MATH-1150 (DUPRÉ) SPRING 2011 TEST 3 ANSWERS

Wednesday 13 April 2011

DIRECTIONS

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF EACH SHEET TURNED IN.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME ON EACH SHEET TURNED IN.

THIRD: WRITE YOUR CORRECT SPRING 2011 MATH-1150 SECTION NUMBER DIRECTLY UNDERNEATH YOU FIRST NAME ON EACH SHEET TURNED IN.

FOURTH: Write NEATLY and CLEARLY, putting your answers in the space provided. If I cannot read it you do not get credit.

FIFTH: Any failure to follow any part of any of the above directions can result in additional loss of credit.

Suppose that a bee travels a path through the air from his beehive and that a stop watch and high speed movie camera is used to record the bee's flight. The results show that the distance s along his path of the bee from his beehive after t seconds is

$$s(t) = t(8t - t^2),\ 0 \leq t \leq 8.$$

1. What is ds/dt as a function of time t?

SOLUTION: We can use the product rule here,

$$\frac{ds}{dt} = 1 \cdot (8t - t^2) + t \cdot (8 - 2t),$$

or, we can remove the parenthesis in the expression for s(t) using the distributive law:

$$s(t) = t(8t - t^2) = 8t^2 - t^3,$$

so now to find the derivative we just use the power rule on each term:

$$\frac{ds}{dt} = 16t - 3t^2.$$

You notice the two answers appear different but on closer examination they are seen to be algebraically equivalent:

$$(8t - t^2) + t(8 - 2t) = 8t - t^2 + 8t - 2t^2 = 16t - 3t^2,$$

that is, both answers are actually the same.

2. What is the velocity of the bee along his path when $t = 2$?

SOLUTION: Denoting the velocity at time t by $v(t)$, we have

$$v(t) = \frac{ds}{dt} = 8t - t^2 + t(8 - 2t^2) = 16t - 3t^2,$$

so when $t = 2$ we have velocity

$$v(2) = 8 \cdot 2 - 2^2 + 2(8 - 2 \cdot 2) = 16 - 4 + 2(4) = 12 + 8 = 20,$$

or using $v(t) = 16t - 3t^2$, we have more quickly

$$v(2) = 16 \cdot 2 - 3 \cdot 2^2 = 32 - 12 = 20.$$

Thus either way, the result is the same, at $t = 2$ the velocity of the bee is 20.

FINAL ANSWER: 20

3. What is the acceleration of the bee along his path when $t = 2$?

SOLUTION: The acceleration $a(t)$ is the derivative of the velocity $v(t)$,

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(16t - 3t^2) = 16 - 6t,$$

so at $t = 2$ the velocity is

$$a(2) = 16 - 6 \cdot 2 = 16 - 12 = 4.$$

Using the expression $v(t) = (8t - t^2) + t(8 - 2t)$, we have

$$a(t) = 8 - 2t + 1 \cdot (8 - 2t) + t \cdot (-2) = 16 - 4t + t(-2) = 16 - 6t,$$

so we get the same result, $a(2) = 4$.

FINAL ANSWER: 4

4. What is the average rate of change of s over the time interval $[1, 3]$?

SOLUTION: The average rate of change of s over the interval $[1, 3]$ is

$$m = \frac{s(3) - s(1)}{3 - 1} = \frac{3(8 \cdot 3 - 3^2) - 1 \cdot (8 \cdot 1 - 1^2)}{3 - 1} = \frac{45 - 7}{2} = \frac{38}{2} = 19.$$

FINAL ANSWER: 19

Calculate the limits.

5. $\lim_{x\to 0} \frac{\sin(x^4)}{x^3}$

SOLUTION: The quick way here is to recall that for any very small number, say u, we have we have $\sin(u)$ is essentially the same as u itself, so the expression in the limit as $x \to 0$ will be the same as the expression $x^4/x^3 = x$ which certainly goes to zero as $x \to 0$.

