MATH-1230 (DUPRÉ) FALL 2010 TEST 6 ANSWERS

FIRST: PRINT YOUR LAST NAME IN LARGE CAPITAL LETTERS ON THE UPPER RIGHT CORNER OF THIS SHEET.

SECOND: PRINT YOUR FIRST NAME IN CAPITAL LETTERS DIRECTLY UNDERNEATH YOUR LAST NAME.

THIRD: WRITE YOUR FALL 2010 MATH-1230 LAB DAY DIRECTLY UNDERNEATH YOU FIRST NAME.

DIRECTIONS: WRITE YOUR FINAL ANSWERS IN THE SPACE PROVIDED ON THE TEST SHEET. WRITE YOUR FULL SOLUTION TO EACH PROBLEM ON A SHEET OF PLAIN WHITE PAPER SHOWING ALL YOUR WORK WITH EACH SOLUTION ON A SEPARATE SHEET OF PAPER USING ONE SIDE ONLY. DO NOT WRITE ON THE BACK OF ANY SHEET RURNED IN. FOLLOW STEPS ABOVE FOR EACH SHEET TURNED IN FOR IDENTIFICATION PURPOSES. EACH PROBLEM IS WORTH 5 POINTS. THERE ARE 20 PROBLEMS.

Suppose that $X_1, X_2, X_3, ..., X_n, ...$ **is a sequence of uncorrelated random variables all having mean** μ **and standard deviation** σ**. Let**

$$T_n = \sum_{k=1}^{n} X_k, \; n = 1, 2, 3, ...$$

and let

$$\bar{X}_n = \frac{1}{n} T_n, \; n = 1, 2, 3, ...,$$

so we are assuming

$$Cov(X_i, X_j) = \sigma^2, \; i = j, \textbf{and } Cov(X_i, X_j) = 0, \; i \neq j.$$

1. What is $Cov(X_3, \bar{X}_2)$?

ANSWER: Since X_3 is uncorrelated with both X_1 and X_2, it follows that X_3 is uncorrelated with $T_2 = X_1 + X_2$, as

$$Cov(X_3, T_2) = Cov(X_3, X_1) + Cov(X_3, X_2) = 0 + 0 = 0,$$

and therefore

$$Cov(X_3, \bar{X}_2) = Cov(X_3, (1/2)T_2) = \frac{1}{2} \cdot Cov(X_3, T_2) = 0.$$

Thus,

$$Cov(X_3, \bar{X}_2) = 0.$$

FINAL ANSWER: 0

2. What is $Cov(X_2, \bar{X}_3)$?

ANSWER: Since $T_3 = X_1 + X_2 + X_3$, therefore

$$Cov(X_2, T_3) = Cov(X_2, X_1) + Cov(X_2, X_2) + Cov(X_2, X_3) = 0 + \sigma^2 + 0 = \sigma^2,$$

and therefore as $\bar{X}_3 = (1/3)T_3$,

$$Cov(X_2, \bar{X}_3) = \frac{1}{3} \cdot Cov(X_2, T_3) = \frac{\sigma^2}{3}.$$

FINAL ANSWER: $\frac{\sigma^2}{3}$

3. What is $Cov(T_2, \bar{X}_3)$?

ANSWER: At this point you should realize that

$$Cov(X_k, T_l) = \sigma^2, \text{ if } k \leq l, \text{ and } Cov(X_k, T_l) = 0, \ k > l.$$

Consequently,

$$Cov(X_k, \bar{X}_l) = \frac{\sigma^2}{l}, \text{ if } k \leq l, \text{ and } Cov(X_k, \bar{X}_l) = 0, \ k > l.$$

Thus, in particular,

$$Cov((X_1, \bar{X}_3) = \frac{\sigma^2}{3} = Cov(X_2, \bar{X}_3),$$

so

$$Cov(T_2, \bar{X}_3) = \frac{2\sigma^2}{3}.$$

FINAL ANSWER: $\frac{2\sigma^2}{3}$

4. What is $Cov(\bar{X}_2, \bar{X}_3)$?

ANSWER: Since $\bar{X}_2 = (1/2)T_2$, it follows that

$$Cov(\bar{X}_2, \bar{X}_3) = \frac{1}{2} \cdot Cov(T_2, \bar{X}_3) = \frac{1}{2} \cdot \frac{2\sigma^2}{3} = \frac{\sigma^2}{3}.$$

