MATH-1220 (DUPRÉ) SPRING 2011 TEST 1 ANSWERS

Wednesday 29 February 2012

DIRECTIONS

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF EACH SHEET TURNED IN.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME ON EACH SHEET TURNED IN.

THIRD: WRITE YOUR CORRECT SPRING 2011 MATH-1150 SECTION NUMBER DIRECTLY UNDERNEATH YOU FIRST NAME ON EACH SHEET TURNED IN.

FOURTH: Write NEATLY and CLEARLY, putting your answers in the space provided. If I cannot read it you do not get credit.

FIFTH: Any failure to follow any part of any of the above directions can result in additional loss of credit.

1. (10 POINTS) Calculate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{24 \cdot e^x}{1+e^{2x}} dx$

ANSWER: Substitute $u = e^x$, then $du = e^x dx,\ u(0) = 1,\ u(\ln\sqrt{3}) = \sqrt{3}$, and the result is

$$\int_0^{\ln\sqrt{3}} \frac{24 \cdot e^x}{1+e^{2x}} dx = 24 \int_1^{\sqrt{3}} \frac{du}{1+u^2}.$$

Since the antiderivative of $(1+u^2)^{-1}$ is $\arctan u$, the result is

$$\int_0^{\ln\sqrt{3}} \frac{24 \cdot e^x}{1+e^{2x}} dx = 24[\arctan(\sqrt{3}) - \arctan(1)] = 24\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = 2\pi.$$

In the following problems give the indicated antiderivative or indefinite integral

2. (10 POINTS) $\int \frac{dz}{\sqrt{9-4z^2}}$

ANSWER: Just substitute $u = 2z/3$, then $dz = (3/2)du$, $9u^2 = 4z^2$, and the integral becomes

$$\int \frac{dz}{\sqrt{9-4z^2}} = \int \frac{(3/2)du}{\sqrt{9-9u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin(u) = \frac{1}{2}\arcsin\left(\frac{2}{3}z\right) + C$$

3. (20 POINTS) $\int x\tan^2 x\ dx$

ANSWER: There is more than one way to handle this, but probably the simplest is to begin by using the trig identity $1 + \tan^2 = \sec^2$, so the integral becomes

$$\int x\tan^2 x\ dx = \int x[\sec^2(x) - 1]dx = \int x\sec^2(x)\ dx - \int x\ dx.$$

Of course the second integral is a trivial application of the power rule and now the first is easily done by parts: set $u = x$ and $dv = \sec^2(x)dx$, so $du = dx$ and $v = \tan x$.
Therefore

$$\int x\sec^2 x\ dx = uv - \int v\ du = x\tan x - \int \tan x\ dx = x\tan x - \ln|\sec x|$$

so

$$\int x\tan^2 x dx = x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$$

4. (20 POINTS) $\int \sin^4 x\ \cos^5 x\ dx$

ANSWER: Begin by substituting $u = \sin x$, so $du = \cos x\ dx$, and after using the Pythagorean identity, which here means $\cos^2 x = 1 - u^2$, the integral becomes

$$\int \sin^4 x\ \cos^5 x\ dx = \int u^4(1-u^2)^2\ du = \int u^4[1 - 2u^2 + u^4]\ du$$
$$= \int [u^4 - 2u^6 + u^8]\ du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9},$$

which is a trivial application of the power rule. We therefore find after replacing $\sin x$ for u, that

$$\int \sin^4 x\ \cos^5 x\ dx = \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.$$

5. (15 POINTS) $\int \sin^2 x \, dx$

ANSWER: Here we need to apply the double angle relation in the form

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Therefore

$$\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{x}{2} - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

