Covariant energy-momentum and an uncertainty principle for general relativity

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Abstract

We recognize the natural covariant extension for energy-momentum in general relativity: energy-momentum in spacetime as opposed to space. The key indicator is the Tolman energy integral for stationary systems. The demand that the general expression for arbitrary dynamic systems reduce to the Tolman integral in the case of stationary bounded distributions leads to the matter-localized Ricci integral for energy-momentum in support of the energy localization hypothesis. The role of the observer is addressed and it is recognized that the field of freely-falling observers extract the global Tolman energy. It is suggested that in the extreme of strong gravity, the Heisenberg Uncertainty Principle be generalized in terms of spacetime energy-momentum.

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The unresolved problems of energy and the issue of its localization within general relativity were debated in the early years of the theory's formulation and remain subjects of considerable interest today. In the 1920s, some authors including Einstein and Eddington held the view that while one could work usefully with energy in the traditional sense as a global concept in general relativity,

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no satisfactory meaning could be attached to its localization. Their belief was based on the manner in which energy-momentum for the gravitational field was incorporated into general relativity. Rather than having a *bona fide* energy-momentum tensor T_i^k [1] in general relativity to incorporate energy-momentum as in the rest of physics, these authors relied upon an energy-momentum pseudotensor t_i^k, first introduced by Einstein, to play the equivalent role for gravity. Unlike true tensors, this pseudotensor could be made to vanish at any pre-assigned point by an appropriate transformation of coordinates, rendering its status rather nebulous.

Through the years, other pseudotensors performing the same function as that of Einstein's pseudotensor were introduced but they all carried the stigma of being non-covariant objects. In addition, they were not symmetric and hence did not lend themselves to forming an angular momentum construct as does the symmetric energy-momentum tensor T^{ik} of special relativity. Landau and Lifshitz [1] were able to produce a symmetric energy-momentum peudotensor but their construct was not Lagrangian-based. Moreover, like all pseudotensors, their's suffered from a lack of general covariance and could be made to vanish at any pre-assigned point by an appropriate choice of coordinate system.

Over the years, the debate over energy-momentum in general relativity took some interesting turns. Various authors including Bondi [2] argued that gravitational energy must be localizable while Misner and Sharp [3] accepted that energy was localizable for cases of spherical symmetry but not otherwise. Related to the energy issue was that of the nature of gravitational radiation, waves of gravity presumably carrying energy off to infinity in analogy with electromagnetic radiation. Many authors had calculated energy fluxes via gravitational radiation using energy-momentum pseudotensors and it was widely believed that the process was placed on a secure footing by Bondi [4] who developed a "news function" to describe the flux. However, it was later shown that the news function is related to the pseudotensor and hence it carries the drawbacks and limitations of the latter [5].

As a result of this history, the prevailing general feeling would appear to be that the energy of the gravitational field itself is not localizable and for total energy of a system including gravity one must take a so-called "quasi-local" approach. This has largely grown out of the ADM analysis of energy and momentum for spatially asymptotically flat models which in turn generalizes the Komar mass and momentum for models admitting timelike Killing fields. The Komar mass and momentum are defined as 2-surface integrals which for boundaries coincide with the Tolman integral via Gauss' Theorem. The proofs of the Positive Energy Theorem for the ADM mass and momentum under the assumption of the dominant energy condition, first provided by Schoen and Yau [6] and the simpler spinor proof given by Witten [7] have led to several different definitions of quasi-local mass having various differing properties, advantages and disadvantages. Later authors have made the Witten spinor proof rigorous (e.g. [8]) and generalized it to include the Bondi mass [9] in the asymptotically

[1] Latin indices range from 0 to 3 and Greek indices range from 1 to 3.

flat case. In spirit, we can say that these approaches attempt to make sense of how an observer sees the energy density in regions away from his location as compared to the energy density at his own location event, and are thus formed using integration. The Positive Energy Theorem itself has been a major factor in the acceptance of the quasi-local approach, but it must be noted that it depends on the assumption of the dominant energy condition. However, recently, all the energy conditions of general relativity have come into question [10]. On the other hand, if we have an acceptable energy-momentum tensor for the total energy including the gravitational field, it would be natural for an observer wishing to evaluate the total energy in a remote region to require observers located at each event in the region to report the energy density they observe and then to integrate all the results. Such an approach would not depend on any assumptions of energy conditions, but of course would depend on the choice of observers throughout the region.

