CALCULUS PRACTICE TEST PROBLEMS

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1. Draw a picture of a non-zero vector and beside it draw a picture of its negative.

2. Draw a picture of two non-zero vectors which are not parallel to each other. Draw the picture of the sum of these two vectors using head to tail addition.

3. Draw a picture of a non-zero vector X and beside it draw a picture of $(.5)X$.

4. Draw a picture of two non-zero vectors X and Y which are not parallel to each other, positioned so their tails are at the same point. Draw the difference vector $X - Y$.

5. A plane is flying in darkness and its airspeed indicator is showing 400 miles/hour and its compass is indicating that the plane is flying due north. The plane is flying in the jet stream which at the instant of interest happens to be flowing due east at 300 miles/hour. What is the plane's true ground speed?

6. A plane is flying in darkness and its airspeed indicator is showing 100 miles/hour and its compass is indicating that the plane is flying due north. The plane is flying in the jet stream which at the instant of interest happens to be flowing due east at 100 miles/hour. What is the plane's true ground speed and true direction relative to the ground?

7. A 20 mile stretch of beach is currently eroding at the rate of 2 miles per century. What is the current rate of land loss due to this erosion in square miles per century?

8. An oil spill is partially contained, but there are two breaks in the containment, one currently being 500 feet long where the oil is spreading out from the spill at the rate of 3 feet per minute, the second break currently being 800 feet long where the oil is spreading out at 2 feet per minute. What is the current rate of increase in the area of the oil spill in square feet per minute?

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9. A rectangle is growing because the sides are changing in length. Currently, one side is 12 inches long and growing at the rate of 3 inches per second and the other side is 5 inches long and growing at the rate of 2 inches per second. What is the rate of increase in the area of the rectangle in square inches per second?

10. Suppose that a rectangle has sides of length $x(t)$ and $y(t)$ at time t and that

$$x(t) = t^2 + t^3 \text{ and } y(t) = t^2 + t^4.$$

If $A(t)$ denotes the rectangle's area at time t, then what is $\dot{A}(t)$ expressed in terms of t?

11. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are both smooth functions and that

$$f(2) = 5,\ g(2) = 7,\ f(7) = 4,\ g(5) = 6,\ f'(2) = 3,\ g'(2) = 8,\ f'(7) = 9,\ g'(5) = 10.$$

Calculate

$(f + g)'(2) =$

$(fg)'(2) =$

$(f \circ g)'(2) =$

$(g \circ f)'(2) =$

$(f/g)'(2) =$

12. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f' = f$ and $f(0) = 1$. Suppose that $f(x) \neq 0$, for all $x \in \mathbb{R}$. Suppose the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth and $g' = g$. Define the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ by the rule

$$h(x) = \frac{g(x)}{f(x)},\ x \in \mathbb{R}.$$

Calculate $h'(5)$.

13. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are given by

$$f(x) = x^3 - 5 \text{ and } g(x) = x^2 + 4, \text{ for any } x \in \mathbb{R}.$$

Calculate

$(f + g)'(2) =$

$(fg)'(2) =$

$(f \circ g)'(2) =$

$(g \circ f)'(2) =$

$(f/g)'(2) =$

14. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f' = f$ and $f(0) = 1$, and suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function with $f(g(x)) = x$, for all $x \in \mathbb{R}$. Calculate $g'(2)$.

15. Suppose that W is the set of all vectors in three dimensional space and that U_1, U_2, U_3 are vectors in W so that each has unit length and all are perpendicular to each other. Suppose that

$$X = 3U_1 - 2U_2 + U_3 \text{ and } Y = 2U_1 + 5U_2 - 2U_3.$$

Calculate

$U_2 \cdot (X + Y) =$

$X \cdot Y =$

16. In the space below, draw a picture of two non-zero vectors which are not parallel to each other. Label one vector X and the other vector Y. As well, draw the pictures of $-Y$, $X + Y$, and $X - Y$. Also draw the picture of $(1/2)(X + Y)$ using the parallelogram picture for $X + Y$.

