MATH-1150 (DUPRÉ) FALL 2011 TEST 3 ANSWERS

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF EACH SHEET TURNED IN.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME ON EACH SHEET TURNED IN.

THIRD: WRITE YOUR SPRING 2011 MATH-1150 SECTION NUMBER DIRECTLY UNDERNEATH YOU FIRST NAME ON EACH SHEET TURNED IN.

FOURTH: Write NEATLY and CLEARLY, putting your answers in the space provided. If I cannot read it you do not get credit.

Suppose that

$$f(u,v) = 28uv^2 - 9u^2v^3.$$

We can easily see that

$$f(2,1) = (28)(2) - (9)(4) = 56 - 36 = 20.$$

Therefore the point $(2,1)$ **is on the curve with equation** $f(u,v) = 20$ **which is a curve in the** $(u,v)-$**plane.**

1. Suppose that u and v are both functions of time and the function z of time is given by $z = f(u,v)$. Use the chain rule to express dz/dt in terms of $u, v, du/dt$, and dv/dt.

We just use the differentiation rules to differentiate $z = 28uv^2 - 9u^2v^3$, with respect to t, since u and v are functions of t. The result is

$$\frac{dz}{dt} = 28\frac{du}{dt} \cdot v^2 + 28u \cdot 2v\frac{dv}{dt} - 9 \cdot 2u\frac{du}{dt} \cdot v^3 - 9u^2 \cdot 3v^2\frac{dv}{dt}.$$

2. Suppose that a beetle is crawling along the graph of the curve $z = f(u,v) = 20$ so that the rate of increase of u for the beetle is 3 units per second at the instant the beetle passes through the point $(2,1)$. What is the rate of increase of z for the beetle at the instant the beetle passes through the point $(2,1)$?

Since the beetle is crawling on the curve $z = 20$, the value of z is constant in time and therefore

$$\frac{dz}{dt} = 0.$$

3. Suppose that a beetle is crawling along the graph of the curve $z = f(u, v) = 20$ so that the rate of increase of u for the beetle is 3 units per second at the instant the beetle passes through the point $(2, 1)$. What is the rate of increase of v for the beetle at the instant the beetle passes through the point $(2, 1)$?

We can now combine the equations from the previous two problems and see that

$$0 = \frac{dz}{dt} = 28\frac{du}{dt} \cdot v^2 + 28u \cdot 2v\frac{dv}{dt} - 9 \cdot 2u\frac{du}{dt} \cdot v^3 - 9u^2 \cdot 3v^2\frac{dv}{dt}.$$

Into this equation we can then put the information that $u = 2,\ v = 1$, and $du/dt = 3$. This results in the equation

$$0 = 28 \cdot 3 + 28 \cdot 2 \cdot 2 \cdot \frac{dv}{dt} - 9 \cdot 2 \cdot 2 \cdot 3 - 9 \cdot 2^2 \cdot 3 \cdot \frac{dv}{dt}$$

or

$$0 = 84 + 112\frac{dv}{dt} - 108 - 108\frac{dv}{dt}$$

which is equivalent to

$$4\frac{dv}{dt} = 108 - 84 = 24,$$

or

$$\frac{dv}{dt} = \frac{24}{4} = 6,$$

so finally,

$$\frac{dv}{dt} = 6.$$

4. What is $\dfrac{dv}{du}$ at the point $(2, 1)$ on the curve $z = f(u, v) = 20$?

Since the chain rule tells us

$$\frac{dv}{dt} = \frac{dv}{du} \cdot \frac{du}{dt},$$

we have

$$6 = \frac{dv}{du} \cdot 3,$$

and therefore

$$\frac{dv}{du} = 2.$$

Suppose that a spherical balloon is being blown up so that the rate of increase of volume is 20π cubic centimeters per second at the instant that the surface area is 100π square centimeters.

5. At the instant the surface area of the balloon is 100π square centimeters, what is the rate of increase of the balloon's radius, in centimeters per second?

As a general principle, if a solid increases its volume, V due to a moving boundary, and if the surface area of the moving boundary is A at a specific instant and v is the normal or outward velocity of the moving boundary at that same instant, then the rate of increase of volume at that instant is

$$\frac{dV}{dt} = A \cdot v.$$

In our problem, the outward velocity is the rate of change of the radius, since the radius is perpendicular to the surface of a sphere at each point on that surface. That is, here we have

$$\frac{dr}{dt} = v.$$

Since we are told that at the instant of interest the volume is increasing at the rate of 20π cubic centimeters per second and the area is $A = 100\pi$, we have

$$20\pi = \frac{dV}{dt} = A \cdot v = 100\pi \cdot \frac{dr}{dt},$$

so

$$\frac{dr}{dt} = \frac{20\pi}{100\pi} = \frac{1}{5} = 0.2 \text{ cm per sec.}$$

6. Using the fact that the surface area is $A = 4\pi r^2$, for a sphere of radius r, what is the rate of increase of the balloon's surface area at the instant that it is 100π square centimeters?

