SETS & FUNCTIONS

MAURICE J. DUPRÉ

1. SETS

1.1. **SET:.** an undefined term. A set S can also be called a **Collection**.

1.2. **SET MEMBERSHIP:.** The statement that x is a member of set S is denoted $x \in S$. We also say x is in S to mean $x \in S$. We write $x \notin S$ to mean that x is not a member of S.

2. FUNCTIONS

2.1. **FUNCTION:.** For sets R and S we say f is a **function** from R into S to mean that f is a rule which assigns a member $f(r) \in S$ to each member $r \in R$. **NOTE:** $f(r)$ is usually NOT multiplication.

2.2. **FUNCTION DIAGRAM:.** We write $f : R \longrightarrow S$ to mean f is a function from set R into set S, and call R the **Domain** of f and S the **Codomain** of f.

2.3. **FUNCTION EQUALITY:.** For functions $f : P \longrightarrow Q$ and $g : R \longrightarrow S$ the statement $f = g$ means that $P = R$ and $Q = S$ and for every $x \in P$ also $f(x) = g(x)$. That is to say, equal functions must have the same domain, the same codomain, and the same rule.

2.4. **IDENTITY FUNCTION:.** For any set S we denote by $Id_S : S \longrightarrow S$ the **Identity Function** given by the rule $Id_S(x) = x$, for every $x \in S$.

2.5. **FUNCTION COMPOSITION:.** For any sets Q, R, S, and any functions

$$f : Q \longrightarrow R \text{ and } g : R \longrightarrow S$$

the **Composition** f followed by g, denoted $g \circ f$, is the function

$$g \circ f : Q \longrightarrow S$$

given by the rule

$$(g \circ f)(x) = g(f(x)), \text{ for every } x \in Q.$$

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2.6. **ASSOCIATIVE LAW OF FUNCTION COMPOSITION:.** For sets Q, R, S, T and any functions

$$f : Q \longrightarrow R,\ g : R \longrightarrow S,\ h : S \longrightarrow T,$$

it is true that

$$(h \circ g) \circ f = h \circ (g \circ f).$$

2.7. **IDENTITY COMPOSITION:.** For any function $f : R \longrightarrow S$,

$$Id_S \circ f = f = f \circ Id_R.$$

3. SUBSETS

3.1. **SUBSET:.** We say R is a **Subset** of set S, denoted $R \subset S$, to mean every member of R is also a member of S.

3.2. **SET CONTAINMENT:.** We say the set R **Contains** S, denoted $R \supset S$, to mean $S \subset R$.

3.3. **EMPTY SET:.** We denote by $\emptyset$, the set which has no members and thus if R is any set, then

$$\emptyset \subset R.$$

3.4. **SET BUILDER NOTATION:.** If $A(x)$ is a statement for each x in set U, then the subset of U consisting of all members x of U for which $A(x)$ is true is denoted

$$\{x \in U : A(x)\} \text{ or } \{x \in U \mid A(x)\},$$

or simply by $\{x : A(x)\} = \{x \mid A(x)\}$, when U is understood. If $a_1, a_2, ..., a_n$ all belong to U, then

$$\{a_1, a_2, a_3, ..., a_n\} = \{x \in U \mid x = a_k,\ \text{for some } k,\ 1 \leq k \leq n\}.$$

In particular, $\emptyset = \{x \in U \mid x \neq x\}$.

3.5. **SET DIFFERENCE:.** If A and B are sets, then $A \setminus B$ called the **Set Difference** of A and B is defined by

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

3.6. **SET INTERSECTION:.** If A and B are sets, then their **Intersection**, denoted $A \cap B$, is the set

$$A \cap B = \{x \in A : x \in B\} = \{x : x \in A \ \& \ x \in B\} = \{x \in B : x \in A\},$$

and thus consists of their common membership. The symbol $\cap$ is called the **Cap** symbol.

