CALCULUS PRACTICE TEST PROBLEM ANSWERS

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1. Draw a picture of a non-zero vector and beside it draw a picture of its negative.

ANSWER: $\longrightarrow$ and $\longleftarrow$

2. Draw a picture of two non-zero vectors which are not parallel to each other. Draw the picture of the sum of these two vectors using head to tail addition.

ANSWER: Shifiting it parallel to itself, draw a copy of the second vector with its tail at the tip of the first vector. Then draw a straight line from the tail of the first to the tip of the new copy of the second vector and put the arrowhead on the end which touches the tip of the new copy of the second vector. The resulting arrow is the head to tail addition of the two vectors.

3. Draw a picture of a non-zero vector X and beside it draw a picture of $(.5)X$.

ANSWER: $\longrightarrow$ and $\rightarrow$

4. Draw a picture of two non-zero vectors X and Y which are not parallel to each other, positioned so their tails are at the same point. Draw the difference vector $X - Y$.

ANSWER: Simply draw the line segment connecting the tips of the two vectors and put the arrowhead on the end of the segment touching the tip of X.

5. A plane is flying in darkness and its airspeed indicator is showing 400 miles/hour and its compass is indicating that the plane is flying due north. The plane is flying in the jet stream which at the instant of interest happens to be flowing due east at 300 miles/hour. What is the plane's true ground speed?

ANSWER: The airspeed indicator tells the pilot the speed of the plane relative to the air and the compass heading tells the pilot the direction the nose of the plane is pointed. The plane is in the jet stream so the plane's velocity relative to the ground is the vector sum of the velocity relative to the air and the velocity relative to the ground. Relative to the air, the plane's velocity is due north, so if $V_{p,a}$ is the velocity vector of the plane relative to the air, it is pointing due north and has length 400. If $V_{a,g}$ denotes the velocity of the air in

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the jet stream relative to the ground, then that is a vector pointing due west of length 300. Therefore, if $V_{p,g}$ denotes the plane's velocity relative to the ground, then

$$V_{p,g} = V_{p,a} + V_{a,g}.$$

Using head to tail addition, we see that the vector sum $V_{p,g}$ is the arrow forming the hypotenuse of a right triangle of sides 300 and 400, so by the Pythagorean Theorem, the speed of the plane relative to the ground is 500 miles per hour,

$$\text{Ground speed} = \|V_{p,g}\| = \sqrt{300^2 + 400^2} = 500.$$

6. A plane is flying in darkness and its airspeed indicator is showing 100 miles/hour and its compass is indicating that the plane is flying due north. The plane is flying in the jet stream which at the instant of interest happens to be flowing due east at 100 miles/hour. What is the plane's true ground speed and true direction relative to the ground?

ANSWER: As in the previous problem, we have in this case the vector sum picture forming a right triangle with equal short sides, so now the plane must be moving northeast at

$$\text{Groundspeed} = 100\sqrt{2} \text{ miles per hour},$$

or approximately 141 miles per hour.

7. A 20 mile stretch of beach is currently eroding at the rate of 2 miles per century. What is the current rate of land loss due to this erosion in square miles per century?

ANSWER: The rate of change of area due to a moving boundary is simply the length of the moving boundary multiplied by its outward velocity. As the erosion moves the boundary into the land, the outward velocity is negative 2 miles per century, so the rate of increase of land area is $(-2)(20 = -40$ square miles per century. The negative of the rate of increase is the rate of land loss, so the rate of land loss is 40 square miles per century.

8. An oil spill is partially contained, but there are two breaks in the containment, one currently being 500 feet long where the oil is spreading out from the spill at the rate of 3 feet per minute, the second break currently being 800 feet long where the oil is spreading out at 2 feet per minute. What is the current rate of increase in the area of the oil spill in square feet per minute?

ANSWER: The total rate of increase in area is the sum of the rates due to each of the moving boundaries, so if the area is A, then $\dot{A}$ is the rate of increase of area of the oil spill, so

$$\dot{A} = (500)(3) + (800)(2) = 1500 + 1600 = 3100 \text{ square feet per minute}.$$

9. A rectangle is growing because the sides are changing in length. Currently, one side is 12 inches long and growing at the rate of 3 inches per second and the other side is 5 inches long and growing at the rate of 2 inches per second. What is the rate of increase in the area of the rectangle in square inches per second?

10. Suppose that a rectangle has sides of length $x(t)$ and $y(t)$ at time t and that

$$x(t) = t^2 + t^3 \text{ and } y(t) = t^2 + t^4.$$

If $A(t)$ denotes the rectangle's area at time t, then what is $\dot{A}(t)$ expressed in terms of t?

ANSWER: Since $A = xy$, we have $A(t) = x(t)y(t)$, so

$$\dot{A}(t) = \dot{x}(t)y(t) + x(t)\dot{y}(t) = (2t + 3t^2)(t^2 + t^4) + (t^2 + t^3)(2t + 4t^3).$$

Alternately,

$$A(t) = x(t)y(t) = (t^2 + t^3)(t^2 + t^4) = t^4 + t^5 + t^6 + t^7,$$

so

$$\dot{A}(t) = 4t^3 + 5t^4 + 6t^5 + 7t^6.$$

This may appear to be a different answer at first, but

$$(2t + 3t^2)(t^2 + t^4) + (t^2 + t^3)(2t + 4t^3) = (2t^3 + 3t^2 + 2t^5 + 3t^6) + (2t^3 + 2t^4 + 4t^5 + 4t^6)$$

$$= 4t^3 + 5t^4 + 6t^5 + 7t^6,$$

so they are actually the same answer.

11. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are both smooth functions and that

$$f(2) = 5,\ g(2) = 7,\ f(7) = 4,\ g(5) = 6,\ f'(2) = 3,\ g'(2) = 8,\ f'(7) = 9,\ g'(5) = 10.$$

Calculate

$(f + g)'(2) = f'(2) + g'(2) = 3 + 8 = 11$

$(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (3)(7) + (5)(8) = 21 + 40 = 61$

$(f \circ g)'(2) = f'(g(2))g'(2) = f'(7)g'(2) = (9)(8) = 72$

$(g \circ f)'(2) = g'(f(2))f'(2) = g'(5)f'(2) = (10)(3) = 30$

$$(f/g)'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(3)(7) - (5)(8)}{7^2} = \frac{21 - 40}{49} = -\frac{19}{49}$$

12. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f' = f$ and $f(0) = 1$. Suppose that $f(x) \neq 0$, for all $x \in \mathbb{R}$. Suppose the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth and $g' = g$. Define the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ by the rule

$$h(x) = \frac{g(x)}{f(x)}, \ x \in \mathbb{R}.$$

Calculate $h'(5)$.

