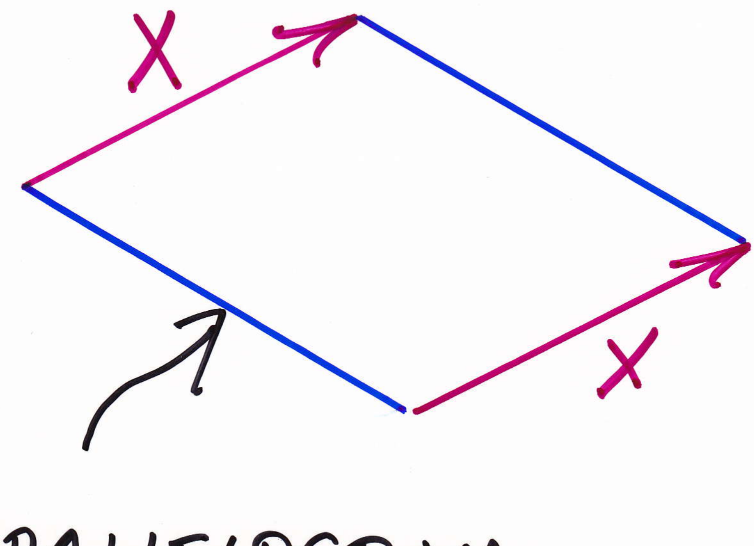
VECTORS

MAURICE J DUPRE'

ARROWS IN SPACE

VECTOR REPRESENTATION

PARALLEL ARROWS WITH SAME LENGTH AND SAME DIRECTION PICTURE THE SAME VECTOR.

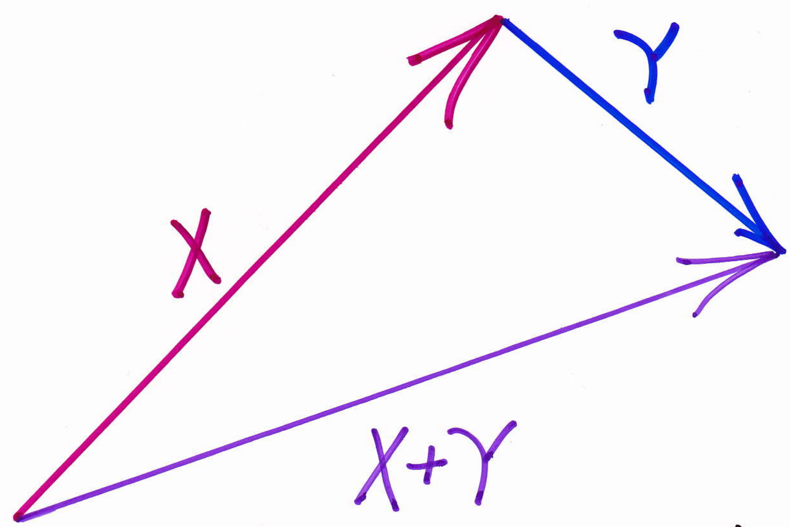


PARALLELOGRAM IN SPACE

$\boxed{\|X\| = \text{LENGTH of } X}$

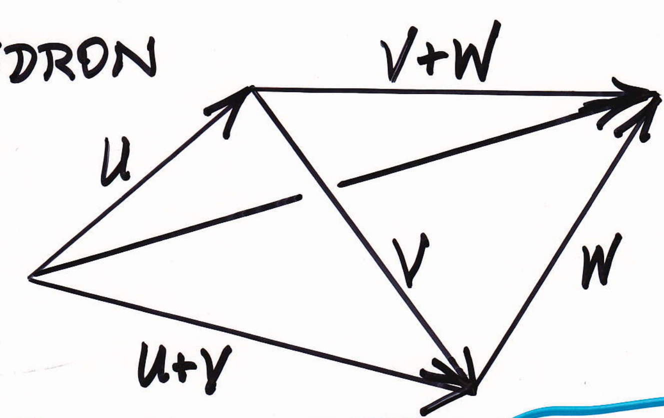
VECTOR ADDITION

HEAD TO TAIL



ASSOCIATIVE LAW

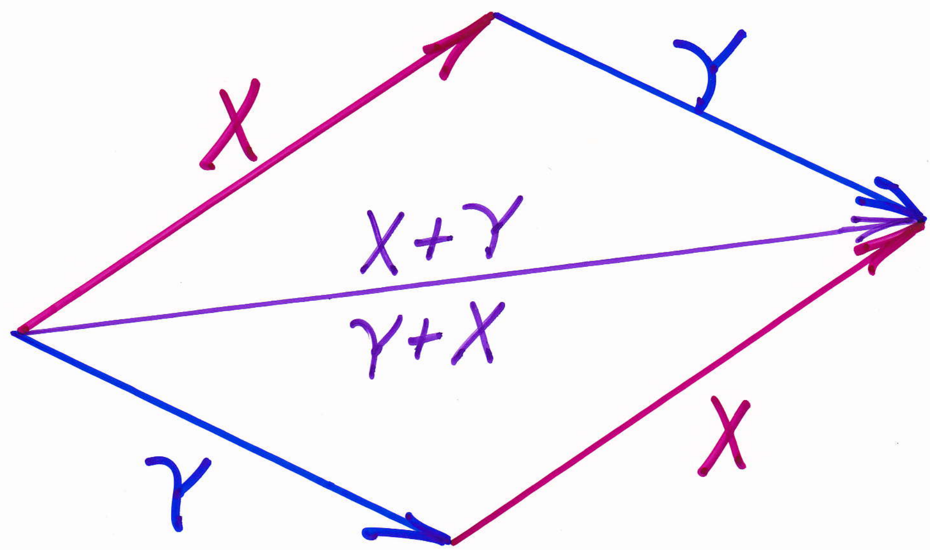
TETRAHEDRON IN SPACE



$$\boxed{(U+V)+W = U+(V+W)}$$

VECTOR ADDITION

PARALLELOGRAM & COMMUTATIVE LAWS



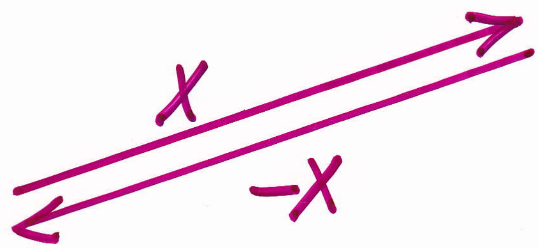
$$\boxed{X + Y = Y + X}$$

ZERO VECTOR

ANY POINT IN SPACE IS A PICTURE OF THE ZERO VECTOR, 0. THE ZERO VECTOR HAS ZERO LENGTH AND NO DIRECTION (OR ALL DIRECTIONS SIMULTANEOUSLY).