3.7. **INTERSECTION PROPERTY:.** If A, B, C are sets, then

$$B \cap C \subset B \text{ and } B \cap C \subset C,$$

and moreover,

$$[(A \subset B) \ \& \ (A \subset C)] \text{ implies } [A \subset B \cap C].$$

Thus $B \cap C$ is the largest subset common to both sets B and C.

3.8. **UNION AXIOM:.** Given any two sets there is a set which contains both of them as subsets. Consequently, for any finite number of sets there is a set which contains all of them as subsets. More generally, we assume axiomatically, that for any collection all of whose members are sets, there is a set containing all the sets in the collection as subsets.

3.9. **SET BUILDING AXIOM:.** Every set is itself the member of some set. Thus for any set A, we can form the set $\{A\}$ which is the set having exactly one member, namely the set A, itself. Thus, if $A_1, A_2, A_3, ..., A_n$ are all sets, then so is $\{A_1, A_2, A_3, ..., A_n\}$.

3.10. **SET UNION:.** Given any two sets A and B, choose a set U containing both A and B as subsets, and define their **Union**, denoted $A \cup B$, as the subset of U given by

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}.$$

The symbol $\cup$ is called the **Cup** symbol.

3.11. **UNION PROPERTY:.** If A, B, C are all sets, then

$$A \subset A \cup B \text{ and } B \subset A \cup B,$$

and moreover

$$[(A \subset C) \ \& \ (B \subset C)] \text{ implies } [A \cup B \subset C].$$

Thus $A \cup B$ is the smallest set containing both A and B as subsets.

3.12. **GENERAL SET UNION:.** If C is a collection of sets, choose a set U which contains every set in the collection as a subset, and define the union of the collection, denoted $\bigcup C$, as the subset of U given by

$$\bigcup C = \{x \in U \mid x \in A, \text{ for some } A \in C\}.$$

Thus, in particular,

$$A \cup B = \bigcup \{A, B\}.$$

3.13. **GENERAL UNION PROPERTY:.** If C is any collection of sets, then

$$A \subset \bigcup C, \quad \text{for every } A \in C,$$

and moreover, for any set U,

$$\text{if } A \subset U, \text{ for every } A \in C, \text{ then } \bigcup C \subset U.$$

Thus $\bigcup C$ is the smallest set containing all the sets in C as subsets.

23. GENERAL SET INTERSECTION: If C is a collection of sets, then the intersection of the collection is the intersection of all the sets in the collection, denoted $\bigcap C$ and, defined by

$$\bigcap C = \{x \in \bigcup C \mid x \in A, \text{ for every } A \in C\}.$$

Thus, in particular,

$$A \cap B = \bigcap \{A, B\}.$$

3.14. **GENERAL INTERSECTION PROPERTY:.** If C is a collection of sets, then

$$\bigcap C \subset A, \text{ for every } A \in C,$$

and moreover, for any set B,

$$\text{if } B \subset A, \text{ for every } A \in C, \text{ then } B \subset \bigcap C.$$

Thus, $\bigcap C$ is the largest set which is a subset of every member of C

4. INFINITE SETS

4.1. **NATURAL NUMBERS:.** We define the **natural number zero** as the empty set, $0 = \emptyset$. We define the **natural number one** as the set containing one member which is the empty set, in symbols, $1 = \{0\}$. We define the **natural number 2** as the set containing zero and one, in symbols, $2 = \{0, 1\}$. **And So On.** In this way, every **Natural Number** n becomes a set with exactly n members.

4.2. **AXIOM ON INFINITY:.** There is a set which contains every natural number.