ANSWER: We can use the quotient rule to differentiate h, and use $f' = f$ with $g' = g$ to get

$$h'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{[f(x)]^2} = \frac{g(x)f(x) - g(x)f(x)}{[f(x)]^2} = 0, \text{ for all } x \in \mathbb{R}.$$

In particular, then with $x = 5$, we have

$$h'(5) = 0.$$

13. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are given by

$$f(x) = x^3 - 5 \text{ and } g(x) = x^2 + 4, \text{ for any } x \in \mathbb{R}.$$

Calculate

ANSWERS: To calculate the answers to the problems below, we need to calculate

$$f(2), \ g(2), \ f'(2), \ g'(2), \ f'(g(2)), \ g'(f(2)).$$

We see that

$$f(x) = x^3 - 5, \ \text{ so } f(2) = 8 - 5 = 3 \ \text{ and } g(x) = x^2 + 4, \text{ so } g(2) = 4 + 4 = 8.$$

Differentiating f and g we have

$$f'(x) = 3x^2 \text{ and } g'(x) = 2x,$$

so

$$f'(2) = (3)(4) = 12, \ g'(2) = (2)(2) = 4, \ f'(g(2)) = f'(8) = (3)(64) = 192,$$

and

$$g'(f(2)) = g'(3) = (2)(3) = 6.$$

To summarize, we have

$$f(2) = 3,\ g(2) = 8,\ f'(2) = 12,\ g'(2) = 4, f'(g(2)) = 192,\ g'(f(2)) = 6.$$

Therefore

$(f + g)'(2) = f'(2) + g'(2) = 12 + 4 = 16$

$(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (12)(8) + (3)(4) = 96 + 12 = 108$

$(f \circ g)'(2) = f'(g(2))g'(2) = (192)(4) = 768$

$(g \circ f)'(2) = g'(f(2))f'(2) = (6)(12) = 72$

$(f/g)'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \dfrac{(12)(8) - (3)(4)}{[8]^2} = \dfrac{84}{64} = \dfrac{21}{16}$

Alternately, we can differentiate the combinations of the functions using the rules for arbitrary x and substitute $x = 2$ after the differntiation is done. As

$$f(x) = x^3 - 5 \text{ and } g(x) = x^2 + 4,$$

it follows that

$$f'(x) = 3x^2 \text{ and } g'(x) = 2x.$$

Therefore

$(f + g)'(x) = 3x^2 + 2x,$ so $(f + g)'(2) = (3)(4) + (2)(2) = 12 + 4 = 16$

$(fg)'(x) = (3x^2)(x^2 + 4) + (x^3 - 5)(2x),$ so $(fg)'(2) = (12)(8) + (3)(4) = 108$

$(f \circ g)'(x) = f'(g(x))g'(x) = 3[g(x)]^2(2x) = 3[x^2 + 4]^2(2x)$, so

$$(f \circ g)'(2) = (3)(8)^2(4) = (12)(64) = 768$$

$(g \circ f)'(x) = g'(f(x))f'(x) = 2f(x)f'(x) = 2(x^3 - 5)(3x^2)$, so

$$(g \circ f)'(2) = (2)(3)(12) = 72$$

$(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \dfrac{3x^2(x^2 + 4) - (x^3 - 5)(2x)}{[x^2 + 4]^2}$, so

$$(f/g)'(2) = \frac{(12)(8) - (3)(4)}{64} = \frac{21}{16}$$

14. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f' = f$ and $f(0) = 1$, and suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f(g(x)) = x$, for all $x \in \mathbb{R}$. Calculate $g'(2)$.

ANSWER: Look at the equation

$$f(g(x)) = x.$$

Differentiate both sides, using the chain rule on the left side to get

$$f'(g(x))g'(x) = 1.$$

Now substitute $f' = f$ in this last equation to get

$$f(g(x))g'(x) = 1,$$

and substitute $x = f(g(x))$ from the equation we started with into the equation above to get

$$xg'(x) = 1$$

and therefore

$$g'(x) = \frac{1}{x},\ x \neq 0.$$

In particular, as $2 \neq 0$, we have

$$g'(2) = \frac{1}{2}.$$

15. Suppose that W is the set of all vectors in three dimensional space and that U_1, U_2, U_3 are vectors in W so that each has unit length and all are perpendicular to each other. Suppose that

$$X = 3U_1 - 2U_2 + U_3 \text{ and } Y = 2U_1 + 5U_2 - 2U_3.$$

Calculate

$U_2 \cdot (X + Y) = -2 + 5 = 3$

$X \cdot Y = (3)(2) + (-2)(5) + (1)(-2) = 6 - 12 = -6$

ANSWER: Since U_1, U_2, U_3 are all mutually perpendicular and of unit length, it follows that

$$U_j \cdot U_j = \|U_j\|^2 = 1^2 = 1 \text{ and } U_j \cdot U_k = 0, \text{ if } j \neq k, \text{ for } 1 \leq j \leq 3,\ 1 \leq k \leq 3.$$

Therefore

$U_2 \cot X = 3(U_2 \cdot U_1) - 2(U_2 \cdot U_2) + (U_2 \cdot U_3) = (3)(0) - (2)(1) + 0 = -2.$

Notice that likewise we would have $U_1 \cdot X = 3$ and $U_3 \cdot X = 1$.

More generally,

$$\text{for } W = aU_1 + bU_2 + cU_3, \text{ we have } U_1 \cdot W = a,\ U_2 \cdot W = b, \text{ and } U_3 \cdot W = c,$$

so always,

$$W = (W \cdot U_1)U_1 + (W \cdot U_2)U_2 + (W \cdot U_3)U_3.$$

In particular, as $X = 3U_1 - 2U_2 + U_3$,
we have

$$X \cdot W = 3(U_1 \cdot W) - 2(U_2 \cdot W) + U_3 \cdot W = 3a - 2b + c.$$

Therefore

$$X \cdot Y = (3)(2) + (-2)(5) + (1)(-2) = 6 - 10 - 2 = -6.$$

In general,

$$\text{if } X = x_1U_1 + x_2U_2 + x_3U_3 \text{ and } Y = y_1U_1 + y_2U_2 + y_3U_3,$$

then

$$X \cdot Y = x_1y_1 + x_2y_2 + x_3y_3,$$

$$X + Y = (x_1 + y_1)U_1 + (x_2 + y_2)U_2 + (x_3 + y_3)U_3,$$

$$\text{and for any real number } r, \text{ we have } rX = (rx_1)U_1 + (rx_2)U_2 + (rx_3)U_3.$$

If we agree that we fix the mutually perpendicular unit vectors U_1, U_2, U_3, in three dimensional space, then we can simply write $X = (x_1, x_2, x_3)$ in place of $X = x_1U_1 + x_2U_2 + x_3U_3$, and we have the rules

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3),\ r(x_1, x_2, x_3) = (rx_1, rx_2, rx_3),$$

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3.$$

16. In the space below, draw a picture of two non-zero vectors which are not parallel to each other. Label one vector X and the other vector Y. As well, draw the pictures of $-Y$, $X + Y$, and $X - Y$. Also draw the picture of $(1/2)(X + Y)$ using the parallelogram picture for $X + Y$.

ANSWER: Once you have drawn the two vectors, simply reverse the arrow labeled Y and label it $-Y$ then draw X and Y with their tails together and complete the parallelogram. The diagonal arrow of the parallelogram with tail at the point where the tails of X and Y meet and tip at the opposite vertex of the parallelogram is $X + Y$. Now draw the arrow with tail at the tip of Y and tip at the tip of X making the other diagonal of the parallelogram. This arrow is a picture of $X - Y$. The point where the diagonals meet gives the tip of $(1/2)(X + Y)$ and the tail is again the same point of the parallelogram which coincides with the tails of X and Y.

17. A swimmer is swimming in a river. At a given instant, the swimmer is moving at 4 feet per second relative to the water. At that same instant, the water is moving at 3 feet per second relative to the Earth. Relative to the water, the direction the swimmer moves is in the direction from his feet to his head, which is the direction of his body. Give the swimmer's speed relative to the Earth in the following three cases for the direction of his body: with the water flow, against the water flow, and across the water flow.