To be more precise here, we can also quickly use L'Hopital's Rule, since both the numerator and denominator have limit zero individually. Thus this limit is of the form $0/0$. This means we can differentiate the numerator and denominator individually and take the limit of the quotient of the derivatives

$$\lim_{x\to 0} \frac{\sin(x^4)}{x^3} = \lim_{x\to 0} \frac{[\cos(x^4)][4x^3]}{3x^2} = \lim_{x\to 0} \frac{4x\cos(x^4)}{3} = \frac{4\cdot 0^4 \cdot 1}{3} = 0.$$

This problem is also easy simply using the fact from Chapter 2 in the textbook where we learned that

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1,$$

so using $\theta = x^4$, we have by the product rule for limits and substitution,

$$\lim_{x\to 0} \frac{\sin(x^4)}{x^3} = [\lim_{x\to 0} x][\lim_{x\to 0} \frac{\sin(x^4)}{x^4}] = [\lim_{x\to 0}][\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}] = 0 \cdot 1 = 0.$$

FINAL ANSWER: 0

6. $\lim_{x\to 1} \dfrac{x^4 - 7x + 6}{x^3 - 4x + 3}$

SOLUTION: Again, this limit is of the form 0/0, so L'Hopital's Rule applies for the fastest result.

$$\lim_{x\to 1} \frac{x^4 - 7x + 6}{x^3 - 4x + 3} = \lim_{x\to 1} \frac{4x^3 - 7}{3x^2 - 4} = \frac{4-7}{3-4} = 3.$$

Of course, the expression can be factored. Since both the numerator and denominator vanish when $x = 1$, it follows that both numerator and denominator have $x-1$ as a common factor. The other factor for the factorization in each case can be found using long division of polynomials. Also, in the case at hand, we can just multiply out:

$$x^4 = ([x-1]+1)^4 = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$$

and therefore

$$\begin{aligned} x^4 - 7x + 6 &= (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1 - 7[(x-1)+1] + 6 \\ &= (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 - 3(x-1) \\ &= (x-1)[(x-1)^3 + 4(x-1)^2 + 6(x-1) - 3]. \end{aligned}$$

Likewise,

$$x^3 = ([x-1]+1)^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1,$$

so

$$\begin{aligned} x^3 - 4x + 3 &= (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 - 4[(x-1)+1] + 3 \\ &= (x-1)^3 + 3(x-1)^2 - (x-1) \\ &= (x-1)[(x-1)^2 + 3(x-1) - 1] \end{aligned}$$

and therefore

$$\frac{x^4 - 7x + 6}{x^3 - 4x + 3} = \frac{(x-1)^3 + 4(x-1)^2 + 6(x-1) - 3}{(x-1)^2 + 3(x-1) - 1},$$

making it obvious that

$$\lim_{x\to 1} \frac{x^4 - 7x + 6}{x^3 - 4x + 3} = \frac{-3}{-1} = 3.$$

FINAL ANSWER: 3

Suppose that u and v are both functions with domain $(-100, 100)$ and that both are twice differentiable on their domain. Suppose also that

$$e^u - \tan(u^2 + v) = uv^2.$$

7. Give the equation relating $u,\ v,\ du/dt$, and dv/dt.

SOLUTION: Just differentiate the equation on both sides with respect to t, using the chain rule and the power, sum and product rules for differentiation. We get

$$e^u \cdot \frac{du}{dt} - [\sec^2(u^2+v)]\left[2u\frac{du}{dt} + \frac{dv}{dt}\right] = \frac{du}{dt} \cdot v^2 + u \cdot \left(2v\frac{dv}{dt}\right)$$

8. If $du/dt = 4$ when $u = 0$, then what is dv/dt when $u = 0$?

SOLUTION: First, we need to find the numerical value of v when $u = 0$. When we substitute $u = 0$ in the original equation we have right away $1 - \tan(v) = 0$, and therefore $\tan(v) = 1$, which means as an angle v would be a 45 degree angle making $v = \pi/4$. In any case, we have $v = \arctan(1) = \pi/4$.