4.3. **THE SET OF NATURAL NUMBERS:.** We define $\mathbb{N}$ to be the **Set of all Natural Numbers**. To actually make $\mathbb{N}$ simply choose any set U containing all the natural numbers and define

$$\mathbb{N} = \{n \in U \mid n \text{ is a natural number}\}.$$

4.4. **THE SET OF RATIONAL NUMBERS:.** We define the **Set of all Rational Numbers**, denoted by $\mathbb{Q}$ as the set of all fractions of natural numbers:

$$\mathbb{Q} = \{p/q \mid p, q \in \mathbb{N}, \ q \neq 0\}.$$

4.5. **THE SET OF REAL NUMBERS:.** We denote the **Set of all Real Numbers** with the symbol $\mathbb{R}$, so $\mathbb{N} \subset \mathbb{R}$. The construction of the real numbers as a set involves technicallities beyond the scope of these notes. However, in brief, one possible way to construct a real number is to define the concept of a **Cut**, a construction due to Dedekind. We can say that $J \subset \mathbb{Q}$ is a **Cut** provided that it has the property that if $r \leq s$ and $s \in J$, then $r \in J$. If J is a cut and $K = \mathbb{Q} \setminus J$, and if $r \leq s$ with $r \in K$, then $s \in K$, since otherwise $s \in J$ which would mean $r \in J$, a contradiction. If $r \in J$ and $s \in K = \mathbb{Q} \setminus J$, then either $r < s$ or $s < r$, but the latter would imply that $s \in J$, a contradiction, so it must be the case that $r < s$ here. One then defines $\mathbb{R}$ as the set of all such cuts of rational numbers. Next it must be shown that such cuts behave like numbers as we think of them, namely, we need to be able to add and multiply with the usual laws holding. For instance, if H and J are cuts, we say $H \leq J$ to mean $H \subset J$. If H and J are cuts, then the sum and product of cuts to produce new cuts must be defined. For instance,

$$H + J = \{r + s \mid r \in H,\ s \in J\},$$

which is not hard to see must be a cut of rational numbers. Likewise,

$$HJ = \{rs \mid r \in H,\ s \in J\}.$$

If H and J are cuts, and if H is not contained in J, then there is $h \in H$ which is not in J, so and if $j \in J$ were to have $j > h$, then it would be the case that $h \in J$, as J is a cut, so it must be that $j < h$, and therefore $j \in H$, as H is a cut. This means that for any two cuts H and J, either $H \subset J$, or $J \subset H$. If we define $H \leq J$ to mean that $H \subset J$, then the set of all cuts has an order which behaves like the order for real numbers. In particular, if S is a set of cuts, then $\bigcup S$ is also a cut which is the least upper bound of the cuts in S, so we say the set of cuts is **Order Complete**, which is the crucial property of the set of real numbers which allows us to think of the set of all real numbers as being continuous like a line.

5. SET CONSTRUCTIONS WITH FUNCTIONS

5.1. **FUNCTION SETS AXIOM:.** If R and S are sets, then there is a set which contains all functions with domain R and codomain S.

5.2. **SET EXPONENTIATION:.** If R and S are sets, then choosing any set U which contains all functions with domain R and codomain S, we define S^R as the set

$$S^R = \{f \in U \mid f : R \longrightarrow S\}.$$

5.3. **PAIRING:.** If U is a set and $x, y \in U$, then we can define a function $f : 2 \longrightarrow U$ by the rule $f(0) = x$ and $f(1) = y$. Notice that as $2 = \{0, 1\}$, the rule is completely specified on the domain 2. We henceforth denote this function by (x, y) and call it an **Ordered Pair**. Notice that for ordered pairs (a, b) and (c, d), equality, $(a, b) = (c, d)$ means both $a = c$ and $b = d$.

5.4. **LISTING:.** If $x_0, x_1, x_2, ..., x_{m-1}$, are all members of a set U, then we can form the **Ordered List**, denoted $x = (x_0, x_2, x_3, ..., x_{m-1})$, called a **Tuple** or, more precisely, an $m-$tuple, defined as the unique function $f : m \longrightarrow U$ whose rule is $f(k) = x_k,\ 0 \leq k < m$. Therefore, for any two tuples $x = (x_0, x_1, x_2, ..., x_{m-1})$ and $y = (y_0, y_1, y_2, ..., y_{n-1})$ we have $x = y$ means that $m = n$ and for every k with $0 \leq k < m$, we also have $x_k = y_k$. That is two lists are the same if and only if they have the same length and the same entries.

