MATH-1150 (DUPRÉ) SPRING 2011 TAKE HOME TEST 1

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF EACH SHEET TURNED IN.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME ON EACH SHEET TURNED IN.

THIRD: WRITE YOUR SPRING 2011 MATH-1150 SECTION NUMBER DIRECTLY UNDERNEATH YOU FIRST NAME ON EACH SHEET TURNED IN.

1. DEFINITIONS AND THEOREMS

You may only use the following theorems in each problem in the **PROBLEMS** section.

Theorem 1.1. *Any constant function is continuous. If $D \subset \mathbb{R}$ and if $h : D \longrightarrow \mathbb{R}$ is the rule given by $h(x) = x$, for all $x \in D$, then h is continuous. If f and g are continuous functions then so are $f+g$, $f \cdot g$, f/g, and if $f(x) \geq 0$, for all $x \in D$, and if $p \in \mathbb{R}$, then f^p is continuous.*

Definition 1.1. If $B \subset \mathbb{R}$, then $b \in \mathbb{R}$ is in the **Closure** of B if and only if every open interval which contains b must intersect B. If $A \subset \mathbb{R}$, and $b \in \mathbb{R}$, then b is a **Limit Point** of A if and only if b is in the closure of $A \setminus \{b\}$.

Theorem 1.2. *Suppose that $D \subset E \subset \mathbb{R}$ and $c \in E$ is a limit point of D. Suppose that $f : D \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$ and that*

$$f(x) = g(x), \text{ for all } x \in D.$$

Further, suppose that g is continuous at c. Then f has a limit as x approaches c, and it is given by

$$\lim_{x \to c} f(x) = g(c).$$

2. WORKED EXAMPLES

In each of the following problems justify your answer using the theorems and definitions above.

Example 2.1. Suppose that D is the open interval with endpoints 3 and 5. Show that the closure of D is the closed interval $[3, 5]$.

Solution: If the open interval J contains the point $x = 3$, then there is a positive number $r > 0$ such that the open interval $(3-r, 3+r)$ is contained in J. But, $3+r/2$ then must belong to both J and D. Therefore every open interval which contains $x = 3$ must intersect D. Similarly, replacing 3 by 5 and 3+r/2 by 5-r/2, we conclude every open interval which contains 5 must intersect D. Therefore 3 and 5 are in the closure of D. On the other hand, if b is in $\mathbb{R} \setminus [3, 5]$, then either b is in the open interval $(-\infty, 3)$ or in the open interval $(5, \infty)$ and these are open intervals which do not intersect D, and therefore the closure of D is exactly $[3, 5]$.

Example 2.2. Suppose that D is the open interval

$$D = \{x \in \mathbb{R} \ : \ x > 0\}$$

and $f : D \longrightarrow \mathbb{R}$ is given by

$$f(x) = \frac{x^2 + 4x^{3/2}}{x}, \ x \in D.$$

Give the reason why f has a limit as x approaches 0 and find the limit.

Solution: We can simplify the expression for $f(x)$ and find

$$f(x) = \frac{x^2 + 4x^{3/2}}{x} = x^{1/2} + 4 = 4 + \sqrt{x}, \ x \in D,$$

and we can define the function $g : [0, \infty) \longrightarrow \mathbb{R}$ by the rule

$$g(x) = 4 + \sqrt{x}, \ x \in [0, \infty).$$

According to definition 1.1, 0 is a limit point of $D \subset \mathbb{R}$, and according to theorem 1.1, g is continuous and therefore continuous at 0, since 0 is in its domain. Therefore by theorem 1.2 the limit of f as x approaches 0 is $g(0) = 4$, which is to say

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^3 + 4x^2}{x^2} = g(0) = 4 + \sqrt{0} = 4 + 0 = 4.$$

3. PROBLEMS

FIVE PROBLEMS EACH WORTH TWENTY POINTS

In each of the following problems justify your answer (in the space provided) using the theorems and definitions above.

1. Suppose that D is the result of removing the closed interval $[2, 3]$ from the closed interval $[1, 4]$, in other words, $D = [1, 4] \setminus [2, 3]$. What is the closure of D and why?

2. Suppose that $D = [1, 4] \setminus [2, 3]$ and that $f : D \longrightarrow \mathbb{R}$ is defined by

$$f(x) = x^3 - 5x^2 + 8x + 1, \ \text{if } x \in [1, 2),$$

and

$$f(x) = \sqrt{x}, \ \text{if } x \in (3, 4].$$

Tell why f has limits as x approaches 2 and as x approaches 3 and give

$$\lim_{x \to 2} f(x) \text{ and } \lim_{x \to 3} f(x).$$

3. Suppose that $D = [1,5] \setminus \{4\}$ and that $f : D \longrightarrow \mathbb{R}$ is defined by

$$f(x) = x^2 - 4x + 1, \text{ if } x \in [1,4),$$

and

$$f(x) = \sqrt{x}, \text{ if } x \in (4,5].$$

Tell why f does NOT have a limit as x approaches 4.

4. Explain why the following limit exists and find it.

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{2x - 4}$$

5. Let $\mathbb{Q} \subset \mathbb{R}$ be the set of all rational numbers. We know $\pi \in \mathbb{R}$ is an irrational number, so

$$\pi \in \mathbb{R} \setminus \mathbb{Q}.$$

Is π a limit point of $\mathbb{Q}$? Give the reasons for your answer.