At this point you could observe that if W is any unknown and if $Cov(X_k, W) = b$ is independent of k, that is all $X_1, X_2, ..., X_m$ have the same covariance with W, then their average $\bar{X}_m$ has the same covariance with W. More generally, the covariance of the average with W is the average of the covariances with W. That is, if

$$Cov(X_k, W) = b_k, \ k \leq m,$$

then

$$Cov(\bar{X}_m, W) = \bar{b} = \frac{b_1 + b_2 + ... + b_m}{m}.$$

FINAL ANSWER: $\frac{\sigma^2}{3}$

5. What is $Cov(\bar{X}_4, \bar{X}_7)$?

ANSWER: As pointed out above, at this point you could observe that if W is any unknown and if $Cov(X_k, W) = b$ is independent of k, that is all $X_1, X_2, ..., X_m$ have the same covariance with W, then their average $\bar{X}_m$ has the same covariance with W. More generally, the covariance of the average with W is the average of the covariances with W. That is, if

$$Cov(X_k, W) = b_k,\ k \leq m,$$

then

$$Cov(\bar{X}_m, W) = \bar{b} = \frac{b_1 + b_2 + ... + b_m}{m}.$$

Applying this to $W = \bar{X}_7$, we see that

$$Cov(X_k, \bar{X}_7) = \frac{1}{7} Cov(X_k, T_7) = \frac{\sigma^2}{7},\ k \leq 7,$$

so as $4 \leq 7$, we must have

$$Cov(\bar{X}_4, \bar{X}_7) = \frac{\sigma^2}{7}.$$

FINAL ANSWER: $\frac{\sigma^2}{7}$

6. What is $Cov(\bar{X}_4, X_2 - \bar{X}_7)$?

ANSWER: From the previous problem, $Cov(\bar{X}_4, \bar{X}_7) = (\sigma^2/7)$, so

$$Cov(\bar{X}_4, X_2 - \bar{X}_7) = Cov(\bar{X}_4, X_2) - Cov(\bar{X}_4, \bar{X}_7) = \frac{\sigma^2}{4} - \frac{\sigma^2}{7} = \frac{3\sigma^2}{28}.$$

FINAL ANSWER: $\frac{3\sigma^2}{28}$

7. What is $Cov(X_7 - \bar{X}_6, X_5 - \bar{X}_4)$?

ANSWER: As pointed out above, at this point you could observe that if W is any unknown and if $Cov(X_k, W) = b$ is independent of k, that is all $X_1, X_2, ..., X_m$ have the same covariance with W, then their average $\bar{X}_m$ has the same covariance with W. More generally, the covariance of the average with W is the average of the covariances with W. That is, if

$$Cov(X_k, W) = b_k,\ k \leq m,$$

then

$$Cov(\bar{X}_m, W) = \bar{b} = \frac{b_1 + b_2 + ... + b_m}{m}.$$

Therefore, using the same method as in the previous problem, we find that

$$Cov(\bar{X}_4, \bar{X}_6) = \frac{\sigma^2}{6}.$$

In fact, from this we also see that if m is the larger of the two numbers k, l, then

$$Cov(\bar{X}_k, \bar{X}_l) = \frac{\sigma^2}{m}.$$

Therefore, we have

$$\begin{aligned} Cov(X_7 - \bar{X}_6, X_5 - \bar{X}_4) &= Cov(X_7, \bar{X}_5) + Cov(\bar{X}_6, \bar{X}_4) - Cov(\bar{X}_6, X_5) - Cov(X_7, \bar{X}_4) \\ &= 0 + \frac{\sigma^2}{6} - \frac{\sigma^2}{6} - 0 = 0 \end{aligned}$$

FINAL ANSWER: 0

Suppose that X is an unknown with probability density function (pdf) satisfying

$$f_X(x) = \frac{3}{4}(1 - x^2), \ -1 \leq x \leq 1.$$

8. What is $P(X \leq -2)$?

ANSWER: We first calculate

$$\int_{-1}^{1} f_X(x)dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{3}{4}\left[2 - \frac{2}{3}\right] = \frac{3}{4} \cdot \frac{4}{3} = 1.$$