In a series of papers [11], author FIC presented a new hypothesis that energy, including the contribution from gravity, was localized in the non-vanishing regions of the energy-momentum tensor T_i^k. This carries the significant consequence that gravity waves do not actually convey energy in the course of their propagation through the vacuum. (This is assuming that the waves really do exist and they must exist if the essential element of finite velocity of propagation of information in relativity holds).

Various reasons were advanced in support of the hypothesis. There was the work of Gurses and Gursey [12] showing that an exact gravitational plane wave, the simplest of all wave types, is of the Kerr-Schild class for which one can always find a coordinate system where *all* the components of the gravitational energy-momentum pseudotensor vanish identically everywhere, not just at a pre-selected point. Thus, one would be hard-pressed to believe that energy is flowing if the construct representing its energy can be transformed out of existence in one fell swoop. This is in contrast to the energy-momentum tensor in electromagnetism whose time-time component T_0^0 and time-space components T_0^α which represent respectively the energy density and Poynting vector of energy flux, are non-zero for an electromagnetic wave and remain non-zero for all transformations of the coordinates. Various other reasons were advanced in support of the localization hypothesis but the goal of finding a tensorial construct for general relativity embodying such a localization, remained unrealized.

In a very recent paper [13], author MJD provided arguments in favor of the Ricci tensor R_{ik} as the essential tensor embodying energy-momentum in general relativity, i.e. for the inclusion of the contribution from gravity. Clearly this is in support of the localization hypothesis as the Ricci tensor vanishes in vacuum. The choice of the Ricci tensor has immediate appeal as it is generally covariant, a true tensor unlike the pseudotensor. As well, it is attractive from another aspect: Bondi [14] had noted that since the Riemann tensor characterizes the presence of spacetime curvature, i.e. the essence of gravity, in an invariant manner, it would be natural, he reasoned, if the Riemann tensor were to describe the energy-momentum including the contribution from gravity. Since the second-rank Ricci tensor is formed as a trace of the fourth-rank Riemann tensor, it carries in part the content of the latter and hence could be seen to embody at least the spirit of the Bondi idea.

In this paper, we approach the problem from a different direction. The most reliable identification that we can make with energy for a system that includes the contribution from gravity within the framework of general relativity, stems not from local measure but rather global measure. Tolman [15] was able to show that the total energy of an isolated stationary (i.e. having no explicit dependence on time) distribution of matter could be expressed as

$$E = \frac{c^4}{4\pi G} \int R_0^0 \sqrt{-g} d^3 x \tag{1}$$

where g is the determinant of the four-dimensional spacetime metric and R_0^0 is the mixed time-time component of the Ricci tensor. (Naturally the coordinate system that is chosen for the derivation is one for which the system shows no explicit time dependence.) Its measure as the correct total energy is secured by its connection with the coefficient of the $1/r$ term in the asymptotic form of the g_{00} component of the metric tensor. There is no ambiguity about this measure of total energy. Through the use of the Einstein field equations it is more often expressed in terms of the components of the energy-momentum tensor as

$$E = \int \left(T_0^0 - T_1^1 - T_2^2 - T_3^3 \right) \sqrt{-g} d^3 x. \tag{2}$$

Tolman used the pseudotensor to achieve this result and a more elegant approach that did not rely upon the pseudotensor was later applied in [1] which produced the same answer.

An immediate point to note is that for either expression (1) or (2), while the square root of the metric determinant is for the complete four-dimensional spacetime metric, the integral is over the three-space coordinate volume d^3x. The integral would be one over the desired proper volume if d^3x were replaced by d^4x to connect properly with $\sqrt{-g}$.