ANSWER: The swimmer's velocity vector relative to the Earth is always the vector sum of his velocity relative to the water and the velocity of the water. Therefore when his body is in the direction of the flow, his speed relative to the Earth is the sum of his speed relative to the water and the speed of the water, as the velocity vectors point in the same direction, which is therefore 7 feet per second. When the swimmer's body direction is against the water flow, the velocity vectors point in opposite directions, and therefore in this case the swimmer's speed relative to Earth is $4 - 3 = 1$ foot per second. When the swimmer's body direction is across the water flow, then the two vectors are perpendicular, so the vector sum is an arrow which is the hypotenuse of the right triangle formed by the two vectors, so the speed is the hypotenuse length of a right triangle with sides of length 3 and 4, which is therefore 5 feet per second by the Pythagorean Theorem.

18. Paradise Island has a coast consisting of two beaches separated by stable rocky coast. One beach is 8 miles long and eroding at the rate of 3 miles per century and the other beach is 10 miles long and eroding at the rate of 2 miles per century. What is the rate of land loss in square miles per century due to the erosion along the 8 mile beach? What is the total rate of land loss due to erosion of both beaches combined?

ANSWER: The general principle here is that the rate of change of area due to a moving boundary is the length of the moving boundary multiplied by the outward or normal velocity of that moving boundary, assuming that all points of that moving boundary have the same normal velocity. If there are several parts of the boundary which move, then the total rate of change of area is the sum of the rates due to the individual pieces which move. Thus, the rate of land loss due to the 8 mile beach is $8 \cdot 3 = 24$ square miles per century. The rate of land loss due to the 10 mile beach is likewise $2 \cdot 10 = 20$ square miles per century and therefore the total rate of land loss due to the erosion of both beaches is $24 + 20 = 44$ square miles per century.

19. If $\dot{x} = 1$ and $y = 2x^3 - 5x^2 + 3$, then what is $\dot{y}$ in terms of x?

ANSWER: Using the rules for computing rates we know so far, together with $\dot{x} = 1$ we find

$$\dot{y} = 2 \cdot 3x^2\dot{x} - 5 \cdot 2x\dot{x} + 0 \cdot \dot{x} = 6x^2 - 10x.$$

20. An eagle is coasting in a horizontal circle, by making use of updrafts. The circle has a radius of 50 feet and the eagle moves around this circle with a constant speed of 20 feet per second. What is the eagle's acceleration in feet per second per second?

ANSWER: The velocity vector is always tangent to the circle of motion and has constant length of 20. However, as the eagle travels the circle, his velocity vector changes and is not constant, only his speed is constant. For a circle of radius r travelled at speed v, the length of the acceleration vector is $a = v^2/r$, so here the acceleration of the eagle is $a = (20)^2/50 = 400/50 = 8$ feet per second per second. The direction of the acceleration vector is from the eagle's position to the center of the circle, so likewise, the acceleration vector is not constant, only its length, a, is constant.

21. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are both smooth functions and that

$$f(5) = 3,\ g(5) = 2,\ f(2) = 4,\ g(3) = 7,\ f'(5) = 4,\ g'(3) = 2,\ f'(2) = 6,\ g'(5) = 10.$$

Calculate

ANSWERS:

$$(f - g)'(5) = f'(5) - g'(5) = 4 - 10 = -6$$

$$(fg)'(5) = f'(5)g(5) + f(5)g'(5) = (4)(2) + (3)(10) = 38$$

$$(f \circ g)'(5) = f'(g(5))g'(5) = f'(2)g'(5) = (6)(10) = 60$$

$$(g \circ f)'(5) = g'(f(5))f'(5) = g'(3)f'(5) = (2)(4) = 8$$

$$(f/g)'(5) = \frac{f'(5)g(5) - f(5)g'(5)}{[g(5)]^2} = \frac{(4)(2) - (3)(10)}{2^2} = -\frac{11}{2}$$

22. Calculate $f'(x)$ if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is

$$f(x) = \frac{x^5 - 8}{x^4 + 2x^2 + 3}.$$

ANSWER:

$$f'(x) = \frac{(x^5 - 8)'(x^4 + 2x^2 + 3) - (x^5 - 8)(x^4 + 2x^2 + 3)'}{(x^4 + 2x^2 + 3)^2},$$

therefore,

$$f'(x) = \frac{5x^4(x^4 + 2x^2 + 3) - (x^5 - 8)(4x^3 + 4x)}{(x^4 + 2x^2 + 3)^2}.$$

23. Suppose $g : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth functions with $f' = f$ and $g' = g$.

If $h = (f/g)$, what is $h'(2)$?

ANSWER: Use the quotient rule for differentiation and the equations $f' = f$ and $g' = g$ giving

$$h' = \frac{f'g - fg'}{g^2} = \frac{fg - fg}{g^2} = \frac{0}{g^2} = 0.$$

Therefore $h'(x) = 0$ for any $x \in \mathbb{R}$, so in particular,

$$h'(2) = 0.$$

24. Suppose that

$$f(x) = x^2(6 - x) = 6x^2 - x^3.$$

Give all the values of x at which f has a local maximum.

Give all the values of x at which f has an inflection point.

Find the area of the region above the $x-$axis, under the curve $y = f(x)$, between the vertical lines $x = 2$ and $x = 4$.

ANSWERS:

Give all the values of x at which f has a local maximum.

ANSWER: $f'(x) = 12x - 3x^2 = 3x(4 - x)$, so $f' = 0$ for $x = 0$ and for $x = 4$. Also, $f''(x) = 12 - 6x = 6(2 - x)$, so $f''(0) = 12 > 0$ and $f''(4) = -12 < 0$. Therefore f has a local maximum at $x = 4$ and a local minimum at $x = 0$.

FINAL ANSWER: $x = 4$

Give all the values of x at which f has an inflection point.

ANSWER: Since $f''(x) = 6(2 - x)$ we see that f'' vanishes at $x = 2$, that $f'' > 0$, for $x < 2$ and $f'' < 0$ for $x > 2$, so f is concave up for $x \leq 2$ and concave down for $x \geq 2$. Therefore f has an inflection point at $x = 2$.

FINAL ANSWER: $x = 2$

Find the area of the region above the $x-$axis, under the curve $y = f(x)$, between the vertical lines $x = 2$ and $x = 4$.

ANSWER:

Denoting the area by A, we have

$$A = \int_2^4 [6x^2 - x^3]\, dx = \left[6 \cdot \frac{x^3}{3} - \frac{x^4}{4}\right]_2^4 = \left[\frac{6 \cdot 4^3}{3} - \frac{4^4}{4}\right] - \left[\frac{6 \cdot 2^3}{3} - \frac{2^4}{4}\right] = 4^3 - 12 = 52.$$

FINAL ANSWER: $A = 52$

25. Suppose that a toy store owner wants to fence off a rectangular area along the side of his store for an electric train display. He only needs to have fencing on three sides of the area since the side wall itself will serve as one side of the rectangle. If the total length of fencing he has is 24 feet, what is the area (in square feet) of the rectangle with the maximum area he can make?

ANSWER: Let x be the length of the side of the rectangle which is perpendicular to the wall and let y be the length of the side of the rectangle which is parallel to the wall. Since there have to be two sides of fencing perpendicular to the wall and only one side of fencing parallel to the wall, and we have 24 feet of fencing material, it follows that

$$2x + y = 24.$$

The area of the rectangle is $A = xy$. Since $y = 24 - 2x$, we can substitute this expression for y in the area expression getting area as a function of x alone. Therefore,

$$A(x) = xy = x(24 - 2x) = 24x - 2x^2.$$

To find the value of x where A has maximum value, we calculate $A'(x)$ and set it equal to zero and solve for x. We have

$$A'(x) = 24 - 4x = 4(6 - x),$$

which only vanishes when $x = 6$. If we calculate A'' we find $A''(x) = -4 < 0$, and therefore the graph of $A(x)$ is concave down and A must have a maximum value at $x = 6$. Then $y = 24 - 2x = 24 - 2 \cdot 6 = 12$. Therefore, for maximum area use 12 feet of fence for the side parallel to the wall and 6 feet of fence for each of the two sides perpendicular to the wall, for a maximum area $A = 6 \cdot 12 = 72$ square feet fenced in.