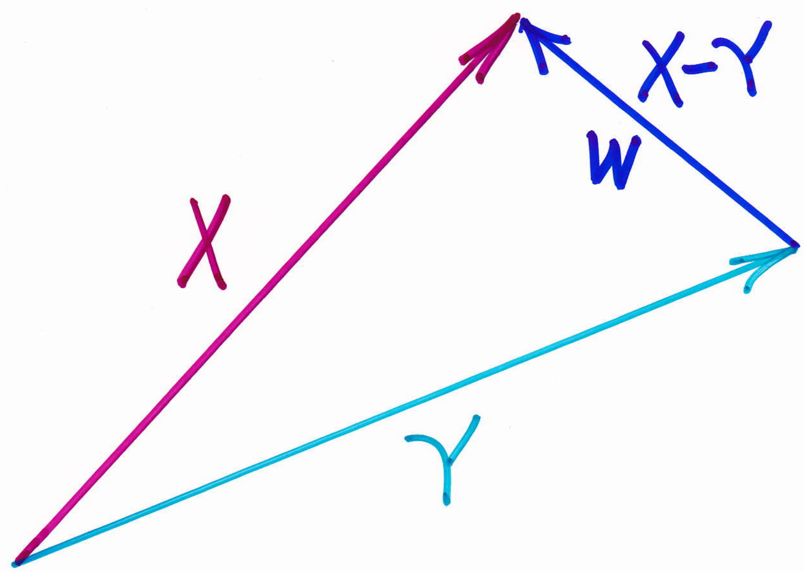
NEGATIVE of VECTOR

TO NEGATE A VECTOR REVERSE ITS DIRECTION.