This means that $f_X(x) = 0$, if $x < -1$ or if $x > 1$.
Therefore

$$P(X \leq -2) = \int_{-\infty}^{-2} f_X(x)dx = \int_{-\infty}^{-2} 0 \cdot dx = 0.$$

FINAL ANSWER: 0

9. What is $P(X \leq 0)$?

ANSWER: Since $f_X(x) = 0$, if $x < -1$, it follows that

$$P(X \leq 0) = \int_{-\infty}^{0} f_X(x)dx = \int_{-1}^{0} \frac{3}{4}[1 - x^2]dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{0} = \frac{3}{4}\left[1 - \frac{1}{3}\right] = \frac{1}{2}.$$

FINAL ANSWER: 1/2

10. What is $P(X \geq .4)$?

ANSWER: Since $f_X(x) = 0$, for $x > 1$, it follows that

$$P(X \geq .4) = \int_{.4}^{\infty} f_X(x)dx = \int_{.4}^{1} \frac{3}{4}[1 - x^2]dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{.4}^{1} = \frac{1}{2} - \left[\frac{3}{10} - \frac{16}{1000}\right] = .216$$

FINAL ANSWER: .216

11. What is $P(-.2 \leq X \leq .3)$?

ANSWER: Since both $-1 \leq -.2 \leq 1$ and $-1 \leq .3 \leq 1$, and $f_X(x) = (3/4)[1 - x^2]$, for any x in the interval $-1 \leq x \leq 1$, it follows that

$$P(-.2 \leq X \leq .3) = \int_{-.2}^{.3} \frac{3}{4}[1 - x^2]dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-.2}^{.3} = \left[\frac{3x - x^3}{4}\right]_{-.2}^{.3}$$
$$= \left[\frac{.9 - .027}{4}\right] + \left[\frac{.6 - .008}{4}\right]$$
$$= \frac{.873 + .592}{4} = \frac{1.465}{4} = .36625$$

FINAL ANSWER: .36625

12. What is $E(X)$?

ANSWER: Since $f_X(x) = 0$ unless $-1 \leq x \leq 1$, it follows that

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_{-1}^{1} x \cdot \frac{3}{4}[1 - x^2]dx = \int_{-1}^{1} \frac{3}{4}[x - x^3]dx = \frac{3}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1} = 0$$

FINAL ANSWER: 0

13. What is $E(X^2)$?

ANSWER: Since $f_X(x) = 0$ unless $-1 \leq x \leq 1$, it follows that

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx = \int_{-1}^{1} x^2 \cdot \frac{3}{4}[1 - x^2]dx = \int_{-1}^{1} \frac{3}{4}[x^2 - x^4]dx = \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$$
$$= 2 \cdot \frac{3}{4}\left[\frac{1}{3} - \frac{1}{5}\right] = 2 \cdot \frac{3}{4} \cdot \frac{2}{15} = \frac{1}{5}$$

FINAL ANSWER: 1/5

14. What is σ_X^2, the variance of X?

ANSWER: For any X we have

$$Var(X) = E(X^2) - [E(X)]^2,$$

so as here $E(X) = 0$, it follows that

$$Var(X) = E(X^2) - 0^2 = \frac{1}{5}.$$

FINAL ANSWER: 1/5

15. If T is the total of 5 random observations of X, then what is $E(T)$?

ANSWER: In general, if T is the total of n observations of X, then $E(T) = nE(X)$, so here, as $E(X) = 0$, we must also have $E(T) = 0$.

FINAL ANSWER: 0

16. If T is the total of 5 independent random observations of X, then what is σ_T^2, the variance of T?

ANSWER: In general, if T is the total of n independent observations of X, then

$$Var(T) = nVar(X),$$

so here we have

$$Var(T) = 5 \cdot \frac{1}{5} = 1.$$

FINAL ANSWER: 1

Suppose that A is a statement with $p = P(A) = E(I_A)$, where I_A is the indicator unknown of A. Let T_n be the total of n independent observations of I_A.