FINAL ANSWER: $A = 72$ **square feet**

26. Find the following antiderivatives.

ANSWERS:

$$\int e^x \, dx = e^x + C$$

$$\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C$$

$$\int x^{\pi} \, dx = \frac{x^{\pi+1}}{\pi + 1} + C$$

$$\int \ln x \, dx = [x \ln x] - x + C$$

27. Suppose that a ladybug is walking along the curve $4x^2+2y^2=6$. Let $\dot{x}$ denote the rate of change of the ladybug's $x-$coordinate and $\dot{y}$ denote the rate of change of the ladybug's $y-$coordinate.

Give an equation relating $x,\ y,\ \dot{x}$, and $\dot{y}$.

ANSWER:

Since both x and y are time dependent, we can differentiate the equation with respect to time on both sides to get

$$0=\dot{6}=(4x^2+2y^2\dot{)}=4\cdot 2x\dot{x}+2\cdot 2y\dot{y},$$

and therefore

FINAL ANSWER: $8x\dot{x}+4y\dot{y}=0$

At the instant the ladybug is at the point $(1,1)$, what is $\dot{y}$ if $\dot{x}=2$?

ANSWER: All we need to do is to put $x=1,\ \dot{x}=2$ and $y=1$ in the equation found in the previous problem. The result is

$$16+4\dot{y}=0,$$

and therefore

FINAL ANSWER: $\dot{y}=-4$

Notice that if $f:W\longrightarrow\mathbb{R}$ where $W\subset\mathbb{R}^2$, so $z=f(x,y)$, then

$$\dot{z}=M(x,y)\dot{x}+N(x,y)\dot{y}$$

for some functions $M,N:W\longrightarrow\mathbb{R}$. For instance, in the example above,

$$z=f(x,y)=4x^2+2y^2,$$

and the result is

$$M(x,y)=8x,\ \text{and } N(x,y)=4y.$$

If C is a constant and we consider the curve with equation $f(x,y)=C$, then we must have $\dot{z}=\dot{C}=0$.

This gives the equation

$$M(x,y)\dot{x}+N(x,y)\dot{y}=0,$$

so knowing x,y,M,N and one out of two of $\dot{x},\dot{y}$, then the other is also determined. If we consider the equation

$$\dot{z}=M(x,y)\dot{x}+N(x,y)\dot{y},$$

this holds no matter how x and y change as long as (x,y) stays in W, the domain of f.

Thus we can hold y constant and let x increase at rate $\dot{x} = 1$. This amounts to differentiating $f(x, y)$ treating y as constant and x as the independent variable. We denote the result $\partial_x f$. Then $\dot{x} = 1$ and $\dot{y} = 0$, since y is constant, so the equation

$$\dot{z} = M(x, y)\dot{x} + N(x, y)\dot{y}$$

gives

$$\partial_x f(x, y) = \dot{z} = M(x, y)(1) + N(x, y)(0) = M(x, y).$$

Similarly, holding x constant and differentiating $f(x, y)$ with respect to y, treated as the independent variable, we denote the result by $\partial_y f$, so in this case $\dot{x} = 0$ and $\dot{y} = 1$, and

$$\partial_y f(x, y) = \dot{z} = M(x, y)(0) + N(x, y)(1) = N(x, y).$$

We therefore always have

$$\dot{z} = [\partial_x f(x, y)]\dot{x} + [\partial_y f(x, y)]\dot{y}.$$

For instance, if

$$z = f(x, y) = 4x^2 + 2y^2,$$

then

$$\partial_x f(x, y) = 8x \text{ and } \partial_y f(x, y) = 4y,$$

and

$$\dot{z} = (8x)\dot{x} + 4y\dot{y}.$$

For the ladybug travelling on $4x^2 + 2y^2 = 6$, we have $z = 6$, so z is constant and therefore

$$\dot{z} = 0,$$

so

$$(8x)\dot{x} + (4y)\dot{y} = 0$$

is the final answer.

28. Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by the rule

$$f(x) = ax^3 + bx^2 + cx + d, \; x \in \mathbb{R}.$$

Here $a, b, c, d \in \mathbb{R}$ are fixed constants and $a \neq 0$.

Use the quadratic formula to find the critical points of f in terms of the coefficients a, b, c, d.

Give the condition on the coefficients necessary for the expression in the radical of the quadratic formula to be non-negative.

Give the inflection point of f in terms of the coefficients a, b, c, d and.

Show that the average of the two critical points is the inflection point.

Show that

$$f(x) = a\left[x + \frac{b}{3a}\right]^3 + \left[c - \frac{b^2}{3a}\right]x + d - a\left[\frac{b^3}{(3a)^3}\right], \; x \in \mathbb{R},$$

Show using the modified coefficients setting

$$\beta = -\frac{b}{3a}, \; \gamma = c + \beta \; \text{ and } \delta = d + a\beta^3,$$

we have

$$f(x) = a(x - \beta)^3 + \gamma x + \delta,$$

which effectively does away with the quadratic term.

Show that the graph of f is symmetric about its inflection point, that is, if $x = B$ is the inflection point, show that there is a function F so that

$$f(B \pm u) = f(B) \pm F(u), \; \text{for all } u \in \mathbb{R}.$$

This means that $(B, f(B))$ is the midpoint of the line segment joining $(B - u, f(B - u))$ to the point $(B, f(B + u))$, for any $u \in \mathbb{R}$, showing that the graph of f is symmetric about the point $(B, f(B))$.

Show that the slope of the line connecting the local maximum point to the local minimum point on the graph of f is $(2/3)f'(x_0)$, where x_0 is the inflection point of f.

ANSWERS:

The derivative of f is

$$f'(x) = 3ax^2 + 2bx + c.$$

To find the critical points here, we must solve the equation $f'(x) = 0$, which in the case at hand is a quadratic equation

$$3ax^2 + 2bx + c = 0.$$

The quadratic formula gives the two solutions of the quadratic equation

$$Ax^2 + Bx + C = 0$$

as

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

and for the equation

$$3ax^2 + 2bx + c$$

we have

$$A = 3a,\ B = 2b,\ \text{ and } C = c,$$

giving critical points

$$x = \frac{-(2b) \pm \sqrt{(2b)^2 - 4(3a)(c)}}{2(3a)} = \frac{2(-b) \pm \sqrt{2^2b^2 - 2^2(3ac)}}{2(3a)}.$$

so

$$x = \frac{2(-b) \pm 2\sqrt{b^2 - 3ac}}{2(3a)} = \frac{(-b) \pm \sqrt{b^2 - 3ac}}{(3a)},$$

and finally,

$$\text{critical points are } x = \frac{(-b) \pm \sqrt{b^2 - 3ac}}{(3a)} = -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{(3a)},$$

provided that

$$b^2 \geq 3ac,$$

to guarantee that the number in the square root radical is not negative.

In particular, if $b^2 < 3ac$, then there are no critical points.