Using $A = 4\pi r^2$, we must have

$$\frac{dA}{dt} = 4\pi 2r \cdot \frac{dr}{dt} = 8\pi \cdot \frac{1}{5} r.$$

We need to find r. But at the instant we are interested in,

$$100\pi = A = 4\pi r^2,$$

so

$$r^2 = 25,$$

which means

$$r = 5.$$

Therfore,

$$\frac{dA}{dt} = 8\pi \frac{1}{5} \cdot 5 = 8\pi.$$

Suppose that the function f with domain $[0,6]$ is defined by

$$f(x) = x(6-x),\ \ 0 \leq x \leq 6.$$

Suppose that $R(x)$ is a rectangle with two sides on the coordinate axes and with a vertex at the origin $(0,0)$ and with the vertex opposite the origin at the point $(x, f(x)),\ \ 0 \leq x \leq 6$. Let A denote the area of the recatangle as a function of x, so $A(x)$ denotes the area of $R(x)$ for $0 \leq x \leq 6$.

7. What is the expression for $A(x)$ in terms of x alone, for $0 \leq x \leq 6$?

$$A(x) = x \cdot f(x) = x \cdot x(6-x) = x^2(6-x) = 6x^2 - x^3.$$

8. What is the derivative of $A(x)$ for $0 < x < 6$?

$$A'(x) = \frac{dA}{dx} = 12x - 3x^2 = 3x(4-x).$$

9. What are the critical points of A on the interval $0 < x < 6$?

To find the critical points, we look for points in the interval $0 < x < 6$ where either A' fails to exist or else does exist but equals zero. Since A' exists for all x in the given interval, we look for solutions of $A'(x) = 0$, that is we begin by solving

$$3x(4-x) = 0.$$

The solutions are $x = 0$ and $x = 4$, but only $x = 4$ is in the interval $0 < x < 6$, so the answer is only $x = 4$.

10. What is the second derivative of A on the interval?

Since $A'(x) = 12x - 3x^2$, it follows that

$$A''(x) = 12 - 6x = 6(2-x).$$

11. What are the inflection points of A on the interval $0 < x < 6$?

We have $A''(x) = 6(2-x)$, and this is clearly zero when $x = 2$, is positive for $x < 2$, and negative for $x > 2$. This means A is concave up when $x < 2$ and concave down when $x > 2$, so $x = 2$ is the only value of x at which A has an inflection point. Since it is in the interval $0 < x < 6$, the answer is $(2, A(2))$ and as $A(2) = 2^2(6-2) = 4 \cdot 4 = 16$, we see that A has an inflection point at $(2, 16)$, that is at $x = 2$ the function A has an inflection point in $0 < x < 6$.

12. What is the value of x for which the area $A(x)$ is maximum for $0 \leq x \leq 6$?

Since A is continuous on the interval $[0,6]$ and $x = 4$ is the only critical point in $0 < x < 6$ we need only check and compare $A(0)$, $A(4)$, and $A(6)$. The largest value is the maximum area. Since $A(0) = 0 = A(6)$, and $A(x) \geq 0$ on the interval $[0,6]$ it follows that the maximum must be when $x = 4$. Also, $A''(4) = 6(2-4) = -12 < 0$, so by the second derivative test A has a local maximum at $x = 4$. The maximum area is therefore

$$A(4) = 4^2(6-4) = 16 \cdot 2 = 32.$$

As the answer to the question asked is the value of x for which $A(x)$ is maximum, the final answer is simply

$$x = 4.$$

Calclate the following integrals and derivatives.

13. $\displaystyle\int_0^4 \sqrt{x}dx = \int_0^4 x^{1/2}dx = \frac{x^{3/2}}{3/2}|_0^4 = \frac{2}{3}4^{3/2} = \frac{2}{3}2^3 = \frac{16}{3}.$

14. $\displaystyle\int \frac{x^3 - 4x^5}{x^7}dx = \int [x^{-4} - 4x^{-2}]dx = \frac{x^{-3}}{(-3)} - 4 \cdot \frac{x^{-1}}{(-1)} + C = \frac{4}{x} - \frac{1}{3x^3} + C$

15. $\displaystyle\frac{d}{dx}\int_{-2}^{x}\left[\sqrt{t^4 + t^2 + 2}\right] dt = \sqrt{x^4 + x^2 + 2}$

Remember, for any a, and for any antiderivative F for the continuous function f we have

$$\int_a^x f(t)dt = F(x) - F(a),$$

and for F to be an antiderivative of f means simply $F' = f$, so

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x).$$

Remember, $F(a)$ is just a constant, so its derivative is zero.