17. What is $m_{I_A}(t)$, the moment generating function of I_A in terms of p, t?

ANSWER: Since for any X we have

$$m_X(t) = E(e^{tX}),$$

we only need the values and probabilities for $X = I_A$. As the only possible values are 0 and 1, with 1 having probability p and 0 having probability $1 - p$, it follows that the values of e^{tX} are e^t with probability p and $e^0 = 1$ with probability $1 - p$, and therefore

$$m_X(t) = pe^t + (1-p)e^0 = pe^t + q, \ q = 1 - p.$$

FINAL ANSWER: $m_X(t) = pe^t + (1 - p)$

18. What is the formula or expression for the moment generating function $m_{T_n}(t)$ for T_n in terms of t, n, p?

ANSWER: In general, for any X, if T_n is the total of n independent observations of X, then

$$m_{T_n}(t) = [m_X(t)]^n,$$

so here

$$m_{T_n}(t) = [pe^t + q]^n, \ q = 1 - p.$$

FINAL ANSWER: $m_{T_n}(t) = [pe^t + 1 - p]^n$

19. What is the mean and variance of T_n in terms of n and p?

ANSWER: For any random variable X, the mean and variance of the total T_n of n independent observations of X are

$$E(T_n) = nE(X),$$

and

$$Var(T_n) = nVar(X).$$

We have $E(I_A) = P(A) = p$, and

$$Var(X) = E(X^2) - [E(X)]^2,$$

for any unknown X, so for the case $X = I_A$, we have $X^2 = X$, so $E(X^2) = p$ in this case, so

$$Var(I_A) = p - p^2 = p(1-p) = pq, \ q = 1 - p.$$

As a consequence,

$$E(T_n) = nE(I_A) = np$$

and

$$Var(T_n) = npq, \ q = 1 - p.$$

FINAL ANSWER: $E(T_n) = np$, **and** $Var(T_n) = np(1-p)$

20. What is $P(T_n = k)$ in terms of n, k, p, and what is $P(T_n = 4)$ for the case where $n = 10$ and $p = .3$?

ANSWER: In general, for any simple unknown X with values $v_1, v_2, v_3, ..., v_m$ and with

$$P(X = v_k) = p_k, \ k \leq m,$$

we have

$$m_X(t) = \sum_{k=1}^{m} p_k \cdot e^{v_k t}.$$

Notice that if all the values are non-negative whole numbers, when we substitute $u = e^t$, then we get a polynomial $h(u)$ in u where the exponents are the values and the coefficients give the corresponding probabilities,

$$m_X(t) = h(u) = \sum_{k=1}^{m} p_k \cdot u^{v_k}.$$

But for any polynomial, the coefficients can be found by differentiation, the coefficient of u^k is c_k where

$$c_k = \frac{h^{(k)}(0)}{k!}.$$

Thus we can also write

$$m_X(t) = h(u) = \sum_{k=0}^{d} \frac{h^{(k)}(0)}{k!} \cdot u^k.$$

When we do this with $X = T_n$, where T_n is the total of n observations of I_A, we have

$$m_{T_n}(t) = h(u) = (pu + q)^n,\ q = 1 - p,\ u = e^t$$

But,

$$h(u) = (pu + q)^n$$

is easy to differentiate using the chain rule for differentiation:

$$h'(u) = n(pu + q)^{n-1}p,$$
$$h''(u) = n(n-1)(pu + q)^{n-2}p^2,$$
$$h^{(3)}(u) = n(n-1)(n-2)(pu + q)^{n-3}p^3.$$

Using factorial notation we can rewrite these as

$$h'(u) = \frac{n!}{(n-1)!}(pu + q)^{n-1}p^1,$$
$$h''(u) = \frac{n!}{(n-2)!}(pu + q)^{n-2}p^2$$
$$h^{(3)}(u)\frac{n!}{(n-3)!}(pu + q)^{n-3}p^3.$$

From this we see the obvious pattern, the k^{th} derivative should be

$$h^{(k)}(u) = \frac{n!}{(n-k)!}(pu + q)^{n-k} \cdot p^k,$$

and if we differentiate this we get the formula which results from replacing k by $k + 1$. This means that by mathematical induction, the formula must hold for all $k \leq n$. But then,

$$P(T_n = k) = \frac{h^{(k)}(0)}{k!} = \frac{n!}{k! \cdot (n-k)!}q^{n-k}p^k,\ q = 1 - p.$$

In case where $n = 10$ and $p = .3$ and $k = 4$, the result is

$$P(T_{10} = 4) = \frac{10!}{4! \cdot 6!}(.3)^4(.7)^6 = \frac{200120949}{10^{10}} = .200120949.$$

FINAL ANSWER: .200120949