To find the inflection points, we solve $f''(x) = 0$, and

$$f''(x) = (3ax^2 + 2bx + c)' = 2 \cdot 3ax + 2b = 2(3ax + b),$$

and this means there is always a single inflection point at

$$x = -\frac{b}{3a}.$$

Notice that for any numbers K and L the average of $K + L$ with $K - L$ is simply K and the last expression for the critical points

$$\text{critical points are } x = -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{(3a)},$$

gives the two points

$$x = -\frac{b}{3a} + \frac{\sqrt{b^2 - 3ac}}{(3a)} \text{ and } x = -\frac{b}{3a} - \frac{\sqrt{b^2 - 3ac}}{(3a)},$$

which then have average $-b/(3a)$ which is the inflection point.

To see that

$$f(x) = a\left[x + \frac{b}{3a}\right]^3 + \left[c - \frac{b^2}{3a}\right]x + d - a\left[\frac{b^3}{(3a)^3}\right], \ x \in \mathbb{R},$$

we show the right side of the equation simplifies to the original expression for $f(x)$ by cubing the binomial. Always

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3,$$

since

$$(u+v)^3 = (u+v)^2(u+v) = (u^2+2uv+v^2)(u+v) = (u^3+2u^2v+uv^2)+(u^2v+2uv^2+v^3).$$

Therefore

$$\begin{aligned} a\left[x + \frac{b}{3a}\right]^3 &= a\left[x^3 + 3x^2\frac{b}{3a} + 3x\left(\frac{b}{3a}\right)^2 + \left(\frac{b}{3a}\right)^3\right] \\ &= ax^3 + (3a)x^2\frac{b}{3a} + (3a)x\left(\frac{b}{3a}\right)^2 + a\left(\frac{b}{3a}\right)^3 \\ &= ax^3 + bx^2 + \left(\frac{b^2}{3a}\right)x + a\left(\frac{b}{3a}\right)^3. \end{aligned}$$

Substituting this final expression into the right hand side of the equation at the top for the cubic term, we have

$$a\left[x+\frac{b}{3a}\right]^3+\left[c-\frac{b^2}{3a}\right]x+d-a\left[\frac{b^3}{(3a)^3}\right]$$
$$=ax^3+bx^2+\left(\frac{b^2}{3a}\right)x+a\left(\frac{b}{3a}\right)^3+\left[c-\frac{b^2}{3a}\right]x+d-a\left[\frac{b^3}{(3a)^3}\right]$$
$$=ax^3+bx^2+cx+d=f(x), x\in\mathbb{R}.$$

Since

$$f(x)=a\left[x+\frac{b}{3a}\right]^3+\left[c-\frac{b^2}{3a}\right]x+d-a\left[\frac{b^3}{(3a)^3}\right],\ x\in\mathbb{R},$$

if we put

$$\beta=-\frac{b}{3a},\ \gamma=c+\beta,\text{ and }\delta=d+\beta^3=d-a\left[\frac{b}{3a}\right]^3,$$

then substituting these expressions into the expression

$$a(x-\beta)^3+\gamma x+\delta$$

gives

$$f(x)=ax^3+bx^2+cx+d=a(x-\beta)^3+\gamma x+\delta.$$

With

$$f(x)=a(x-\beta)^3+\gamma x+\delta,$$

we have

$$f'(x)=3a(x-\beta)^2+\gamma,$$

so the critical points are solutions of

$$3a(x-\beta)^2+\gamma=0,$$

an easy equation to solve, as then

$$(x-\beta)^2=-\frac{\gamma}{3a},$$

$$x-\beta=\pm\sqrt{\frac{-\gamma}{3a}},$$

$$x=\beta\pm\sqrt{\frac{-\gamma}{3a}},$$

To simplify notation here, let us substitute

$$B = \beta \text{ and } C = \sqrt{\frac{-\gamma}{3a}},$$

so

$$\gamma = -3aC^2$$

and the expression for $f(x)$ becomes

$$f(x) = a(x-B)^3 - 3aC^2x + \delta = a[(x-B)^3 - 3C^2x + D],$$

where

$$D = \frac{\delta}{a}.$$

The critical points are now simply $x = B \pm C$, as easily seen by tracking back through the equations defining B and C, or more easily just by differentiating the new expression for f and solving $f' = 0$.

Suppose now that $u \in \mathbb{R}$ and $x_{\pm} = B \pm u$ is put into f.

$f(x_{\pm}) = f(B \pm u) = a[\pm u]^3 - 3aC^2[B \pm u] + aD = aD - 3aC^2B + (\pm 1)au^3 - 3aC^2[(\pm 1)u]$
so
$f(x_{\pm}) = f(B \pm u) = a[D - 3C^2B] + (\pm 1)a[u^3 - 3C^2u] = f(B \pm u) \pm a[u^3 - 3C^2u]$.

Setting $u = 0$ we see that $f(B) = a[D - 3C^2B] = \delta + \gamma B = \delta + \gamma\beta$,
and

$$f(x_{\pm}) = f(B \pm u) = f(B) \pm a[u^3 - 3C^2u]$$

Define $F : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F(x) = a[x^3 - 3C^2x], \ x \in \mathbb{R}.$$

Then

$$f(x_{\pm}) = f(B \pm u) = f(B) \pm F(u),$$

and this shows that $f(B)$ is the average of $f(B-u)$ and $f(B+u)$ for any $u \in \mathbb{R}$, and therefore $(B, f(B))$ is the midpoint of the line segment joing $(B-u, f(B-u))$ to the point $(B+u, f(B+u))$, so all these points are on the line through the inflection point on the graph of f which is the midpoint of the points where the line intersects the graph. That is, if you draw any line through $(B, f(B))$, the inflection point on the graph of f, and if the line intersects the graph, then it intersects in two points and the inflection point is the midpoint of the line segment connecting those two points. Actually, you might think that it is possible for a line through $(B, f(B))$ to intersect at a points not of the form $(x \pm u, f(x \pm u))$, but if the line intersects in points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then puting $u = x_1 - B$, we have $x_1 = B + u$, so that means $(x_1 - u, f(x_1 - u))$ is a point on the graph, so if this point is not $(x_2, f(x_2))$, then letting $y = g(x)$ be the equation of this line, we have $f(x) - g(x)$ has value zero at 4 different points, namely $x_1, B, B - u, x_2$, and this impossible for a third degree polynomial, as it can have at most 3 roots.

We have

$$F(x) = a[x^3 - 3C^2 x],\ x \in \mathbb{R},$$

therefore

$$F(C) = a[C^3 - 3C^2 C] = a[C^3 - 3C^3] = -2aC^3$$

We can now see that as the critical points are $x_\pm = B \pm C$, that the critical values are

$$f(B \pm C) = f(B) \pm F(C) = f(B) \pm (-2aC^3) = a[D - 3C^2 B] \mp 2aC^3.$$

The slope of the line segment connecting the two local extreme points on the graph of f is the same as the slope of the line segment connecting either of these two points to the inflection point since that is the midpoint. The slope of this line is therefore the slope of a line through the two points $(B, f(B))$ and $(B + C, f(B + C) = (B + C, f(B) + F(C))$, so that is m where

$$m = \frac{(f(B) + F(C)) - f(B)}{C} = \frac{F(C)}{C} = \frac{-2aC^3}{C} = -2aC^2.$$

On the other hand, the slope of the tangent line through the inflection point has slope $f'(B)$ and as

$$f(x) = a[[(x - B)^3 - 3C^2 x + D]$$

and therfore

$$f'(x) = a[3(x - B)^2 - 3C^2]$$

so

$$f'(B) = -3aC^2 = (3/2)(-2C^2) = (3/2)m.$$

Therefore

$$m = (2/3)f'(B) = (2/3)f'(x_0),\ \text{if } x_0 \text{ is the inflection point of } f.$$

29. Suppose that an isosceles triangle is inscribed in a circle of radius R. Find the maximum possible area and the shape of the triangle which achieves that maximum area.