Next, we can substitute these values for u and v as well as the value for du/dt from the previous problem and find

$$1 \cdot 4 - [\sec^2(v)]\frac{dv}{dt} = 4\left(\frac{\pi}{4}\right)^2.$$

Since $\sec^2 = 1 + \tan^2$, we must have $\sec^2(v) = 1 + \tan^2(v) = 1 + 1 = 2$, so

$$4 - 2\frac{dv}{dt} = 4\left(\frac{\pi}{4}\right)^2,$$

and therefore

$$\frac{dv}{dt} = 2 \cdot \left[\left(\frac{\pi}{4}\right)^2 - 1\right] = \frac{\pi^2 - 16}{8}.$$

Kate is flying her kite. At 10am, her kite was 400 feet above the ground with 500 feet of string out stretched straight from her spool of string to her kite. At 10am, the wind pulling the kite caused the kite to move horizontally away from her at 3 feet per second and to gain altitude at 4 feet per second. In fact, at 10am, the wind caused the kite to accelerate horizontally away from Kate at 15 feet per second per second and accelerate vertically at the rate of 10 feet per second per second.

SOLUTION NOTE: From the description of the relative positions of Kate, the kite, and the ground, it should be clear that our picture sould be of a right triangle with one side horizontal, one side verticle, and with the hypoteneuse connecting Kate to the kite. Let the horizontal side have length x, and the verticle side have length y, and let the hypoteneuse have length z. Then by the Pythagorean Theorem,

$$x^2 + y^2 = z^2.$$

At 10am, we have $y = 400$ and $z = 500$, so our equation tells us $x = 300$ at 10am.

9. How fast (in feet per second) was her kite string going out of her spool of string at 10am?

SOLUTION: We can simply differentiate the equation with respect to time t to find

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

Cancelling, we have then

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

Since the kite is moving horizontally away from Kate at 3 feet per second at 10am, we have $dx/dt = 3$. Since the kite is gaining altitude at 4 feet per second at 10am, we have $dy/dt = 4$. We can therefore put $x = 300,\ y = 400,\ dx/dt = 3,\ dy/dt = 4$, and $z = 500$ into the equation for the derivatives and find

$$300 \cdot 3 + 400 \cdot 4 = 500 \cdot \frac{dz}{dt}.$$

Dividing through by 100, we have

$$9 + 16 = 5\frac{dv}{dt},$$

or

$$\frac{dv}{dt} = 5,$$

which means that at 10am, the string on Kate's spool is going out at the rate of 5 feet per second.

FINAL ANSWER: 5

10. How fast was the string accelerating (in feet per second per second) out of her string spool at 10 am?

SOLUTION: The information given tells us that at 10 am we have

$$\frac{d^2x}{dt^2} = 15,$$

and

$$\frac{d^2y}{dt^2} = 10.$$

From the previous problem, we have

$$x^2 + y^2 = z^2.$$

and

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

To find the relationship between the second derivatives, we just differentiate the last equation again. Using the product rule on each term, the result is

$$\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2} + \left(\frac{d^2y}{dt}\right)^2 + y\frac{d^2y}{dt^2} = \left(\frac{dz}{dt}\right)^2 + z\frac{d^2z}{dt^2}.$$

From the previous problem, we know that at 10am we have

$$x = 300,\ y = 400,\ z = 500,\ \frac{dx}{dt} = 3,\ \frac{dy}{dt} = 4,\ \text{and}\ \frac{dz}{dt} = 5.$$

Putting all these numbers into the last equation gives

$$3^2 + 300 \cdot 15 + 4^2 + 400 \cdot 10 = 5^2 + 500 \cdot \frac{dz}{dt},$$

and since $3^2 + 4^2 = 5^2$, we have

$$300 \cdot 15 + 400 \cdot 10 = 500 \cdot \frac{dz}{dt}.$$

Dividing through both sides of the equation by 100, we have

$$3 \cdot 15 + 4 \cdot 10 = 5 \cdot \frac{dz}{dt},$$

which means

$$\frac{dz}{dt} = \frac{3 \cdot 15 + 4 \cdot 10}{5} = 3 \cdot 3 + 4 \cdot 2 = 9 + 8 = 17.$$

Therefore, at 10 am, string is accelerating out of Kate's string spool at 17 feet per second per second.

FINAL ANSWER: 17