$$\boxed{X + (-X) = 0}$$

VECTOR SUBTRACTION



FROM HEAD TO TAIL ADDITION

$$X = Y + W$$

$$\therefore \boxed{W = X - Y}$$

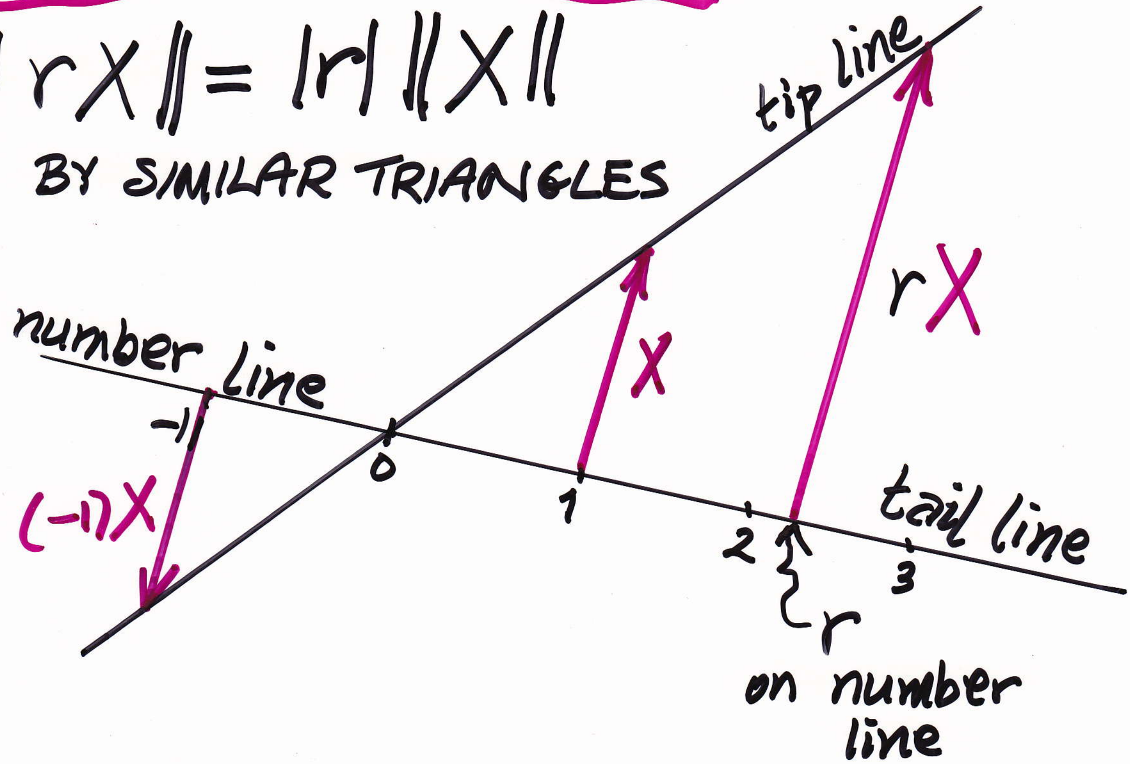
MULTIPLY A VECTOR BY A NUMBER

$$\boxed{\|X\| = \text{LENGTH of } X}$$

$|r| = \sqrt{r^2}$

$$\|rX\| = |r|\,\|X\|$$

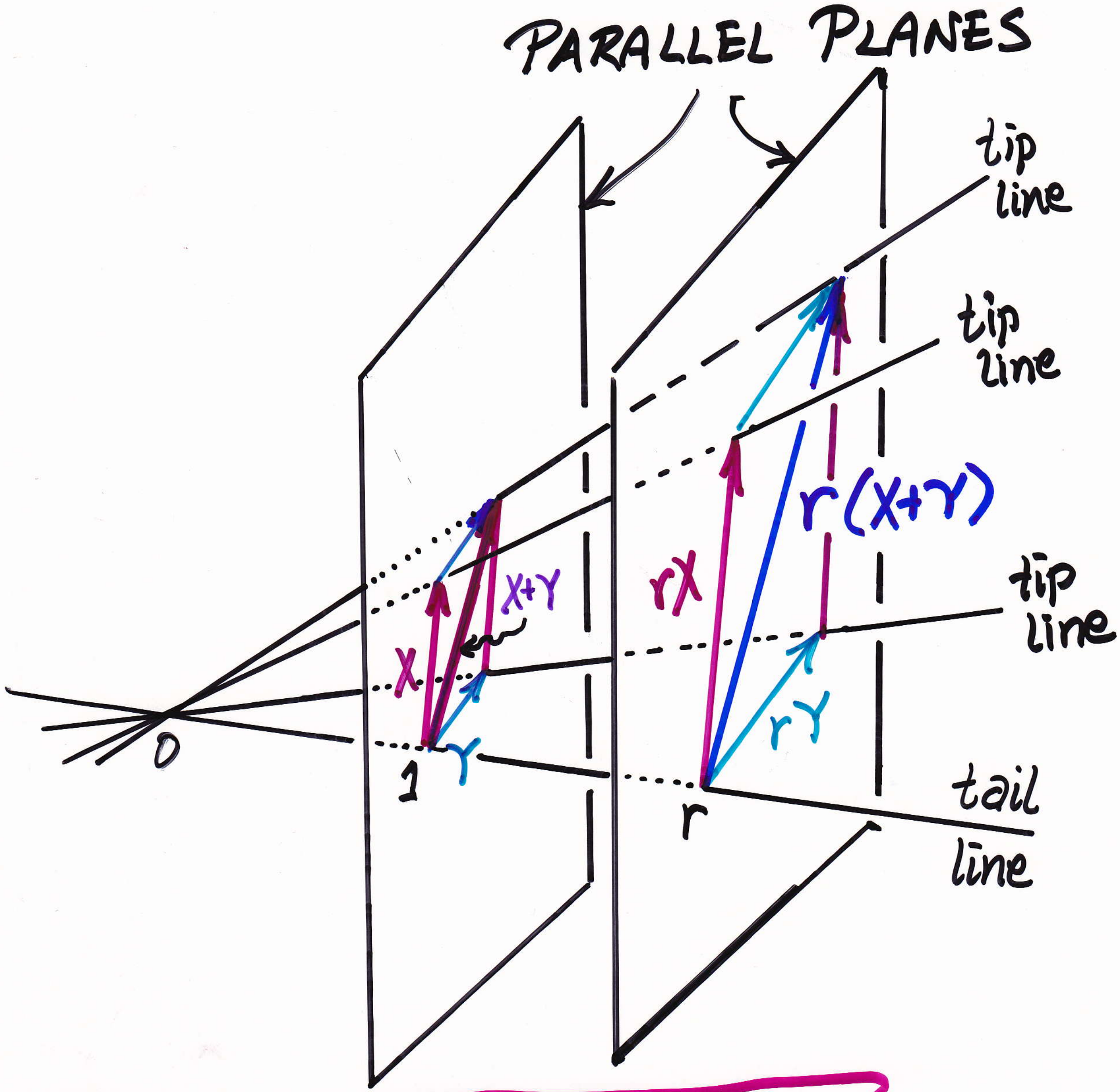
BY SIMILAR TRIANGLES



NOTICE MULTIPLYING X BY A NEGATIVE NUMBER REVERSES ITS DIRECTION, $(-1)X = -X$

SO $\therefore (-1)^2 = 1$.

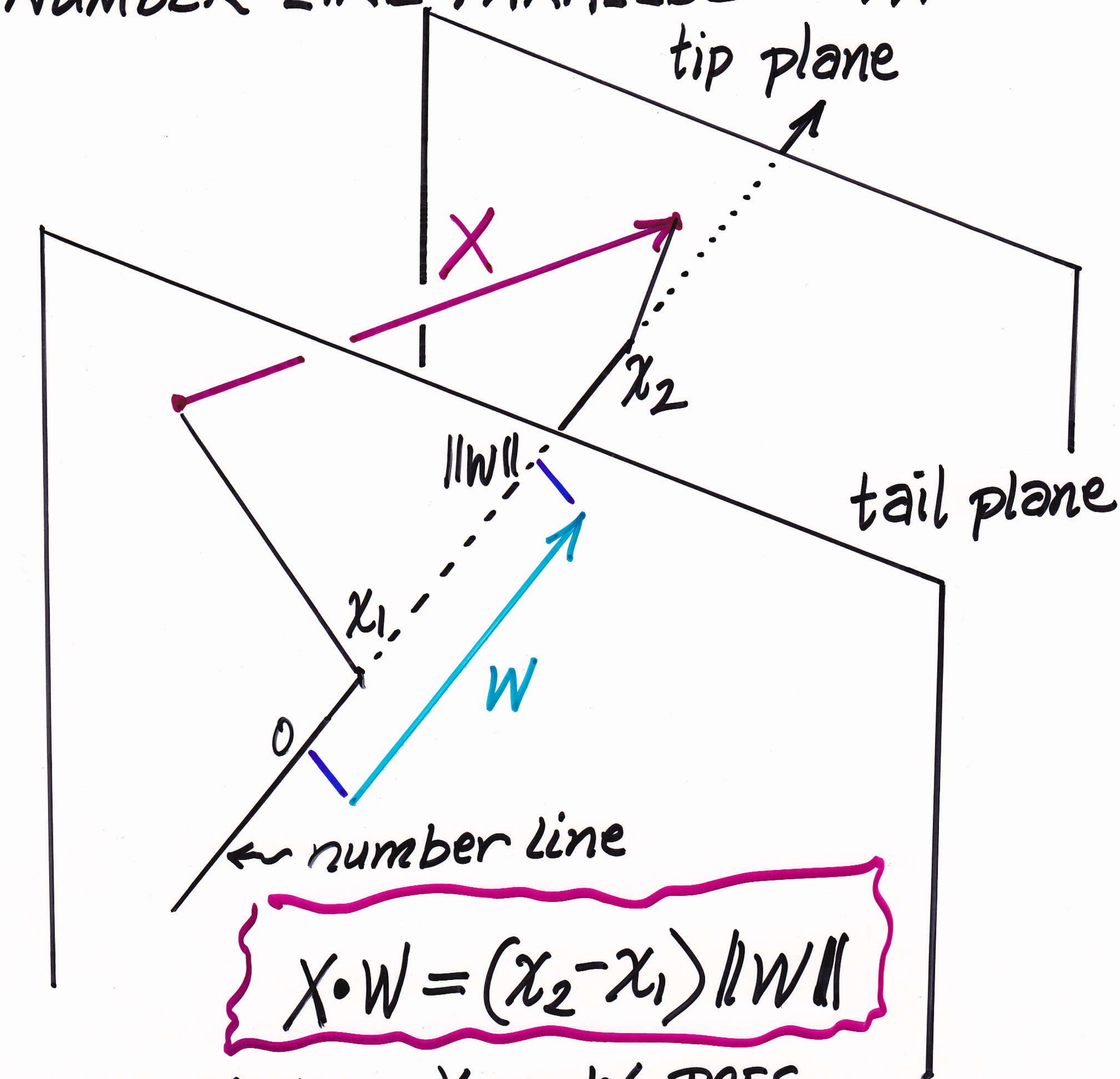
DISTRIBUTIVE LAW
FOR
MULTIPLICATION of a VECTOR BY ANY REAL NUMBER



$$\boxed{r(X+Y) = rX + rY}$$

DOT PRODUCT of TWO VECTORS IS A REAL NUMBER

TWO PLANES PERPENDICULAR TO NUMBER LINE PARALLEL TO W.