6. (20 POINTS) $\int \cos^4 x \, dx$

ANSWER: Again we use the double angle relation, this time in the form

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

so

$$\int \cos^4 x \, dx = \frac{1}{4} \int [1 + 2\cos 2x + \cos^2 2x] \, dx = \frac{1}{4} \int \left[1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right] \, dx$$
$$= \frac{1}{4}\left[x + \sin 2x + \frac{x}{2} + \frac{1}{2}\frac{\sin 4x}{4}\right] + C$$
$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C,$$

so finally we have

$$\int \cos^4 x \, dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

7. (20 POINTS) $\int \frac{2x - 1}{(x - 2)(x - 5)} \, dx$

This is an easy application of the technique of partial fractions. Since the numerator has smaller degree than the denominator, we have

$$\frac{2x - 1}{(x - 2)(x - 5)} = \frac{A}{x - 5} + \frac{B}{x - 2},$$

for some numbers A and B. Multiplying both sides by $x - 5$, cancelling, and then setting $x = 5$ gives $A = (2 \cdot 5 - 1)/(5 - 2) = 9/3 = 3$, whereas multiplying both sides by $x - 2$, cancelling, and then setting $x = 2$ gives $B = (2 \cdot 2 - 1)(2 - 5) = -1$. Therefore the integral is

$$\int \frac{2x - 1}{(x - 2)(x - 5)} \, dx = 3\ln|x - 5| - \ln|x - 2| + C = \ln\left[\frac{|x - 5|^3}{|x - 2|}\right] + C.$$

Let R be the first quadrant region bounded by the curves $y = 4 - x^2,\ y = 4 + x^2$, and $x = 2$. In each of the following problems, give the definite integral (but do not compute it) which gives the required numerical value.

For these problems you should begin by having a clear picture of the region R in mind. Notice the two parabolas intersect where they cross the $y-$axis, at the point $(0, 4)$. The region R is symmetric about the horizontal line $y = 4$. At the right edge $x = 2$, the upper parabola intersects $x = 2$ at the point $(2, 8)$ and the lower parabola intersects the line $x = 2$ at the point $(2, 0)$. In particular, the entire region R is sandwiched between the two vertical lines $x = 0$ and $x = 2$.

8. (5 POINTS) The area of R.

ANSWER: If A denotes the area of R, then

$$A = \int_0^2 |(4 + x^2) - (4 - x^2)|\ dx = \int_0^2 2x^2\ dx$$

9. (5 POINTS) The volume of the solid of revolution obtained by revolving R around the $x-$axis.

ANSWER: If V_x denotes the volume of the solid of revolution obtained by revolving R around the $x-$axis, then

$$V_x = \int_0^2 \pi[4 + x^2]^2\ dx - \int_0^2 [4 - x^2]^2\ dx = \int_0^2 \pi[(4 + x^2)^2 - (4 - x^2)^2]\ dx = 16\pi \int_0^2 x^2\ dx,$$

since the constant terms and fourth power terms all cancel out.

10. (5 POINTS) The volume of the solid of revolution obtained by revolving R around the $y-$axis.

ANSWER: If V_y denotes the volume of the solid of revolution obtained by revolving R around the $y-$axis, then

$$V_y = \int_0^2 2\pi x|(4 + x^2) - (4 - x^2)|\ dx = 4\pi \int_0^2 x^3\ dx.$$

11. (10 POINTS) The surface area of the surface obtained by revolving R around the $x-$axis.

ANSWER: The boundary of the region R consists of three curves, say the first is $y = 4 + x^2,\ 0 \leq x \leq 2$, the second is $y = 4 - x^2,\ 0 \leq x \leq 2$, and the third is $x = 2,\ 0 \leq y \leq 8$. As R is revolved around the $x-$axis, each of these curves sweeps out a part of the boundary of the solid of revolution formed, so the total surface area of the boundary is the SUM of the contributions from each of the three curves. The last clearly sweeps out a disc of radius 8, so its contribution is simply 64π. The top parabola and the bottom parabola can be denoted $f_{\pm}(x) = 4 \pm x^2$, and we have each contributes a surface area $S_{\pm}$ given by

$$S_{\pm} = \int_0^2 2\pi[f_{\pm}(x)]\sqrt{1 + [f'_{\pm}(x)]^2}\ dx.$$

Now $f'_{\pm}(x) = \pm(2x)$, and therefore $[f'_{\pm}(x)]^2 = 4x^2$. This means the radical term is the same for both surface integrals and

$$S_+ + S_- = \int_0^2 2\pi[(4 + x^2) + (4 - x^2)]\sqrt{1 + 4x^2}\ dx.$$

Therefore the total surface area is S where

$$S = S_+ + S_- + 64\pi = \int_0^2 2\pi\left[16 + 8\sqrt{1 + 4x^2}\ \right]\ dx.$$