see the significance of our data is

$$\text{SIGNIFICANCE} = P(Z \leq z_{data}) = P(Z \leq -1) = 0.1587,$$

from Table 3.

10. What is the significance of this data as evidence that the true mean weight of the ducks in Duckburg is NOT EQUAL to 4 pounds?

Again, we have a hypothesis test, but now our alternate hypothesis is

$$H_{alt}: \ \mu \neq 4,$$

which means it is a two-tail test. After all, a sample mean weight of 4.4 is just as significant as 3.6, if the assumed true mean is 4. In terms of standardized scores, this is the same as saying $z_{data} = -1$ has the same significance on the downside as $z_{data} = 1$ would have on the upside. Therefore, the significance of the data is now

$$\text{SIGNIFICANCE} = P(Z \leq -1) + P(Z \geq 1) = 2 \cdot P(Z \leq -1) = 2(.1587) = 0.3174.$$

11. Suppose that the population standard deviation of duck weight for Duckburg ducks is unkwown to us. Suppose that our sample had a standard deviation of 1.6 pounds. What would be the MARGIN OF ERROR in the 95 percent confidence interval?

When we do not know the population standard deviation, we must use the $t-$distribution in place of the standard normal distribution. The population must be assumed normal in order to use the $t-$distribution and the duck weights are given to be normal. Since the degrees of freedom for the $t-$distribution is $df = n - 1$, here we have the standardization of $\bar{x}$ using s in place of σ must have the $t-$distribution for $df = n - 1 = 16 - 1 = 15$ degrees of freedom. Our margin of error is now

$$\text{ME} = t_C \cdot \frac{s}{\sqrt{n}}.$$

Since the level of confidence is $C = 0.95$, or 95 percent, we must find the $t-$ score which cuts off a right hand tail of area $A = .025$, using the $t-$distribution for 15 degrees of freedom. From Table 4, in the textbook, we see that $t_C = 2.131$, so our margin of error is

$$\text{ME} = (2.131)\frac{1.6}{\sqrt{16}} = (2.131)(.4) = 0.8524.$$

Suppose that we do not know the true percentage of ducks in Duckburg that support Proposition B. Suppose that we ask 20 ducks and only 8 say they support Proposition B.

12. What is the significance of this data as evidence that the true proportion of Duckburg ducks that support Proposition B is LESS than 50 percent?

This is a hypothesis test concerning a true proportion with a small sample, so we must calculate the significance using the binomial distribution directly, as it may not be normal.

The alternate hypothesis is $H_{alt}: \ p < .5$, so under the null hypothesis we assume $p = .5$ in our calculations. The sample size is $n = 20$, and our data value of the total number T of ducks who say yes to supporting proposition B is only 8, and T is govenerned by the binomial distribution with $n = 20$ and $p_0 = .5$. Therefore when we see the alternate hypothesis

$$H_{alt}: \ p < .5,$$

we see that the significance of our data is

$$\text{SIGNIFICANCE} = P(T \leq 8) = 0.252,$$

from Table 1 in the textbook.

13. What is the significance of this data as evidence that the true proportion of Duckburg ducks that support Proposition B is MORE than 30 percent?

Again this is a hypothesis test, but now we assume $p_0 = .3$ for the null hypothesis value of the true proportion. Thus, the yes count T has the binomial distribution with $n = 20$ and $p_0 = .3$. Our alternate hypothesis we want our data to prove (we hope) is

$$H_{alt}: \ p > .3,$$

so the significance of our data is

$$\text{SIGNIFICANCE} = P(T \geq 8) = 1 - P(T < 8) = 1 - P(T \leq 7) = 1 - .772 = 0.228,$$

from Table 1. We can also here notice that proving $p_{yes} > .3$ is the same as proving $p_{no} < .7$, so we could just as well be looking at the alternate hypothesis

$$H_{alt}: \ p_{no} < .7$$

and the significance of our data can be calculated with the distribution of T_{no}, the total count of ducks saying no to support for Proposition B. Under the null hypothesis T_{no} has the binomial distribution with $n = 20$ and $p_0 = .7$, and our data has $T_{no} = 12$. Therefore

$$\text{SIGNIFICANCE} = P(T_{no} \leq 12) = 0.228,$$

using Table 1 for $n = 20$ and $p = .7$.

14. What is the significance of this data as evidence that the true proportion of Duckburg ducks that support Proposition B is NOT EQUAL to 50 percent?

The significance of the data for proving $p < .5$ was 0.252, so the significance for proving $p \neq .5$ is twice that value or

$$\text{SIGNIFICANCE} = 2 \cdot P(T \leq 8) = 2(0.252) = 0.504.$$

15. If we wish to make a 95 percent confidence interval for the true proportion of Duckburg ducks that support Proposition B, and if we need the MARGIN OF ERROR to be no more than .01, that is at most 1 percentage point, then how big must our sample be (that is, what is the smallest it can be and still get the job done)?

The margin of error formula is

$$\text{ME} = z_C \frac{\sigma}{\sqrt{n}},$$

which if we require

$$\text{ME} \leq E$$

means

$$n \geq \left(\frac{z_C \cdot \sigma}{E}\right)^2.$$

We do not know σ, but we do know $\sigma = \sqrt{p(1-p)} \leq 1/2$, for dealing with binomial success counts. For 95 percent confidence we have $z_C = 1.96$, and therefore as here we have $E = .01$, we must have

$$n \geq \left(\frac{(1.96)(1/2)}{.01}\right)^2 = \left(\frac{196}{2}\right)^2 = 98^2 = 9604.$$