17. A swimmer is swimming in a river. At a given instant, the swimmer is moving at 4 feet per second relative to the water. At that same instant, the water is moving at 3 feet per second relative to the Earth. Relative to the water, the direction the swimmer moves is in the direction from his feet to his head, which is the direction of his body. Give the swimmer's speed relative to the Earth in the following three cases for the direction of his body: with the water flow, against the water flow, and across the water flow.

18. Paradise Island has a coast consisting of two beaches separated by stable rocky coast. One beach is 8 miles long and eroding at the rate of 3 miles per century and the other beach is 10 miles long and eroding at the rate of 2 miles per century. What is the rate of land loss in square miles per century due to the erosion along the 8 mile beach? What is the total rate of land loss due to erosion of both beaches combined?

19. If $\dot{x} = 1$ and $y = 2x^3 - 5x^2 + 3$, then what is $\dot{y}$ in terms of x?

20. An eagle is coasting in a horizontal circle, by making use of updrafts. The circle has a radius of 50 feet and the eagle moves around this circle with a constant speed of 20 feet per second. What is the eagle's acceleration in feet per second per second?

21. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are both smooth functions and that

$$f(5) = 3,\ g(5) = 2,\ f(2) = 4,\ g(3) = 7,\ f'(5) = 4,\ g'(3) = 2,\ f'(2) = 6,\ g'(5) = 10.$$

Calculate

$(f - g)'(5) =$

$(fg)'(5) =$

$(f \circ g)'(5) =$

$(g \circ f)'(5) =$

$(f/g)'(5) =$

22. Calculate $f'(x)$ if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is

$$f(x) = \frac{x^5 - 8}{x^4 + 2x^2 + 3}.$$

$f'(x) =$

23. Suppose $g : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ are smooth functions with $f' = f$ and $g' = g$.

If $h = (f/g)$, what is $h'(2)$?

24. Suppose that

$$f(x) = x^2(6 - x) = 6x^2 - x^3.$$

Give all the values of x at which f has a local maximum.

Give all the values of x at which f has an inflection point.

Find the area of the region above the $x-$axis, under the curve $y = f(x)$, between the vertical lines $x = 2$ and $x = 4$.

25. Suppose that a toy store owner wants to fence off a rectangular area along the side of his store for an electric train display. He only needs to have fencing on three sides of the area since the side wall itself will serve as one side of the rectangle. If the total length of fencing he has is 24 feet, what is the area (in square feet) of the rectangle with the maximum area he can make?

26. Find the following antiderivatives.

$$\int e^x \, dx =$$

$$\int \sqrt{x} \, dx =$$

$$\int x^\pi \, dx =$$

$$\int \ln x \, dx =$$

27. Suppose that a ladybug is walking along the curve $4x^2 + 2y^2 = 6$. Let $\dot{x}$ denote the rate of change of the ladybug's $x-$coordinate and $\dot{y}$ denote the rate of change of the ladybug's $y-$coordinate.

Give an equation relating x, y, $\dot{x}$, and $\dot{y}$.

At the instant the ladybug is at the point $(1, 1)$, what is $\dot{y}$ if $\dot{x} = 2$?

28. Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by the rule

$$f(x) = ax^3 + bx^2 + cx + d, \; x \in \mathbb{R}.$$

Here $a, b, c, d \in \mathbb{R}$ are fixed constants and $a \neq 0$.

Use the quadratic formula to find the critical points of f in terms of the coefficients a, b, c, d.

Give the condition on the coefficients necessary for the expression in the radical of the quadratic formula to be non-negative.

Give the inflection point of f in terms of the coefficients a, b, c, d and.

Show that the average of the two critical points is the inflection point.

Show that

$$f(x) = a\left[x + \frac{b}{3a}\right]^3 + \left[c - \frac{b^2}{3a}\right]x + d - a\left[\frac{b^3}{(3a)^3}\right], \; x \in \mathbb{R},$$

Show using the modified coefficients setting

$$\beta = -\frac{b}{3a}, \; \gamma = c + \beta \; \text{ and } \delta = d + a\beta^3,$$

we have

$$f(x) = a(x - \beta)^3 + \gamma x + \delta,$$

which effectively does away with the quadratic term.