ANSWER:

We can by symmetry of the circle position the triangle so that its vertex where the two equal sides meet is at the left most point of the circle so by symmetry we see that if we put the center of the circle at $(0,0)$ in the plane, then the upper most vertex will be (x,y) in the first quadrant, as if it is in the second quadrant the area can obviously be increased by moving it to the first quadrant. Therefore the vertex where the two equal sides meet is $(-R,0)$ and the other two vertices are at (x,y) and $(x,-y)$. The equation of the circle is

$$x^2+y^2=R^2,$$

and the area of the triangle is A where

$$A=\frac{1}{2}(2y)(x+R)=(x+R)y.$$

Thus, $A=0$ if $x=\pm R$, so we must have $0\leq x<R$. Making (x,y) move on the circle, since $x\neq\pm R$, we have $\dot{x}\neq 0$, and

$$\dot{A}=\dot{x}y+(x+R)\dot{y}.$$

As the point (x,y) is constrained to move on the circle $x^2+y^2=R^2$, we have

$$2x\dot{x}+2y\dot{y}=0 \text{ or } x\dot{x}+y\dot{y}=0,$$

so

$$\dot{y}=-\frac{x}{y}\dot{x}.$$

Substituting this into the expression for $\dot{A}$

$$\dot{A}=\dot{x}y+(x+R)(-x/y)\dot{x}=\dot{x}(y-[x(x+R)/y]).$$

Where A is maximum, we have $\dot{A}=0$, and obviously $y>0$, so we get the equation

$$y^2-[x(x+R)]=0.$$

Using $x^2+y^2=R^2$, this becomes

$$R^2-x^2-x^2-Rx=0, \text{ or } 2x^2+Rx-R^2=0.$$

The solutions are

$$x=\frac{-R\pm\sqrt{R^2-4(2)(-R)}}{2(2)}=\frac{-R\pm\sqrt{9R^2}}{4}=\frac{-R\pm 3R}{4}.$$

Using the minus sign would give $x=-R$, putting the all vertices at the same point giving zero area, so the positive sign must give the maximum, that is the optimum choice here is

$$x=\frac{-R+3R}{4}=\frac{R}{2}.$$

Then

$$y = \sqrt{R^2 - (R/2)^2} = \sqrt{(3/4)R^2} = \frac{(\sqrt{3})R}{2}.$$

Thus the triangle of largest area inscribed in the circle of radius R centered at the origin has vertices

$$(-R, 0),\ (R/2, (\sqrt{3})R/2),\ (R/2, -(\sqrt{3})R/2).$$

Obviously the distance between the last two of these vertices is $(\sqrt{3}R)$ and the distance from the first to either of the other two is the same and the triangle is iscoceles. But that distance D satisfies

$$D^2 = [R + (R/2)]^2 + (\sqrt{3}R/2)^2 = [(3/2)R]^2 + (3/4)R^2 = [(9/4) + (3/4)]R^2 = 3R^2,$$

therefore

$$D = (\sqrt{3})R,$$

so the triangle of maximum area is equilateral.
The area is

$$A = (x + R)y = (3/2)R(\sqrt{3})R/2 = \frac{3\sqrt{3}}{4}R^2.$$

30. Suppose That a rectangular corral is to be made with cross fencing parallel to the sides of the rectangle so as to make 12 identical rectangular smaller corrals inside the rectangle, like a three by four "egg box".

ANSWER:

If the total length of fencing available is 1200 yards, what will be the dimensions of the corral with largest area?

What is the area of the whole corral?

Suppose that there is a total length L of fencing available. Suppose that M is the total length of fencing used parallel to one side and N is the total length of fencing used for fencing parallel to the other side of the corral. How do M, N, and L compare?

Suppose that instead of a three by four egg box we want an m by n egg box design so that there are mn identical smaller corrals inside the big rectangular corral. How do M, N, L compare now?

ANSWERS:

Let x be the length of the side with 4 rectangles along it and y be the length of the side with 3 rectangles along it. Thus,

$$L = 4x + 5y.$$

Of course the total area is then simply $A = xy$,

so for maximum area $\dot{A} = 0$, as (x, y) moves along the curve $L = 4x + 5y$ (it's a straight line).

Thus,

$$\dot{A} = \dot{x}y + x\dot{y}, \text{ and } 0 = \dot{L} = 4\dot{x} + 5\dot{y}.$$

Setting $\dot{A} = 0$ gives us now the two equations

$$\dot{x}y + x\dot{y} = 0 \text{ and } 4\dot{x} + 5\dot{y} = 0.$$

If we move the point (x, y) along the line $L = 4x + 5y$ at constant speed so that $\dot{x} = 1$, then we must have $\dot{y} = -4/5$.

Putting $\dot{x} = 1$ and $\dot{y} = -4/5$ in the equation for $\dot{A}$ gives now

$$y + (-4/5)x = 0, \text{ or } 5y - 4x = 0, \text{ or } 4x = 5y.$$

Since $4x = 5y$ and $L = 4x + 5y$, we have $L = 4x + 4x = 8x$, so $x = L/8$.

Alternately, $L = 5y + 5y = 10y$, so $y = L/(10)$.

The maximum area is therefore

$$A = \left(\frac{L}{8}\right)\left(\frac{L}{10}\right) = \frac{L^2}{80}.$$

If we consider the amount of fencing used in the direction of the side of length x, since there a length of $4x$ in fencing in this direction and $x = L/8$, the result is that $4x = L/2$. Likewise, if we consider the amount of fencing used in the directions parallel to the side of length y, that is $5y = 5(L/[10]) = L/2$, again. Thus the simple way to describe the solution is to use half the fencing to go in one direction and half in the other. Thus, we can see that if there are M rectangles along the x direction and N rectangles along the y direction, then there are $M + 1$ equal lengths of fence in the x direction and $N + 1$ equal lengths of fence in the y direction, so if we use $L/2$ for the total in each direction, we have

$$(M + 1)x = \frac{L}{2} \text{ and } (N + 1)y = \frac{L}{2},$$

so the general result should be

$$x = \frac{L}{2(M + 1)},\ y = \frac{L}{2(N + 1)}, \quad \text{and } A = \frac{L^2}{4(M + 1)(N + 1)}.$$

Calculate the numerical values of the integrals indicated to 3 decimal place accuracy. Draw pictures of the indicated areas, express the indicated areas as definite integrals, and then find the areas to 3 decimal place accuracy.

INTEGRAL ANSWERS:

31. $\int_0^2 9x^2\ dx = 9\int_0^2 x^2\ dx = 9\frac{x^3}{3}|_0^2 = 3x^3|_0^2 = 3[2^3 - 0^3] = 3[8] = 24.$

32. $\int_1^e \frac{1}{x}\ dx = \ln x|_1^e = \ln e - \ln 1 = 1 - 0 = 1.$

33. $\int_1^e \ln(x)\ dx = [(x\ln x) - x]_1^e = (0 - [-1]) = 1.$

34. $\int_0^{\ln 2} e^x\ dx = e^x|_0^{\ln 2} = 2 - 1 = 1.$

35. $\int_0^2 x^2e^x\ dx = [e^x(x^2 - 2x + 2)]_0^2 = e^2(2) - 2 = 4e^2 - 2.$

36. $\int_1^e x^2\ln(x)\ dx = \left[\frac{x^3}{3}\ln x\right]_1^e - \int_1^e \frac{x^3}{3}\frac{1}{x}\ dx = \frac{e^3}{3} - \left[\frac{x^3}{9}\right]_1^e = \frac{2e^2+1}{9}.$

37. The area under the curve $y = x^3$, above the $x-$axis, and between the vertical lines $x = 1$ and $x = 2$.

$$\text{AREA} = \int_1^2 x^3\ dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$$

38. The area under the curve $y = 1/x$, above the $x-$axis, and between the vertical lines $x = e$ and $x = e^3$.

$$\text{AREA} = \int_e^{e^3} \frac{1}{x}\ dx = \int_e^{e^3} \frac{dx}{x} = [\ln|x|]_e^{e^3} = 3 - 1 = 2.$$

In the following problems, draw pictures and show your work. All answers must be to 3 decimal place accuracy.