5.5. **CARTESIAN PRODUCT:.** For any two sets A and B, we define their **Cartesian Product**, denoted $A \times B$, by

$$A \times B = \{(x, y) \mid x \in R \text{ and } y \in S\} = \{f \in [A \cup B]^2 \mid f(0) \in A \text{ and } f(1) \in B\}.$$

5.6. **GENERAL CARTESIAN PRODUCT OF A COLLECTION OF SETS:.** If C is a collection of sets set $U = \bigcup C$ and define the **Cartesian Product of the Collection**, denoted $\prod C$, by

$$\prod C = \{f \in U^C \mid f(A) \in A, \text{ for every } A \in C\}.$$

If D is a set and $A : D \longrightarrow C$, we often denote $A(d) = A_d$ for $d \in D$, and call D an **Index Set** and write $A = (A_d)_{d \in D}$, calling this an **Indexed Collection of Sets** in which case we define the cartesian product, denoted $\prod_D A_d$, by

$$\prod_D A_d = \prod_{d \in D} A_d = \{f \in U^D \mid f(d) \in A_d, \text{ for every } d \in D\}.$$

If $n \in \mathbb{N}$ and $A_0, A_1, A_2, ..., A_{n-1}$ are sets, define $C = \{A_0, A_1, A_2, ..., A_{n-1}\}$ and define $f : n \longrightarrow C$ by $f(k) = A_k,\ 0 \leq k < n$. Then the cartesian product of these sets is

$$A_0 \times A_1 \times A_2, ..., A_{n-1} = \prod_{k<n} A_k = \{(x_0, x_1, x_2, ..., x_{n-1}) \mid x_k \in A_k,\ 0 \leq k < n\}.$$

5.7. **AXIOM OF CHOICE:.** If C is a non-empty collection of non-empty sets, then

$$\prod C \neq \emptyset.$$

5.8. **DISTRIBUTIVE LAWS:.** For any sets A, B, C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

5.9. **COMMUTATIVE LAWS:.** For any sets A, B,

$$A \cap B = B \cap A,$$

$$A \cup B = B \cup A.$$

5.10. **SET COMPLEMENT:.** When dealing with sets which are all subsets of a given fixed set U, sometimes called a **Universe** in this setting, for $A \subset U$, we call $U \setminus A$ the complement of A and denote this by A'. Then,

$$A \setminus B = A \cap B', \text{ if } A \subset U \text{ and } B \subset U.$$

5.11. **DEMORGAN LAWS:.** If U is a set and if C is a non-empty collection of subsets of U, then

$$\left[\bigcup C\right]' = \bigcap \{A' \mid A \in C\}$$

$$\left[\bigcap C\right]' = \bigcup \{A' \mid A \in C\}.$$

For instance, if A and B are subsets of U, then

$$[A \cup B]' = A' \cap B',$$

$$[A \cap B]' = A' \cup B'.$$

5.12. **IMAGES:.** If $f : R \longrightarrow S$, and if $A \subset R$, then we define the **Image** of A under f, denoted $f(A) \subset S$ by

$$f(A) = \{f(x) \mid x \in A\} = \{y \in S \mid y = f(x) \text{ for some } x \in A\}.$$

We say that f is **Surjective** or **Onto** provided $f(R) = S$. A surjective function is called a **Surjection**.

5.13. **COMPOSITION OF IMAGES:.** If $f : Q \longrightarrow R$ and $g : R \longrightarrow S$ and $A \subset Q$, then

$$[g \circ f](A) = g(f(A)).$$

In particular, the **Composition of Surjective functions is again Surjective**.