Show that the graph of f is symmetric about its inflection point, that is, if $x = B$ is the inflection point, show that there is a function F so that

$$f(B \pm u) = f(B) \pm F(u), \; \text{for all } u \in \mathbb{R}.$$

This means that $(B, f(B))$ is the midpoint of the line segment joining $(B - u, f(B - u))$ to the point $(B, f(B + u))$, for any $u \in \mathbb{R}$, showing that the graph of f is symmetric about the point $(B, f(B))$.

29. Suppose that an isosceles triangle is inscribed in a circle of radius R. Find the maximum possible area and the shape of the triangle which achieves that maximum area.

30. Suppose That a rectangular corral is to be made with cross fencing parallel to the sides of the rectangle so as to make 12 identical rectangular smaller corrals inside the rectangle, like a three by four "egg box".

If the total length of fencing available is 1200 yards, what will be the dimensions of the corral with largest area?

What is the area of the whole corral?

Suppose that there is a total length L of fencing available. Suppose that M is the total length of fencing used parallel to one side and N is the total length of fencing used for fencing parallel to the other side of the corral. How do M, N, and L compare?

Suppose that instead of a three by four egg box we want an m by n egg box design so that there are mn identical smaller corrals inside the big rectangular corral. How do M, N, L compare now?

Calculate the numerical values of the integrals indicated to 3 decimal place accuracy.

31. $\int_0^2 9x^2\ dx =$

32. $\int_1^e \frac{1}{x}\ dx =$

33. $\int_0^e \ln(x)\ dx =$

34. $\int_0^{\ln 2} e^x\ dx =$

35. $\int_0^2 x^2 e^x\ dx =$

36. $\int_0^e x^2 \ln(x)\ dx =$

Draw pictures of the indicated areas, express the indicated areas as definite integrals, and then find the areas to 3 decimal place accuracy.

37. The area under the curve $y = x^3$, above the $x-$axis, and between the vertical lines $x = 1$ and $x = 2$.

38. The area under the curve $y = 1/x$, above the $x-$axis, and between the vertical lines $x = e$ and $x = e^3$.

In the following problems, draw pictures and show your work. All answers must be to 3 decimal place accuracy.

39. The tropical island of Koolau has a 235 mile coast, a river flowing down from the mountains emptying into the ocean at the town of Riverton and ten miles of beach from Riverton to Cape Cone. Some parts of beach are eroding, but near the mouth of the river, river silt is adding to the beach. The outward velocity (in miles per century) of points along the beach is given by the formula

$$v(x) = -\frac{(x-4)^3}{50},\ 0 \leq x \leq 10.$$

Here x is the distance of points from the Riverton end of the beach toward Cape Cone in miles. The rest of the coast of Koolau is stable. What is the overall rate of increase in land area of Koolau due to these erosion processes in square miles per century? Is the island increasing in area or decreasing in area overall?

40. A spherical water balloon is being filled with water at a variable rate. When the radius is R inches, the volume is $V = (4/3)\pi R^3$ and the surface area is $A = 4\pi R^2$. At a given instant the surface area is 1200 square inches and water is flowing in at the rate of 600 cubic inches per second. At this instant, what is the rate of increase of R? At a second instant, the water is flowing in at the rate of 700 cubic inches per second and the radius is 10 inches. What is the rate of increase of surface area at this instant.