39. The tropical island of Koolau has a 235 mile coast, a river flowing down from the mountains emptying into the ocean at the town of Riverton and ten miles of beach from Riverton to Cape Cone. Some parts of beach are eroding, but near the mouth of the river, river silt is adding to the beach. The outward velocity (in miles per century) of points along the beach is given by the formula

$$v(x) = -\frac{(x-4)^3}{50}, \ 0 \leq x \leq 10.$$

Here x is the distance of points from the Riverton end of the beach toward Cape Cone in miles. The rest of the coast of Koolau is stable. What is the overall rate of increase in land area of Koolau due to these erosion processes in square miles per century? Is the island increasing in area or decreasing in area overall?

$$\text{RATE} = \int_0^{10} v(x)\ dx = \int_0^{10} \left[-\frac{(x-4)^3}{50}\right]\ dx = \left[-\frac{(x-4)^4}{4\cdot 50}\right]_0^{10} = \frac{4^4-6^4}{4\cdot 50} = \frac{2^4(2^4-3^4)}{4\cdot 50}$$

$$= \frac{8(16-81)}{100} = 8(-0.65) = -5.2$$

The island is decreasing in area overall at the rate of 5.2 square miles per century.

40. A spherical water balloon is being filled with water at a variable rate. When the radius is R inches, the volume is $V = (4/3)\pi R^3$ and the surface area is $A = 4\pi R^2$. At a given instant the surface area is 1200 square inches and water is flowing in at the rate of 600 cubic inches per second. At this instant, what is the rate of increase of R? At a second instant, the water is flowing in at the rate of 700 cubic inches per second and the radius is 10 inches. What is the rate of increase of surface area at this instant.

At each instant,

$$\dot{V} = A\dot{R},$$

so at the first instant we have

$$600 = \dot{V} = A\dot{R} = 1200\dot{R},$$

and therefore

$$\dot{R} = 0.5,$$

which means that at the first instant considered, the radius R is increasing at the rate of half an inch per second.

For the second instant, since we know $\dot{V} = A\dot{R}$, it follows that

$$\dot{R} = \frac{\dot{V}}{A}.$$

$$A = 4\pi R^2, \quad \text{so by the chain rule, } \dot{A} = 4\pi 2R\dot{R}.$$

Therefore,

$$\dot{A} = 8\pi R \cdot \frac{\dot{V}}{A} = \dot{V} \cdot \frac{8\pi R}{4\pi R^2} = \frac{2\dot{V}}{R} = \frac{2(700)}{20} = 140.$$

This means that the area A is changeing at the rate of 140 square inches per second at the other instant.

This last part can be done alternately by noticing that

$$V = \frac{4}{3}\pi R^3 = \frac{1}{3}AR,$$

or simply,

$$3V = AR$$

so by the product rule for differentiation, we must have

$$3\dot{V} = \dot{A}R + A\dot{R}.$$

Since always, $\dot{V} = A\dot{R}$, this means always

$$3\dot{V} = \dot{A}R + \dot{V},$$

and therefore,

$$2\dot{V} = \dot{A}R,$$

and therefore at each instant,

$$\dot{A} = \frac{2\dot{V}}{R}.$$

For a particle of mass m and velocity vector v the momentum vector, denoted by p is given by the equation

$$p = mv.$$

Newton's Law of Motion says that the force vector, F, acting on the particle at each instant satisfies

$$F = \dot{p}.$$

If the mass m of the particle is constant, then $\dot{p} = m\dot{v} = ma$, so Newton's Law of Motion becomes

$$F = ma,$$

where a denotes the acceleration vector, $a = \dot{v}$. The Kinetic energy of the particle is defined to be

$$K = \frac{1}{2}m\|v\|^2 = \frac{1}{2}m(v \cdot v).$$

In general, the force acting on a particle may vary from place to place along the particles path of motion and may also change with time. Suppose that x is the particle's position vector relative to a fixed reference point, as a function of time, so its velocity vector is $v = \dot{x}$ at each instant of time. We say that the function V depending on position x is a potential energy function for F provided that at each point x,

$$F = -\mathbf{grad}\ V.$$

Then by the chain rule,

$$\dot{V} = (V(x))\dot{} = [\mathbf{grad} V(x)] \cdot \dot{x} = [\mathbf{grad}\ V(x)] \cdot v = -F \cdot v.$$

The total energy, E of the particle at each instant is defined by the equation

$$E = K + V.$$

41. Show that $\dot{K} = m(a \cdot v)$. (Hint: recall that for vector functions of time,

$$(X \cdot Y)\dot{} = \dot{X} \cdot Y + X \cdot \dot{Y}$$

and apply this to differentiate $\|v\|^2 = v \cdot v$, keeping in mind that $\dot{v} = a$.)

ANSWER: By definition,

$$K = \frac{1}{2}m\|v\|^2 = \frac{1}{2}m(v \cdot v),$$

so by the product rule for differentiation, since $\dot{v} = a$.

$$\dot{K} = \frac{1}{2}m[\dot{v} \cdot v + v \cdot \dot{v}] = \frac{1}{2}m[2a \cdot v] = m(a \cdot v).$$

42. Show that $\dot{E} = 0$, which is the Principle of Conservation of Energy, since $\dot{E} = 0$ means E cannot change. (Hint: use the equation $\dot{V} = -F \cdot v$ together with the previous result for $\dot{K}$ and Newton's Law of Motion.)

ANSWER:

By Newton's Law of Motion, we have

$$F = ma,$$

but from the preceding problem, we know

$$\dot{K} = m(a \cdot v) = (ma) \cdot v = F \cdot v.$$

From the discussion preceding the previous problem we know that

$$\dot{V} = -F \cdot v.$$

Therefore, as $E = K + V$, we have

$$\dot{E} = \dot{K} + \dot{V} = F \cdot v - F \cdot v = 0.$$

43. Suppose that a roller coaster is to be built on a piece of land which is a horizontal flat plane. Suppose that the function f gives the height of the track above the ground, so $z = f(s)$ is the height of the track at the point s feet from the start of the track, where s is measured along the shadow of the track on the flat plane on which it is built. Thus, $f'(s)$ gives the slope of the track at the point above s on the track. The gravitational acceleration is constant near the surface of the Earth with absolute value g so the potential energy per unit mass is $V = gz$ at height z above the ground. For the loaded roller coaster cart of mass m at height z, its potential energy is $mV = mgz$. Its kinetic energy is $(0.5)mv^2$, where v is the speed. As the roller coaster cart moves, its shadow on the ground moves, and when the shadow has moved a distance s from the start, then the height of the cart is $z = f(s)$. Assume the roller coaster cart moves without friction.

Give the total energy E of the cart in terms of $v,\ m,\ g$, and f after the cart has gone s feet from the start.