5.14. **UNION OF IMAGES IS IMAGE OF UNION:.** If $f : R \longrightarrow S$, if C is a collection of subsets of R, then

$$f\left(\bigcup C\right) = \bigcup\{f(A) \mid A \in C\}.$$

5.15. **INVERSE IMAGE:.** If $f : R \longrightarrow S$, and if $B \subset S$, we define the **Inverse Image** of B under f, denoted $f^{-1}(B) \subset R$, by

$$f^{-1}(B) = \{x \in R \mid f(x) \in B\}.$$

5.16. **COMPOSITION OF INVERSE IMAGES:.** If $f : Q \longrightarrow R$, if $g : R \longrightarrow S$, if $B \subset S$, then

$$[g \circ f]^{-1}(B) = f^{-1}(g^{-1}(B)).$$

5.17. **INJECTIVE FUNCTIONS:.** If $f : R \longrightarrow S$, then f is **Injective** or **One to One** if and only if for each $y \in S$ the subset $f^{-1}(y)$ has at most one member. Equivalently, f is one to one if and only if for any $x_1, x_2 \in R$, the equation $f(x_1) = f(x_2)$ always implies that $x_1 = x_2$. An injective function is called an **Injection**.

5.18. **COMPOSITION OF INJECTIVES IS INJECTIVE:.** If $f : Q \longrightarrow R$ and $g : R \longrightarrow S$ are both injective, then so is their composition $g \circ f$.

5.19. **INVERSE IMAGE AND SET OPERATIONS:.** If $f : R \longrightarrow S$ and if C is a collection of subsets of S and if B is a subset of S, then

$$f^{-1}\left(\bigcup C\right) = \bigcup\{f^{-1}(K) \mid K \in C\},$$

$$f^{-1}\left(\bigcap C\right) = \bigcap\{f^{-1}(K) \mid K \in C\},$$

$$f^{-1}(S \setminus B) = R \setminus f^{-1}(B).$$

5.20. **MUTUALLY INVERSE FUNCTIONS:.** We say that f and g are **Mutually Inverse Functions** provided that $g \circ f$ is the identity on the domain of f and $f \circ g$ is the identity on the domain of g. Thus $g \circ f$ and $f \circ g$ are identity functions. In more detail, if $f : R \longrightarrow S$ and $g : S \longrightarrow R$, then f and g are mutually inverse functions if and only if $g \circ f = Id_R$ and $f \circ g = Id_S$. Thus, if f and g are mutually inverse funtions, $f : R \longrightarrow S,\ g : S \longrightarrow R$, then

$$g(f(x)) = x, \text{ for every } x \in R \text{ and } f(g(y)) = y, \text{ for every } y \in S.$$

In particular, we notice that if f and g are mutually inverse functions, then both must be injective or **One to One**, and moreover, f and g are both also surjective. A function which is injective and surjective is **Bijective**. Thus, f and g are both bijective. Conversely, if f is bijective, then there is a function g so that f and g are mutually inverse, in which case, g is also bijective. A bijective function is also called a **Bijection**.

5.21. **INVERSE FUNCTIONS:.** We say $f : R \longrightarrow S$ is **Invertible** or has an inverse provided there is $g : S \longrightarrow R$ so that f and g are mutually inverse. Thus f is invertible if and only if f is bijective. Then g is uniquely determined by f and we call g the **Inverse** of f and denote this by writing $g = f^{-1}$. This can sometimes be a confusing notation in case that S is a set of real numbers, so be careful with this notation. If $f : R \longrightarrow S$ and $h : Q \longrightarrow R$ are both invertible functions, then so is $f \circ h$, and

$$(f \circ h)^{-1} = h^{-1} \circ f^{-1}.$$

Also be careful to notice that inversion reverses the order of composition. This is significant because generally composition of functions is not commutive, even if $Q = R = S$.

5.22. **POWER SET AXIOM:.** If A is a set, then there is a set which has every subset of A as a member.