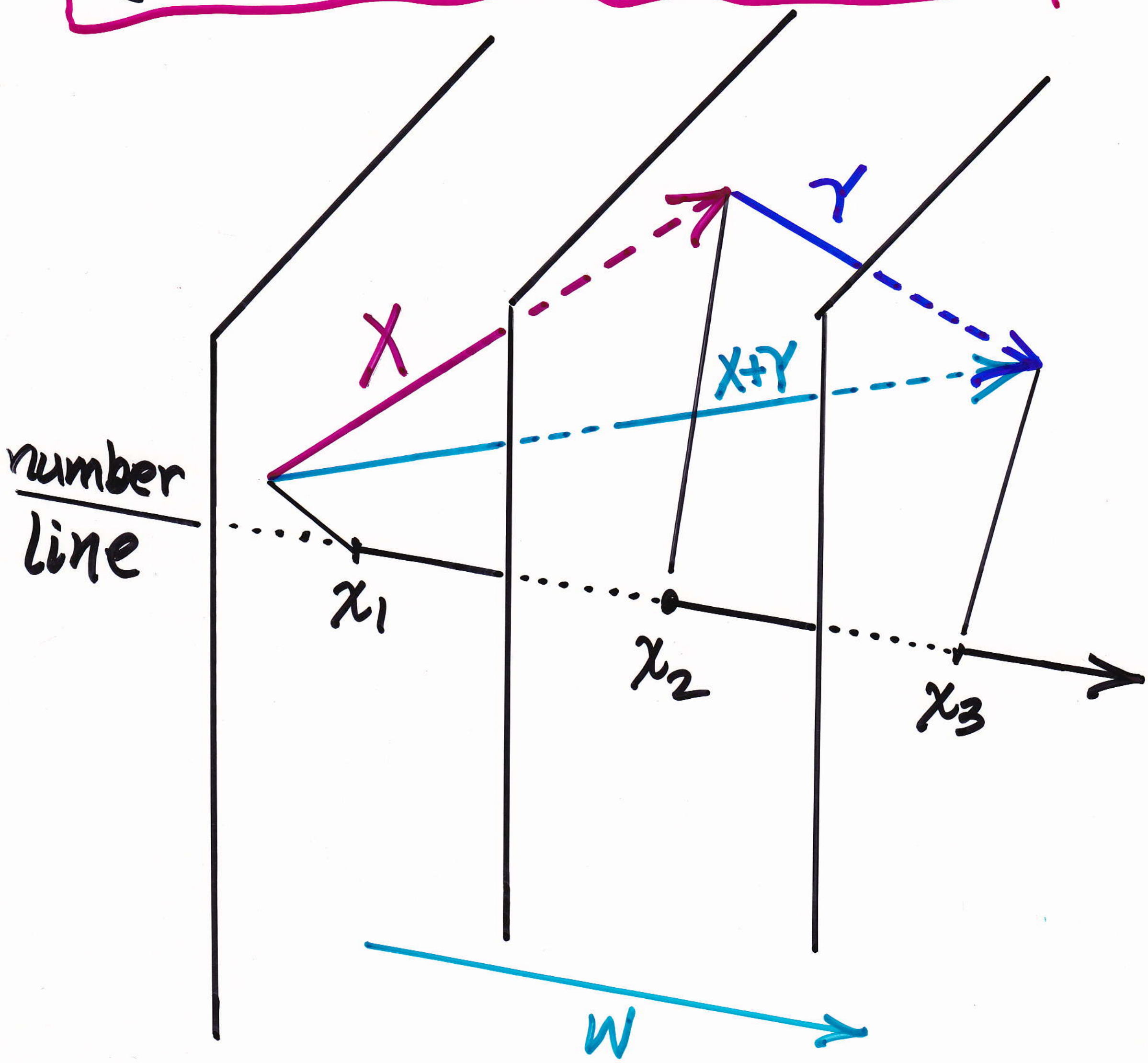


$$\boxed{X \cdot W = (x_2 - x_1)\|W\|}$$

NOTE MOVING X OR W DOES NOT CHANGE $x_2 - x_1$.

DISTRIBUTIVE LAW
FOR
DOT PRODUCT of VECTORS

$$\boxed{(X+Y)\cdot W = X\cdot W + Y\cdot W}$$



$$X\cdot W = (x_2 - x_1)\,\|W\|$$

$$Y\cdot W = (x_3 - x_2)\,\|W\|$$

$$X\cdot W + Y\cdot W \overset{\therefore}{=} (x_3 - x_1)\,\|W\| = (X+Y)\cdot W$$