At this point, we consider what is the minimum modification that we can make to (1) that would render an expression with wholly covariant elements for energy-momentum distributions including the contribution from gravity in the case of *arbitrary* systems, systems that may have intrinsic time-dependence, while incorporating the demand that its energy component reduce to (1) [2] in the case of a bounded stationary distribution. As a minimum, we must change to a *spacetime* integral (i.e. replace d^3x with d^4x) to have a *proper* volume element. As well, we must replace the R_0^0 component with the *complete* Ricci tensor in the integrand to have covariant elements. We designate the resulting structure E_i^k, [3]

$$E_i^k = \frac{c^4}{4\pi G} \int R_i^k \sqrt{-g} d^4 x. \tag{3}$$

[2] That is apart from a necessary multiplicative factor of t, the time interval that it is being observed.

[3] While the integral is not a tensor and hence is not "covariant" in the sense that physicists use the term (see however below regarding the manner in which mathematicians interpret "covariant tensors"), it is emphasized that a change in coordinates changes the integral in

This expands the original expression into several components. In the case of bounded stationary systems, its $0-0$ component does give the correct answer multiplied by the time over which the system is analyzed. We propose that the integral (3) is the necessary generalization for energy-momentum measure in general relativity. We will refer to it as "spacetime energy-momentum". It is the entirely natural generalization of the concept of energy-momentum for general relativity as we are, by necessity, engaged with a curved four-dimensional spacetime in general relativity. It is an expression of the inextricable link between space and time in general relativity. While the change is not dramatic for stationary systems as it simply multiplies the traditional value by the time interval being measured, it is of considerable interest and complexity for intrinsically dynamic systems. In the latter, the admixture with time carries through the Ricci tensor and in a non-trivial manner in the metric determinant. We are familiar with integrals of the form of (3) in field theory, integrals of tensors over proper spacetime volume. Thus it should come as no surprise that general-relativistic energy-momentum should require such a structure. In particular, such integrals over spacelike hypersurfaces and over 2-surfaces are also employed in the quasi-local approach. For instance, using the Witten integral [9] in the case where R is a region foliated by asymptotically flat spacelike hypersurfaces Σ_t for t in the interval J of real numbers, we can compare the ADM mass to our spacetime energy- momentum. Let $u^a = u^{AA'}$ be the unit timelike normal vector field to the foliation. We take α_t to be an asymptotically constant solution of the Dirac-Weyl neutrino equation $D_{AA'}\alpha^A = 0$ on Σ_t and form the null vector field $K_a = \alpha_A \bar{\alpha_{A'}}$. Here D denotes the spacetime Dirac operator restricted to operate on vector fields tangent to the foliation hypersurfaces. Let $K^a(t,\infty)$ be the asymptotically constant value of K^a on Σ_t and let $P_a^{ADM}(t)$ denote the ADM energy-momentum of Σ_t.. Witten's technique of proof for the

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a uniquely determined manner. Indeed this is the most that could be expected for such a measure over an extended distribution. As an explicit example of the uniquely determined transformed expression, consider the x^1 component of linear momentum in frame A and its transformation: Its density in frame A is R_0^1. The total spacetime x^1 value is its integral over proper spacetime volume. We find its unique value in frame A' as follows: The local density in A' is $R_0^{1'} = (R_i^k)(\partial x^i/\partial x'^0)(\partial x'^1/\partial x^k)$. This follows here because of the tensorial character of the local value of energy-momentum. The *uniquely determined* spacetime x'^1 component of momentum in A' is the integral $\int R_0^{1'}\sqrt{-g'}d^4x'$ where g' is the 4-metric determinant in A'.

Actually, mathematicians by contrast regard the integral as being a covariant tensor in their manner of interpreting the expression, as follows. The integral of the Ricci tensor here done componentwise can be made mathematically rigorous by considering the integral as defining an operator on pairs of vector fields. If U is any Borel subset of spacetime and TU is the restriction of the tangent bundle of spacetime to the set U, then the set of smooth sections of TU forms a vector space $\Gamma(TU)$ which we denote by Γ for short. For any pair of sections v, w in Γ the integral of $Ricci(v,w) = R_{ij}v^iw^j$ over U with respect to proper spacetime volume then obviously defines a bilinear map with real number values. It is thus a second rank tensor on the vector space Γ. If U happens to be contained in a single coordinate patch, then v and w can be chosen to be coordinate frame fields which results in the formula we have used to define spacetime energy. We can note here that the tensor actually is defined for any Borel subset of the spacetime manifold, and in particular, when restricted to open subsets, it is also naturally defining a bilinear operator on the pre-sheaf of smooth sections of the tangent bundle of spacetime.