5.23. **POWER SET:.** If S is a set, we define $P(S)$ to be the set of all subsets of S and call $P(S)$ the **Power Set** of S. If $f \in 2^S$, then $f : S \longrightarrow 2 = \{0, 1\}$, and we call f an **Indiator**. Then

$$f^{-1}(1) \subset S.$$

On the other hand, if $A \subset S$, we define then **Indicator** of A to be the function $I_A : S \longrightarrow 2$ defined by $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \in S \setminus A$. Notice that each indicator determines a unique subset of S and each subset of S defines a unique indicator, that is, $A = f^{-1}(1)$ if and only if $I_A = f$. Through the correspondence between indicators and subsets, we can regard $P(S)$ as the same as 2^S, which is the reason $P(S)$ is called the power set of S. That is, more precisely, we can define the bijection $I : P(S) \longrightarrow 2^S$ by the rule

$$I(A) = I_A \text{ for } A \in P(S),$$

and then its inverse is the bijection $J : 2^S \longrightarrow P(S)$ given by the rule

$$J(f) = f^{-1}(1), \quad \text{for } f \in 2^S.$$

5.24. **FUNCTION GRAPH:.** If $f : R \longrightarrow S$ is any function, then we can define its **Graph**, denoted Graph(f), as the subset of $R \times S$ given by

$$\text{Graph}(f) = \{(r, s) \in R \times S \mid s = f(r)\}.$$

5.25. **FUNCTION GRAPH PICTURE:.** Suppose that $f : R \longrightarrow S$ is a function. Notice that if R and S are subsets of $\mathbb{R}$, then we can picture the graph of f, as a subset of $R \times S \subset \mathbb{R}^2$, and $\mathbb{R}^2$ is easily pictured as the ordinary two-dimensional coordinate plane. Thus, the graph of f in this case can be pictured as a subset of the plane. In general, for $r \in R$ we set $r \times S = \{r\} \times S$ and think of this as the "vertical slice" of $R \times S$ passing through $r \in R$. Likewise, for $s \in S$, we set $R \times s = R \times \{s\}$ and think of this as the "horizontal slice" of $R \times S$ passing through $s \in S$. Thus, when we "picture" the graph of f, we realize that all vertical slices intersect the graph in exactly a single point. In fact, if $F \subset R \times S$ with the property that all vertical slices intersect F in exactly a single point, then $F =$Graph(f) for a uniquely deterimined function $f : R \longrightarrow S$. To define f, we use the rule that $f(r) = s$ where (r, s) is the unique point of $F \cap [r \times S]$.

We also see that $f : R \longrightarrow S$ is surjective if and only if every horizontal slice intersects the graph of f at least once. Moreover, f is injective if and only if each horizontal slice intersects the graph of f at most once. In particular, f is bijective if and only if each horzontal slice intersects the graph of f exactly once. In this case, we see that

$$\text{Graph}(f^{-1}) = \{(s, r) \in S \times R \mid (r, s) \in \text{Graph}(f)\}, \text{ for } f \text{ invertible .}$$

The graph of $Id_R : R \longrightarrow R$ is the set

$$\text{Graph}(Id_R) = \{(r, s) \in R \times R \mid r = s\},$$

which we call the **Diagonal** of $R \times R$. In case that $R \subset \mathbb{R}$, then the diagonal is contained in the actual diagonal line of the coordinate plane, $\mathbb{R}^2$.

We can define the twist map $Twist_{(R,S)} : R \times S \longrightarrow S \times R$ by the rule

$$Twist_{(R,S)}(r, s) = (s, r), \text{ for } (r, s) \in R \times S.$$

Notice that $Twist_{(R,S)}$ and $Twist_{(S,R)}$ are mutually inverse functions, so each is a bijection. For any $F \subset R \times S$, we define $F^{-1} = Twist(F) \subset S \times R$, so F^{-1} is simply the result of reversing the order of all ordered pairs in the set F. We then have

$$\text{Graph}(f^{-1}) = [\text{Graph}(f)]^{-1}, \text{ if } f : R \longrightarrow S \text{ is invertible.}$$

Department of Mathematics, Tulane University, New Orleans, LA 70118 USA
E-mail address, M. J. Dupré: `mdupre@tulane.edu`