COMMUTATIVE LAW FOR DOT PRODUCT

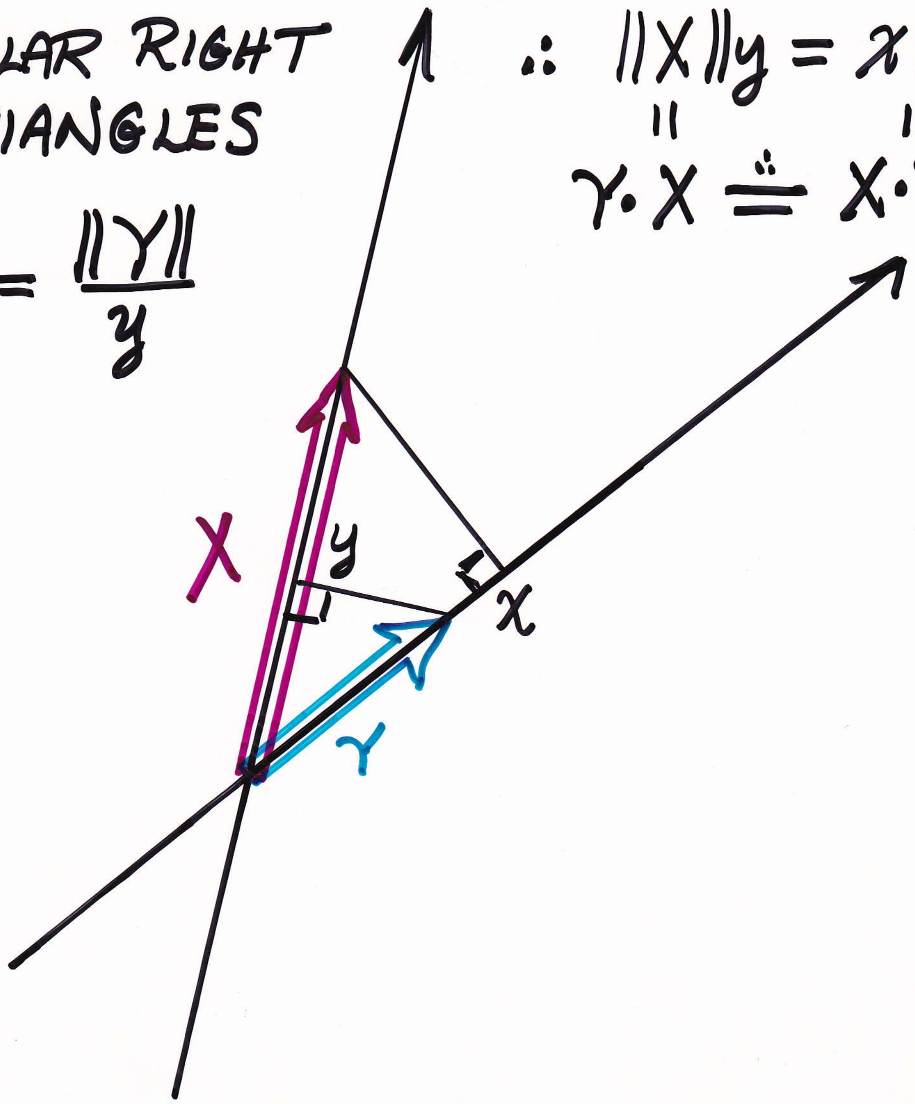
$$\boxed{X \cdot Y = Y \cdot X}$$

PUT TAILS TOGETHER THEN X & Y DETERMINE A PLANE

SIMILAR RIGHT TRIANGLES

$$\frac{\|X\|}{x} = \frac{\|Y\|}{y}$$

$$\therefore \underset{\displaystyle Y \cdot X}{\underset{\shortparallel}{\|X\| y}} = \underset{\displaystyle X \cdot Y}{\underset{\shortparallel}{x \|Y\|}}$$



DOT PRODUCT EXAMPLES

If X and Y point in the same direction then $X \cdot Y = \|X\|\|Y\|$, so just multiply lengths.

$$\boxed{X \cdot Y = 0 \text{ if and only if } X \,\&\, Y \text{ are PERPENDICULAR}}$$

From the picture defining the dot product we see

$$(-X) \cdot Y = -(X \cdot Y)$$

so by the commutative law

$$(-X) \cdot Y = -(X \cdot Y) = X \cdot (-Y)$$

and from the formula defining the dot product then

$$\boxed{(rX) \cdot Y = X \cdot (rY) = r(X \cdot Y)}$$

for any real number r.

MORE DOT PRODUCT EXAMPLES

$$X \cdot X = \|X\|^2 \quad , \quad \|X\| = \sqrt{X \cdot X}$$

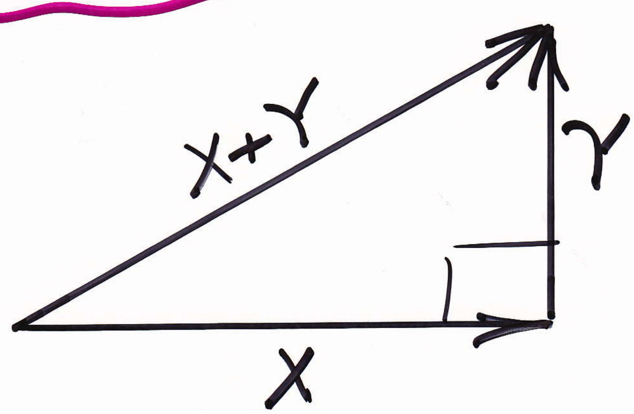
$$\|X+Y\|^2 = (X+Y) \cdot (X+Y)$$

$$= X \cdot X + Y \cdot Y + X \cdot Y + Y \cdot X$$

$$= \|X\|^2 + \|Y\|^2 + 2\,(X \cdot Y)$$

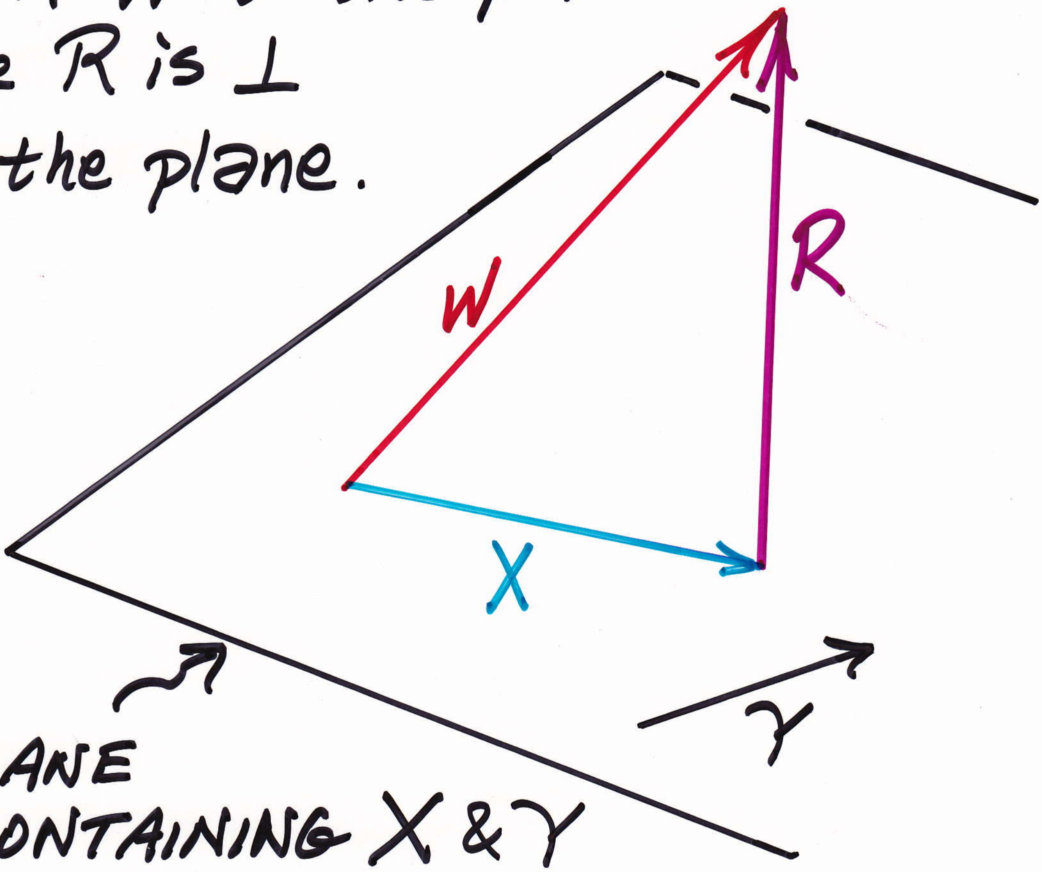
$$\boxed{X \perp Y \Rightarrow \|X+Y\|^2 = \|X\|^2 + \|Y\|^2}$$

(PYTHAGOREAN THEOREM)



STILL MORE DOT PRODUCT EXAMPLES

NOTICE $\|R\|$ is the distance from tip of W to the plane since R is $\perp$ to the plane.