positive energy theorem shows that

$$8\pi P_a^{ADM}(t)K^a(t,\infty) = \int_{\Sigma_t}[-u^{AA'}(D_b\alpha_A)(D^b\bar{\alpha_{A'}}) + 8\pi T_{ab}u^aK^b]d\Sigma_t$$

$$= \int_{\Sigma_t}[-u^{AA'}(D_b\alpha_A)(D^b\bar{\alpha_{A'}}) + (R_{ab} - (1/2)Rg_{ab})u^aK^b]d\Sigma_t,$$

for each time t. Integrating against time on both sides then gives

$$4\pi E[u^a, K_b] - 8\pi\int_J P_a^{ADM}(t)d\tau =$$

$$\int_R [u^{AA'}(D_b\alpha_A)(D^b\bar{\alpha_{A'}}) + (1/2)Rg_{ab}u^aK^b]\sqrt{g}d^4x.$$

The first term in the last integral is always negative, so any scalar curvature or metric condition guaranteeing negativity of the second term would mean that with appropriate consideration of the constant coefficients here that the spacetime energy-momentum does not exceed the proper time integral of the ADM energy-momentum.

Returning to the enlarged structure of spacetime energy-momentum in (3), we recall that this integral satisfies the requirements in a minimal sense. It is well to ask if there is scope for a more complicated expression in the case of an intrinsically dynamic system. We consider the feasibility of developing an extension while maintaining the demand for the construction being composed of purely covariant elements. Any such extension must reduce to the correct Tolman expression in the case of bounded stationary systems. Suppose that for the general case, the Ricci tensor were to be replaced by the Ricci tensor plus an additional tensor (or tensors) of second rank. However, in order to reduce to the correct form for stationarity, the generalization must have partial derivatives with respect to time in such a form so as to reduce to the expression in (1) when the metric is stationary. However, for the maintenance of purely covariant elements, such derivatives must be covariant derivatives. While one could envisage an infinity of such forms, consider the following examples for different add-on tensors S:

$$R^i_k + S^i_{;k}, R^i_k + S^{ij}_{j;k}, R^i_k + S^{imj}_{j;km}, \tag{4}$$

where a semi-colon represents covariant differentiation. Regardless of the chosen form with covariant derivatives, while the desired partial derivatives with respect to time appear as required, extra undesired terms due to spatial derivatives appear as well, terms which persist even in the case of stationary systems, which is unacceptable. Thus, (3) is the only permissible form for spacetime energy-momentum in generality. [4]

[4] Although we have justified the general form in (3) by the demand that it hold for stationary bounded systems with asymptotic flatness, there is no apparent reason why the resulting general expression could not be applied to arbitrary systems, even homogeneous isotropic cosmologies where there are no boundary conditions because there are no boundaries.

The consequences are immediate: since the Ricci tensor is non-zero only in the regions where the energy-momentum tensor is non-zero, gravitational waves, waves of propagating spacetime curvature, are not carriers of energy-momentum through the vacuum, in conformity with the localization hypothesis [11].

At this point, we turn to the important role of the observer in relation to energy-momentum. Recall that in special relativity we express the mass/energy of a body with four-momentum p^i and four-velocity u_i as the inner product $p^i u_i$. However u_i can take on a broader role in the inner product; it can be taken as the four-velocity of an arbitrary observer. With u^i chosen as the four-velocity of the body, the result is the rest mass m. The observer measures the inner product as m if he is comoving with the body. However, we could choose u^i to be the four-velocity of an observer whose speed is v relative to the body at the instant of his intersection with the body. Then the product $p^i u_i$ gives this particular observer's perception of the energy. Its value is $m\gamma$ where γ is the relativistic factor $(1 - v^2/c^2)^{-1/2}$. The crucial role of observer is the lesson of great familiarity for us in special relativity.