Give the equation relating $s,\ \dot{z},\ \dot{s},\ f$, and f'.

Give the equation relating v^2 and $\dot{s}$ and $\dot{z}$.

Give the equation relating E to $m,\ g,\ s,\ \dot{s},\ f,\ f'$.

Suppose that the shadow of the track has length L, so $0 \leq s \leq L$, and suppose that $f(0) - f(L) = H > 0$ is the amount the start point exceeds the finish point in height of the roller coaster. Suppose that the start of the roller coaster is horizontal and the finish is horizontal. If the initial speed of the cart is v_0, what is the final speed?

ANSWERS:
The total energy $E = (0.5)mv^2 + V = (0.5)mv^2 + mgz = (0.5)mv^2 + mgf(s)$, so

$$E = (0.5)mv^2 + mgf(s).$$

Since $z = f(s)$, we have

$$\dot{z} = f'(s)\dot{s}.$$

Let $\mathbf{v}$ denote the velocity vector, so $v = \|\mathbf{v}\|$. The shadow of the cart describes a curve in the plane as the cart moves, and its velocity vector $\mathbf{w}$ is horizontal. Let $\mathbf{u}$ be a fixed vertical vector (that is pointing straight up) of unit length. Let $\mathbf{r}$ be the position vector from a fixed reference point of the cart's shadow, and let $\mathbf{R}$ be the position vector of the cart itself from the same reference point, so $\mathbf{R}$ and $\mathbf{r}$ are functions of s.

Then, since $\mathbf{w} = \dot{\mathbf{r}}$,

$$\mathbf{R} = \mathbf{r} + z\mathbf{u} = \mathbf{r}(s) + [f(s)]\mathbf{u}, \text{ and therefore } |\mathbf{R}\|^2 = \|\mathbf{r}\|^2 + z^2 = \|\mathbf{r}\|^2 + [f(s)]^2,$$

so

$$\mathbf{v} = \dot{\mathbf{R}} = \dot{\mathbf{R}} + \dot{z}\mathbf{u} = \mathbf{w} + \dot{z}\mathbf{u} = \dot{\mathbf{r}} + f'(s)\dot{s}\mathbf{u} = \mathbf{w} + f'(s)\dot{s}\mathbf{u}.$$

As $\mathbf{w}$ is horizontal and $\mathbf{u}$ is vertical, they are perpendicular, so as $\|\mathbf{u}\| = 1$,

$$v^2 = \|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 + [\dot{z}]^2 = \|\mathbf{w}\|^2 + [f'(s)\dot{s}]^2.$$

On the other hand, since s is the distance along the shadow of the cart's path, we must have $\|\mathbf{w}\| = \dot{s}$.

Therefore

$$v^2 = \|\mathbf{v}\|^2 = [\dot{s}]^2 + [\dot{z}]^2 = [\dot{s}]^2(1 + [f'(s)]^2),$$

so finally,

$$v^2 = [\dot{s}]^2(1 + [f'(s)]^2) = [\dot{s}]^2 + [\dot{z}]^2.$$

We can now express the total energy E in terms of $s,\ m,\ \dot{s},\ g\ f$, and f',

$$E = \frac{1}{2}mv^2 + mgz = \frac{1}{2}m([\dot{s}]^2(1 + [f'(s)]^2) + mgf(s).$$

If the initial speed is v_0, and the start and finish of the roller coaster are both horizontal, then $f'(0) = 0 = f'(L)$, so $\dot{s_0} = v_0$ is the start speed and therefore the total energy is

$$E = \frac{1}{2}mv_0^2 + mgf(0) = \frac{1}{2}mv_0^2 + mg[f(L) + H].$$

Since the energy is conserved, we have at the end, the final velocity is horizontal (as $f'(L) = 0$), so the final speed is $v_{final} = \dot{s}_{final}$ and $s_{final} = L$, therefore

$$\frac{1}{2}mv_0^2 + mg[f(L) + H] = E = \frac{1}{2}mv_{final}^2 + mgf(L).$$

Consequently,

$$\frac{1}{2}mv_{final}^2 = \frac{1}{2}mv_0^2 + mgH = m\left[\frac{1}{2}v_0^2 + gH\right],$$

so

$$v_{final}^2 = v_0^2 + 2gH,$$

and

$$v_{final} = \sqrt{v_0^2 + 2gH},$$

a result which is independent of the shape of the roller coaster, only depending on the difference in initial and final heights. Thus, in particular, if $H = 0$, and if $v_0 = 0$, then the final speed at the end is also zero.

44. Suppose W denotes the set of all vectors in three dimensional space and $S \subset \mathbb{R}$. A gnat is flying through the air and the vector function $X : S \longrightarrow W$ gives the gnats position as a function of time t, so $X(t)$ is the position of the gnat at time t, relative to the fixed reference point 0. We set $V = \dot{X}$ and $A = \dot{V}$, so $V : S \longrightarrow W$ gives the velocity vector as a function of time, and $A : S \longrightarrow W$ gives the acceleration vector as a function of time. Let $v : S \longrightarrow \mathbb{R}$ be the real valued function giving the speed at time t, so $v(t) = \|V(t)\|,\ t \in \mathbb{R}$. Let $a : S \longrightarrow \mathbb{R}$ be the real valued function giving the length of the acceleration vector at time t, so $a(t) = \|A(t)\|,\ t \in \mathbb{R}$. Suppose that at a certain instant, t_0, the gnat's acceleration vector is perpendicular to his velocity. Calculate $\dot{v}(t_0)$. Hint: notice that $v^2 = V \cdot V$, and remember the product rule for differentiation of the inner product of vector functions.

Suppose that the gnat is travelling in a circle of radius r at constant speed v. Give a in terms of v and r.

ANSWERS:

We have $v^2 = \|V\|^2 = V \cdot V$, and $\dot{(v^2)} = 2v\dot{v}$, so

$$2v\dot{v} = \dot{(v^2)} = \dot{(V \cdot V)} = \dot{V} \cdot V + V \cdot \dot{V} = A \cdot V + V \cdot A = 2(V \cdot A).$$

Therefore, always,

$$v\dot{v} = V \cdot A.$$

Thus, if V and A are perpendicular at t_0, and if $v(t_0) \neq 0$, then $\dot{v}(t_0) = 0$.

Let R be the position vector of the gnat relative to the center of the circle. Then $\|R\| = r$ is constant, so $\dot{r} = 0$.

Therefore, as $R \cdot R = \|R\|^2 = r^2$ is also constant, we have

$$0 = (R \cdot \dot{R}) = V \cdot R + R \cdot V = 2(V \cdot R),$$

and therefore $V \cdot R = 0$ which means that V and R are always perpendicular. Also, from the previous part, we know that if speed is constant, then $\dot{v} = 0$, so

$$V \cdot A = v\dot{v} = 0,$$

so the acceleration is perpendicular to the velocity and therefore parallel to the position vector R.

Therefore

$$A = kaR/r, \text{ where } k = \pm 1.$$

On the other hand, as $V \cdot R = 0$, if we differentiate this equation, we get

$$A \cdot R + V \cdot V = 0,$$

so

$$A \cdot R + v^2 = 0,$$

and

$$A \cdot R = -v^2.$$

As $v^2 > 0$, it follows that $A \cdot R$ is negative, so the constant $k = -1$ and $A = -aR/r$.

Therefore

$$v^2 = -A \cdot R = (-aR/r) \cdot R = -(-a/r)(R \cdot R) = (a/r)r^2 = -ar,$$

which means that

$$a = \frac{v^2}{r}.$$

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