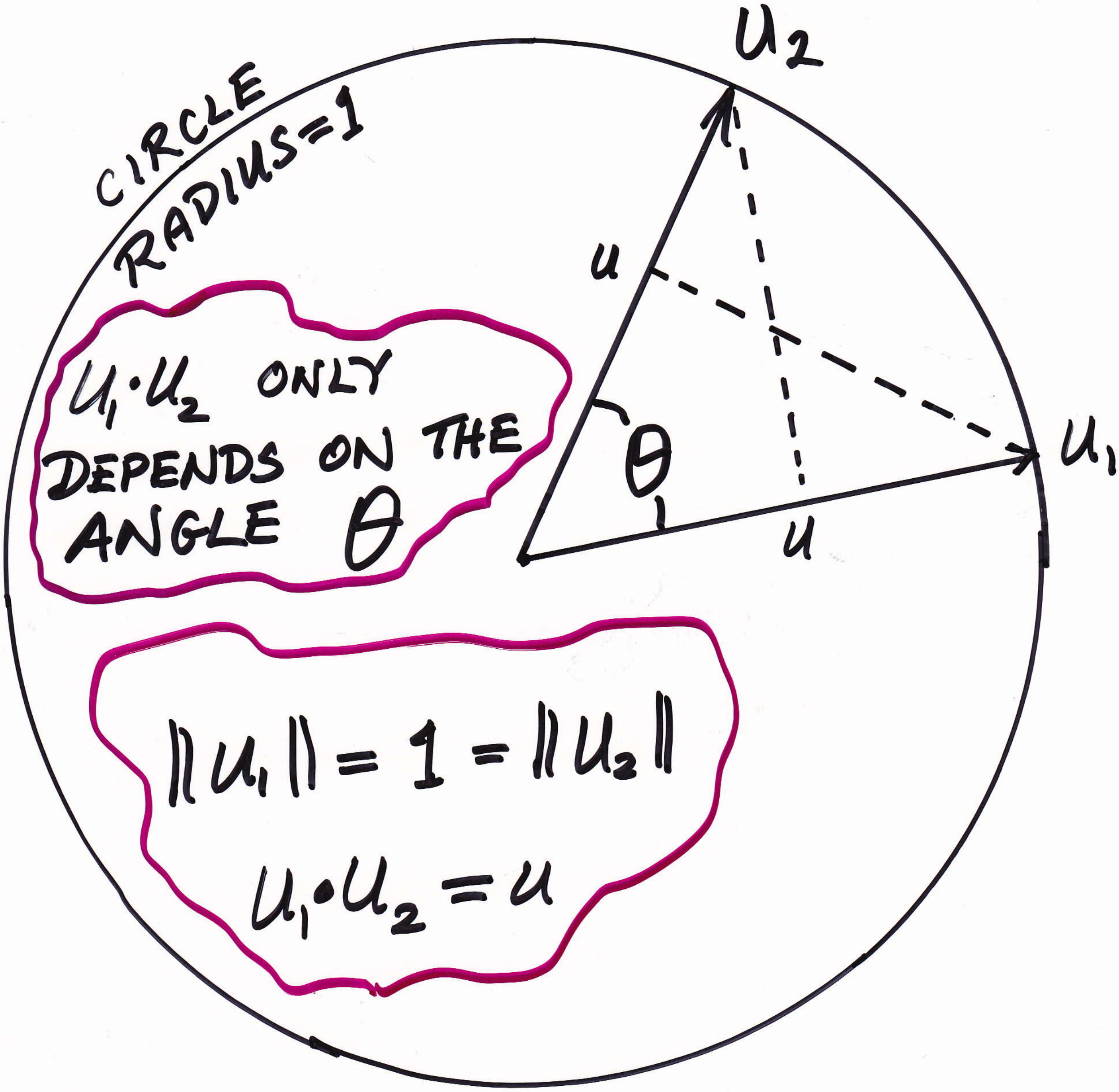
$R \perp$ TO PLANE CONTAINING X & Y

$$X \cdot R = 0 = Y \cdot R$$

$$W \cdot Y = (X+R) \cdot Y = X \cdot Y$$

ALSO $\boxed{\|R\|^2 = \|W\|^2 - \|X\|^2}$

UNIT VECTORS & DOT PRODUCTS



$u = 1$ if and only if $U_1 = U_2$

$u = 0$ if and only if $U_1 \perp U_2$

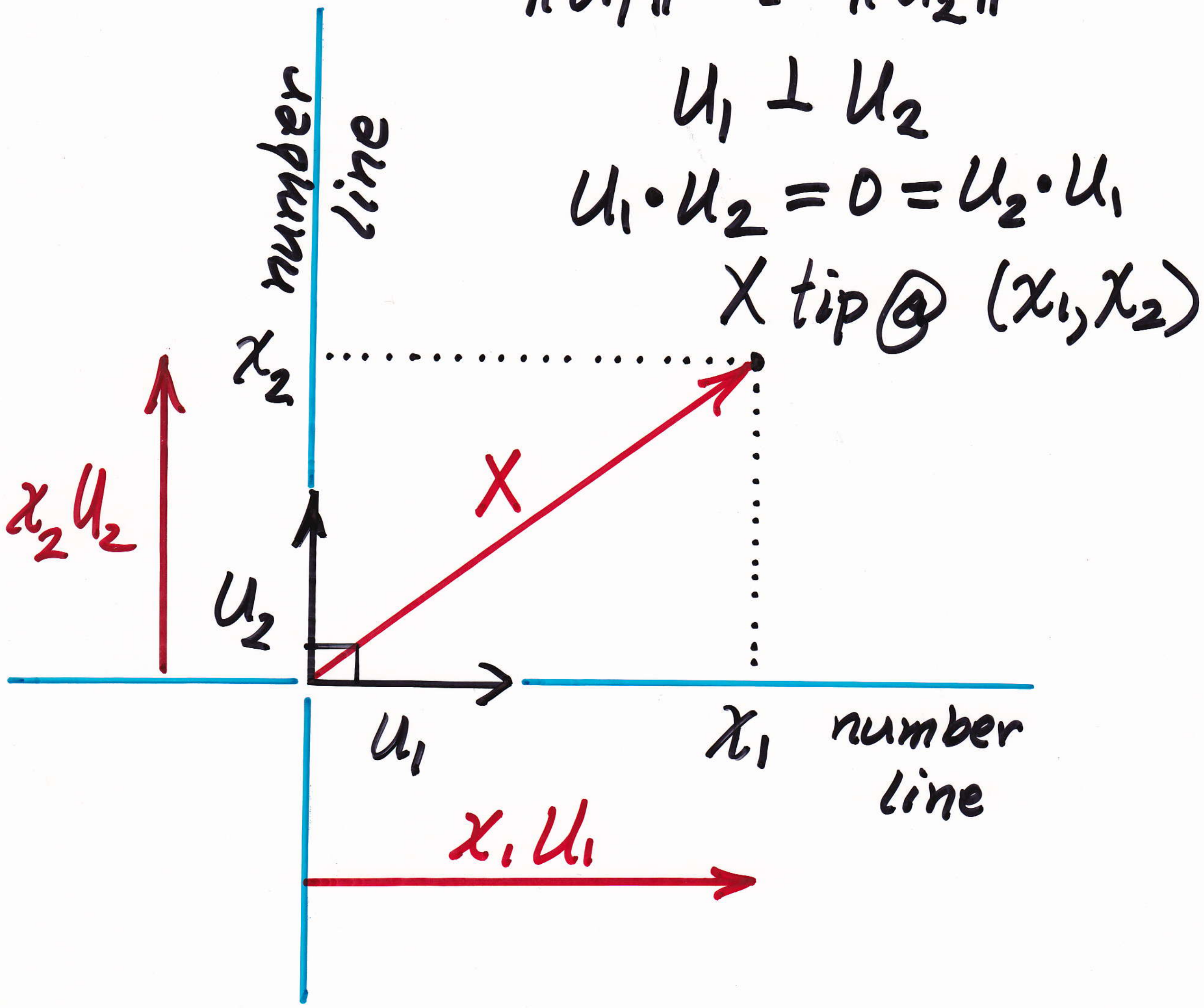
$|u| \leq 1$ and $u = -1$ means $U_1 = -U_2$.