Clearly, to extend this approach to a distribution of energy-momentum and a continuum of observers, it would appear to be natural to express the spacetime energy of a system in general relativity, relative to a field of observers with a corresponding field of four-velocities u^i, in the form of a four-scalar, as

$$E = \frac{c^4}{4\pi G} \int R_i^k u^i u_k \sqrt{-g} d^4x. \tag{5}$$

There is a particularly interesting aspect connected with the reduction of (5) to the Tolman integral (multiplied by the observed time span) by the choice of the field of rest observers. We note that the asymptotic rest observer at spatial infinity computes the Tolman mass m, the intrinsic mass, for a stationary system as it connects with the asymptotic expansion of the Schwarzschild metric [1]. An interesting link-up is as follows: from [1], we realize that it is in the "synchronous" frame where the metric assumes the form

$$ds^2 = dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \alpha, \beta = (1, 2, 3), \tag{6}$$

that the field of "rest" observers have four-velocities *both* contravariant *and* covariant, the same, namely $u^i = u_i = (1, 0, 0, 0)$. These are the required fields of four-velocities which link to the Tolman integral. Moreover it is in this synchronous frame that these "rest" observers are actually *physically* in free-fall. Thus we see that it is the global field of freely-falling observers who compute the Tolman mass, the intrinsic mass of the system. It is often said that by falling freely, one "eliminates" gravity. In fact, it is through free-fall that one is positioned to extract the essence of the energy of a system *including the contribution from gravity*. This is analogous to the previous example in special relativity where one extracts the intrinsic mass of the body by choosing the observer who is at rest relative to the body. One cannot eliminate spacetime curvature by free-fall and it is the curvature that distinguishes true gravity from pseudo-gravity.

While one is free to choose different fields of observers, it is the field of freely-falling observers that is particularly attractive. The added attractive aspect is that the clock of each free-fall observer measures the same interval of proper time by virtue of the structure of (6).

With greater generality, we can view (5) as a special case of

$$E[u^i, v_k] = \frac{c^4}{4\pi G} \int R_i^k u^i v_k \sqrt{-g} d^4 x, \tag{7}$$

a bilinear functional of the fields (u^i, v_k). In the same vein, we can express the observer-related spacetime linear momentum P^α as

$$P^\alpha = \frac{c^4}{4\pi G} \int R_i^\alpha u^i \sqrt{-g} d^4 x \tag{8}$$

where $\alpha = (1, 2, 3)$. The domains of integration are for our choosing according to the physical requirements.

There is a new pathway that opens from the extension of energy-momentum to spacetime. This is suggested by noting that (5) is of the form energy times time, the combination which manifests itself quantum-mechanically as a minimal product of uncertainties in the Heisenberg Uncertainty Principle

$$\Delta E \Delta t \geq \hbar/2, \tag{9}$$

where Δ denotes uncertainty.

If the gravitational contribution to energy is localized, it is most reasonable to assume that gravity must partake in the uncertainty. This directive is amplified by the crucial role of the observer regarding energy measurement, as discussed above. We are thus guided to a generalization of the Heisenberg Uncertainty Principle for energy in the form

$$\Delta \frac{c^4}{4\pi G} \int R_i^k u^i u_k \sqrt{-g} d^4 x \geq \hbar/2. \tag{10}$$

It is only for the regime of very strong gravity that this extended Uncertainty Principle would present a demonstrable difference from the standard Heisenberg form.

Similarly, we generalize the standard Heisenberg expressions for linear momentum

$$\Delta P_x \Delta x \geq \hbar/2, \Delta P_y \Delta y \geq \hbar/2, \Delta P_z \Delta z \geq \hbar/2, \tag{11}$$

to the form

$$\Delta \frac{c^4}{4\pi G} \int R_k^\alpha u^k \sqrt{-g} d^4 x \geq \hbar/2. \tag{12}$$

The explicit role of the observer is evident in (10) and (12) through the presence of the observer four-velocity u^i. We submit that it is in the context of general relativity, where spacetime finds its necessarily unifying character, that these generalized Uncertainty Principles to include gravity, arise so naturally.

It is somewhat ironical to consider that Einstein's original focus was on the Ricci tensor for his field equations. The attention shifted to the Einstein tensor which is the non-gravitational energy-momentum tensor multiplied by constants. Now we are returning to the Ricci tensor, only now as the embodiment of full energy-momentum. It is noteworthy in its simplicity.