For a particle of mass m and velocity vector v the momentum vector, denoted by p is given by the equation

$$p = mv.$$

Newton's Law of Motion says that the force vector, F, acting on the particle at each instant satisfies

$$F = \dot{p}.$$

If the mass m of the particle is constant, then $\dot{p} = m\dot{v} = ma$, so Newton's Law of Motion becomes

$$F = ma,$$

where a denotes the acceleration vector, $a = \dot{v}$. The Kinetic energy of the particle is defined to be

$$K = \frac{1}{2}m\|v\|^2 = \frac{1}{2}m(v \cdot v).$$

In general, the force acting on a particle may vary from place to place along the particles path of motion and may also change with time. Suppose that x is the particle's position vector relative to a fixed reference point, as a function of time, so its velocity vector is $v = \dot{x}$ at each instant of time. We say that the function V depending on position x is a potential energy function for F provided that at each point x,

$$F = -\mathbf{grad}\ V.$$

Then by the chain rule,

$$\dot{V} = (V(x))^{\dot{}} = [\mathbf{grad}V(x)] \cdot \dot{x} = [\mathbf{grad}\ V(x)] \cdot v = -F \cdot v.$$

The total energy, E of the particle at each instant is defined by the equation

$$E = K + V.$$

41. Show that $\dot{K} = m(a \cdot v)$. (Hint: recall that for vector functions of time,

$$(X \cdot Y)^{\dot{}} = \dot{X} \cdot Y + X \cdot \dot{Y}$$

and apply this to differentiate $\|v\|^2 = v \cdot v$, keeping in mind that $\dot{v} = a$.)

42. Show that $\dot{E} = 0$, which is the Principle of Conservation of Energy, since $\dot{E} = 0$ means E cannot change. (Hint: use the equation $\dot{V} = -F \cdot v$ together with the previous result for $\dot{K}$ and Newton's Law of Motion.)

43. Suppose that a roller coaster is to be built on a piece of land which is a horizontal flat plane. Suppose that the function f gives the height of the track above the ground, so $z = f(s)$ is the height of the track at the point s feet from the start of the track, where s is measured along the shadow of the track on the flat plane on which it is built. Thus, $f'(s)$ gives the slope of the track at the point above s on the track. The gravitational acceleration is constant near the surface of the Earth with absolute value g so the potential energy per unit mass is $V = gz$ at height z above the ground. For the loaded roller coaster cart of mass m at height z, its potential energy is $mV = mgz$. Its kinetic energy is $(0.5)mv^2$, where v is the speed. As the roller coaster cart moves, its shadow on the ground moves, and when the shadow has moved a distance s from the start, then the height of the cart is $z = f(s)$.

Give the total energy E of the cart in terms of $v,\ m,\ g$, and f after the cart has gone s feet from the start.

Give the equation relating $s,\ \dot{z},\ \dot{s},\ f$, and f'.

Give the equation relating v^2 and $\dot{s}$ and $\dot{z}$.

Give the equation relating E to $m,\ g,\ s,\ \dot{s},\ f,\ f'$.

Suppose that the shadow of the track has length L, so $0 \leq s \leq L$, and suppose that $f(0) - f(L) = H > 0$ is the amount the start point exceeds the finish point in height of the roller coaster. Suppose that the start of the roller coaster is horizontal and the finish is horizontal. If the initial speed of the cart is v_0, what is the final speed?

44. Suppose W denotes the set of all vectors in three dimensional space and $S \subset \mathbb{R}$. A gnat is flying through the air and the vector function $X : S \longrightarrow W$ gives the gnats position as a function of time t, so $X(t)$ is the position of the gnat at time t, relative to the fixed reference point 0. We set $V = \dot{X}$ and $A = \dot{V}$, so $V : S \longrightarrow W$ gives the velocity vector as a function of time, and $A : S \longrightarrow W$ gives the acceleration vector as a function of time. Let $v : S \longrightarrow \mathbb{R}$ be the real valued function giving the speed at time t, so $v(t) = \|V(t)\|,\ t \in \mathbb{R}$. Let $a : S \longrightarrow \mathbb{R}$ be the real valued function giving the length of the acceleration vector at time t, so $a(t) = \|A(t)\|,\ t \in \mathbb{R}$. Suppose that at a certain instant, t_0, the gnat's acceleration vector is perpendicular to his velocity. Calculate $\dot{v}(t_0)$. Hint: notice that $v^2 = V \cdot V$, and remember the product rule for differentiation of the inner product of vector functions.

Suppose that the gnat is travelling in a circle of radius r at constant speed v. Give a in terms of v and r.

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