Perpendicular Unit Vectors in Plane

$$\|U_1\| = 1 = \|U_2\|$$

$$U_1 \perp U_2$$

$$U_1 \cdot U_2 = 0 = U_2 \cdot U_1$$



$$X = x_1 U_1 + x_2 U_2$$

$$X \cdot U_1 = x_1\, U_1 \cdot U_1 = x_1$$

$$X \cdot U_2 = x_2\, U_2 \cdot U_2 = x_2$$

$$\boxed{X = (X \cdot U_1) U_1 + (X \cdot U_2) U_2}$$

UNKNOWNS & RANDOM VARIABLES PICTURED AS VECTORS

ADDITION OF UNKNOWNS IS ADDITION OF VECTORS

MULTIPLICATION BY REAL NUMBER IS THE SAME FOR UNKNOWNS AND VECTORS

DOT PRODUCT OF UNKNOWNS

$$\boxed{X \cdot Y = E(XY)} \quad \text{SO} \quad \|X\|^2 = E(X^2)$$

IS COMPUTED WITH EXPECTATION

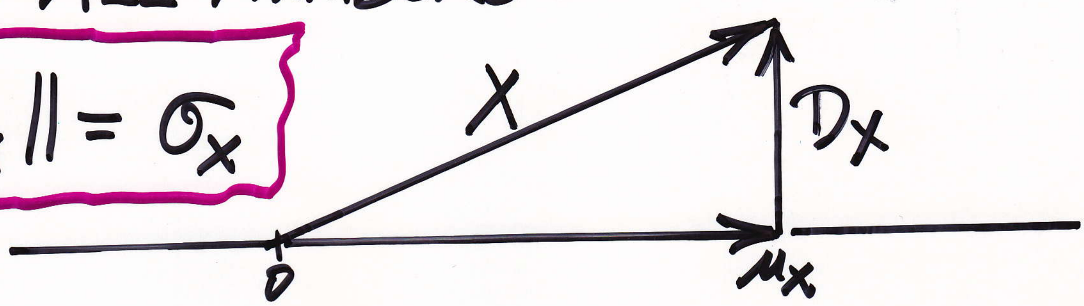
$$X \cdot 1 = E(X1) = E(X) = \mu_X$$

$$D_X = X - \mu_X = \text{DEVIATION}$$

$$r \cdot D_X = E(rD_X) = r\,E(D_X) = 0$$

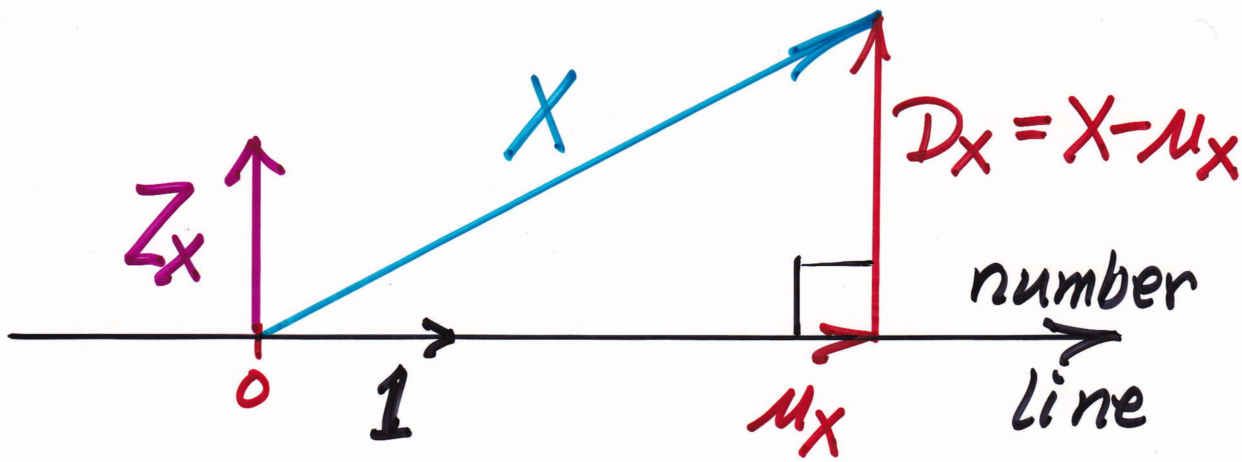
$\therefore$ ALL NUMBERS $\perp$ DEVIATIONS