Gravity waves carry information from their source and hence the question arises as to whether this is inconsistent with these waves not being energy carriers. We know of no reason in principle to preclude the transfer of information in the absence of energy. Indeed Bonnor [16] had noted that Szekeres asymmetric collapsing dust spacetimes have time-varying quadrupole moments and so are presumably energy-emitting according to the old ideas yet since they are asymptotically matched to the energy-conserving Schwarzschild form, cannot be losing energy. Our results are fully consistent with this finding.

In the same vein, it is natural to extend the traditional angular momentum measure to include the contribution from gravity in a new "spacetime angular momentum" construct M^{ikl} in the form[5]

$$M^{ikl} = \frac{c^3}{4\pi G} \int \left(x^i R^{kl} - x^k R^{il} \right) \sqrt{-g} d^4x \tag{13}$$

where the spatial components (x^1, x^2, x^3) of x^i for angular momentum are necessarily the quasi-Cartesian coordinates (x, y, z). As with the spacetime energy-momentum, it is localized within the matter distribution.

One class of dynamic spacetimes is of particular interest because of its simplicity: spherically symmetric spacetimes. This is because the exterior vacuum for such systems is uniquely the static Schwarzschild solution as shown in Birkhoff's theorem. For dynamic spherically symmetric interiors matching to the exterior Schwarzschild solution, the resultant spacetime energy is a simply separable product of the usual mass times the time span of its observation. A simple example drawn from spherical dust collapse illustrates the result.

As shown in [1] and developed further in [17], the essential equations in comoving spherical polar coordinates (τ, R, θ, ϕ) are as follows: The metric is

$$ds^2 = d\tau^2 - e^\lambda dR^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{14}$$

where λ and r are functions of R and τ. For the case in which the dust has been released from rest at infinity in the infinitely distant past, the Einstein field equations yield

$$e^\lambda = (r')^2 \tag{15}$$

$$\dot{r}^2 = \frac{F(R)}{r} \tag{16}$$

$$r = \left(\frac{9F}{4}\right)^{1/3} (R - \tau)^{2/3} \tag{17}$$

[5] This expression is even less "covariant" than the expression for the extended spacetime energy-momentum because it employs the non-tensorial spacetime coordinates (as opposed to their differentials) in the density.

$$8\pi\rho = \frac{F'}{r'r^2} \tag{18}$$

where ρ is the mass density, $F(R)$ is a function of integration, a prime denotes partial differentiation with respect to R and a dot indicates partial differentiation with respect to τ. From (18), it is easy to show with a simple integration [1] that $M(R)$, the total mass including the contribution from gravity, within the comoving radial coordinate R is

$$M(R) = F(R)/2. \tag{19}$$

From the metric (14), the square root of the complete determinant is

$$\sqrt{-g} = e^{\lambda/2}r^2 sin\theta. \tag{20}$$

The 4-volume element in the comoving frame is

$$d\tau d^3x = d\tau dR d\theta d\varphi. \tag{21}$$

Since dust is stress-free, in the comoving frame the only non-vanishing component of the energy-momentum tensor is T_0^0 and hence, from the Einstein field equations,

$$R_0^0 = \frac{8\pi G}{c^4}T_0^0. \tag{22}$$

Thus, from (3),(22),(20),(15),(18),(19) and (21), we find the *spacetime* energy up to the comoving radius R and over a proper time interval τ is

$$\int \sqrt{-g}T_0^0 d^4x = M(R)\tau \tag{23}$$

as expected. In spite of intrinsic time-dependence, for this special case, the result is simply separable in terms of standard energy and time because there are no gravitational waves emitted by the spherically symmetric system.

Forty-two years ago, the late J. L. Synge, one of the most distinguished mathematical physicists of the 20th Century, expressed to author FIC his sentiment that the concept of energy-momentum is simply incompatible with general relativity. His view was influenced by the pseudotensorial constructs for energy-momentum which were an anathema to him. While this view seemed radical at the time, from the present perspective, we see it conveying an essential element of truth: the standard energy-momentum concept does not mesh with general relativity for dynamic systems. However, the extended concept of *spacetime* energy-momentum would appear to fit naturally, enabling us to maintain the concept of energy-momentum in general relativity.

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