$$\boxed{\|D_X\| = \sigma_X}$$



PICTURE ONE UNKOWN
OR
RANDOM VARIABLE

A RIGHT TRIANGLE



$$Z_X = \frac{D_X}{\sigma_X} = \frac{D_X}{\|D_X\|}$$

$$\|Z_X\| = 1 = \|1\| \;\&\; Z_X \perp 1$$

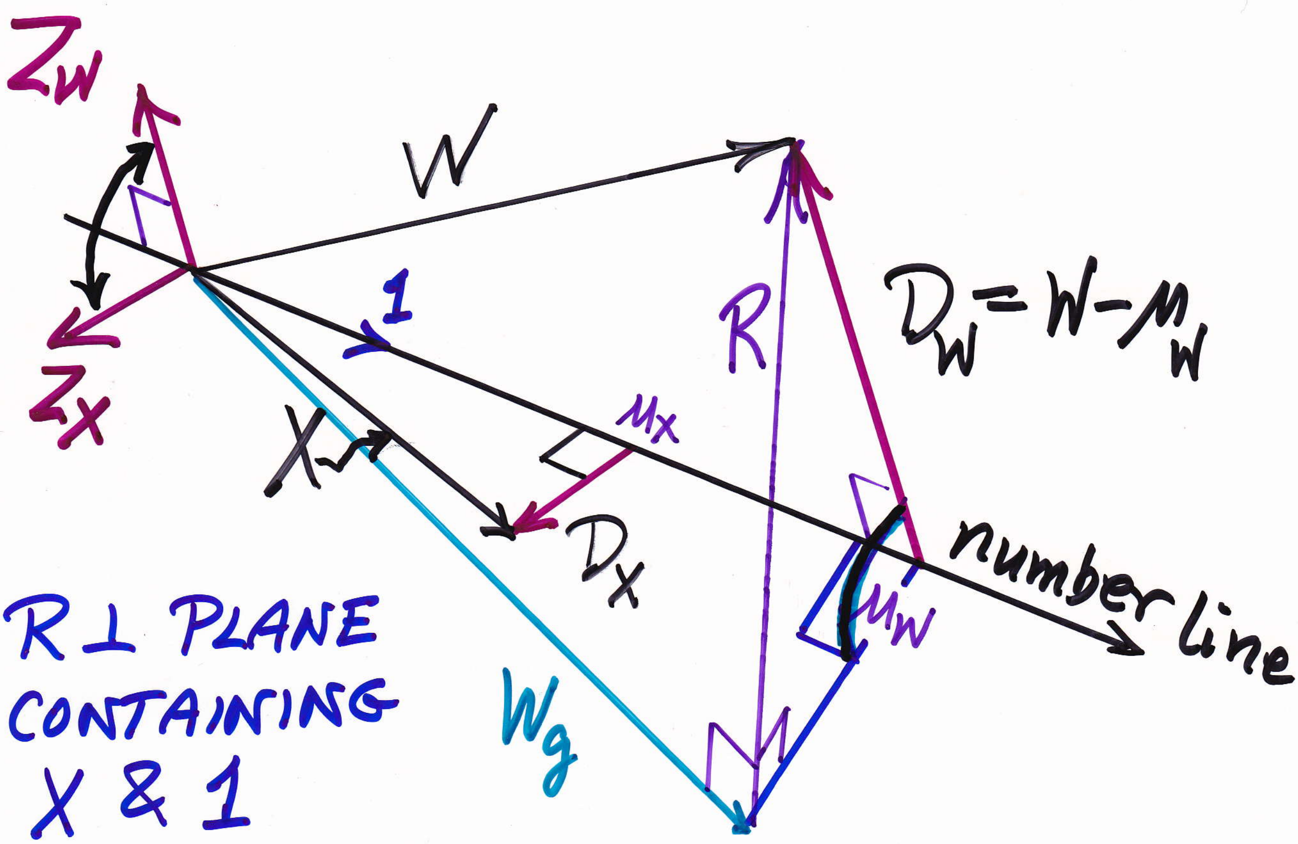
$$X = \mu_X 1 + \sigma_X Z_X$$

$$X \cdot 1 = \mu_X \;\&\; X \cdot Z_X = \sigma_X$$

$$E(X^2) = X \cdot X = \|X\|^2 = \mu_X^2 + \sigma_X^2$$

$$\boxed{E(X^2) = \mu_X^2 + \sigma_X^2}$$

PICTURE TWO UNKNOWNS OR RANDOM VARIABLES



R ⊥ PLANE CONTAINING X & 1

$\therefore\ R \cdot Z_X = 0,\ R \cdot 1 = 0,\ R \cdot W_g = 0$

$W_g \cdot Z_X = W \cdot Z_X \ \&\ W_g \cdot 1 = W \cdot 1$

$W_g = (W_g \cdot 1)1 + (W_g \cdot Z_X) Z_X$

$$\boxed{\therefore\ W_g = (W \cdot 1)1 + (W \cdot Z_X) Z_X}$$

$W \cdot Z_X = D_W \cdot Z_X = \sigma_W (Z_W \cdot Z_X)$

LINEAR REGRESSION USING PICTURE FOR TWO UNKNOWNS OR RANDOM VARIABLES

W_g = "BEST" GUESS FOR VALUE of W GIVEN THE VALUE of X

$E(R^2)$ = ERROR^2 EXPECTED WHEN USING W_g.

$\boxed{Z_W \cdot Z_X = \rho = \text{CORRELATION COEFFICIENT}}$

$$W \cdot 1 = E(W1) = E(W) = \mu_W$$

$$W \cdot Z_X = \sigma_W \, (Z_W \cdot Z_X) = \sigma_W \rho$$

$$W_g = (W \cdot 1)1 + (W \cdot Z_X) Z_X$$

$$\therefore \; W_g = \mu_W + \rho \sigma_W Z_X$$

OR $$W_g = \mu_W + \frac{\rho \sigma_W}{\sigma_X}(X - \mu_X)$$

$$(Z_W)_{guess} = \rho Z_X$$

Expected Error^2 Using Linear Regression

$$W_g = \mu_W \mathbf{1} + \rho \sigma_W Z_X$$

$$\mathbf{1} \perp Z_X \ \& \ \|\mathbf{1}\| = 1 = \|Z_X\|$$

$$\therefore \ \|W_g\|^2 = \mu_W^2 + \rho^2 \sigma_W^2$$

$$R \perp W_g \ \& \ W = W_g + R$$

$$\therefore \ \|W\|^2 = \|W_g\|^2 + \|R\|^2.$$

ALSO $\|W\|^2 = \mu_W^2 + \sigma_W^2.$

$$\therefore \ \mu_W^2 + \sigma_W^2 = \|W\|^2 = \|W_g\|^2 + \|R\|^2$$

$$= \mu_W^2 + \rho^2 \sigma_W^2 + \|R\|^2$$

$$\therefore \ \sigma_W^2 = \rho^2 \sigma_W^2 + \|R\|^2$$

$$\therefore \ E(R^2) = \|R\|^2 = \sigma_W^2 - \rho^2 \sigma_W^2$$

$$\boxed{E(R^2) = (1-\rho^2)\sigma_W^2}$$

VECTORS

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ARROWS IN SPACE