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Curvature of Universal Bundles of Banach Algebras

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For Professor I. Gohberg

Abstract. Given a Banach algebra we construct a principal bundle with connection over the similarity class of projections in the algebra and compute the curvature of the connection. The associated vector bundle and the connection are a universal bundle with attendant connection. When the algebra is the linear operators over a Hilbert module, we establish an analytic diffeomorphism between the similarity class and the space of polarizations of the Hilbert module. Likewise, the geometry of the universal bundle over the latter is studied. Instrumental is an explicit description of the transition maps in each case which leads to the construction of certain functions. These functions are in a sense pre-determinants for the universal bundles in question.

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1. Introduction

The book of Helton et al. [22] outlined a program of operator-analytic techniques using flag manifold models, the theorems of Beurling-Lax-Halmos, Wiener-Hopf factorization and $\mathcal{M}\times\mathcal{M}$-theory, which could be applied to the study of integrable systems (such as the Sato-Segal-Wilson theory [33, 32, 34]) and Lax-Phillips scattering (cf. work of Ball and Vinnikov [2, 3]). Several of the fundamental techniques implemented in this scheme of ideas can be traced back to the remarkable accomplishments of Professor I. Gohberg and his co-workers spanning a period of many years.

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Our interest in this general subject arose from two directions. Initially, the first two authors (with Evard) studied the problem of smooth as well as analytic parametrization of subspaces of a Banach space using global techniques. The work on this problem had been significantly motivated by that of Gohberg and Leiterer [18, 19]. The general results that were obtained appear in [17, 11, 12]. From another direction [14, 15, 16] we have developed an operator-theoretic, Banach algebra approach to the Sato-Segal-Wilson theory, in the setting of Hilbert modules with the extension of the classical Baker and Tau(τ)-functions to types of operator-valued functions. One aspect of this latter work involved looking at the geometry of the space of polarizations of a Hilbert module using a Grassmannian over the Banach algebra A in question, a topic which is developed in this paper. We consider the techniques and results as presented here to be also of independent interest in related areas of operator theory.

If $P(A)$ denotes the space of projections in A, then we consider the geometry of the space $\Lambda = \mathrm{Sim}(p, A)$, namely the similarity class of a given projection $p \in P(A)$. We construct a principal bundle with connection over Λ and compute the curvature of the connection. The transition map for this bundle leads to the construction of a function which we refer to as the *$\mathcal{T}$-function.* If $\mathfrak{P}$ denotes the space of polarizations of a Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ (where $\boldsymbol{\mathcal{A}}$ is a unital C*-algebra), we show that Λ and $\mathfrak{P}$ are analytically diffeomorphic (Theorem 4.1). Related (in the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$) is the $\mathfrak{T}$-function of [28, 39] obtained over $\mathfrak{P}$ via a cross-ratio approach.

To be more specific, let us point out that the $\mathcal{T}$-function is effectively the co-cycle for the universal bundle over the space of restricted polarizations, relating essentially the same two underlying sections, but initially this is viewed in terms of the corresponding principal bundle. Hence the interest is in the calculation of the geometry, connection, and curvature of the principal bundle of the universal bundle using two sections which are each covariantly constant over two complementary subbundles of the tangent bundle of the space of restricted polarizations. Our approach is justified by the fact that, technically, one only needs a single section to trivialize a principal bundle over the domain of the section and hence knowledge of the covariant derivative of that section allows the computation of the horizontal subspace over points of the image of the section, which can then be transferred to any fiber passing through the image of that section using the action of the structure group of the principal bundle. However, if one can find sections known to have zero covariant derivative along certain subbundles of the base tangent bundle, then the computation is certainly simplified, and in the case at hand we have two which suffice.

One main task we describe in this paper is to use the restricted algebra directly. Since the analysis only depends on the fact that the restricted algebra is a Banach algebra, our treatment presents, for any Banach algebra, a representation of the manifolds in question, as those naturally embedded in Banach spaces which provide a natural geometry recovering the exact same geometry that arises in [28, 39] thus leading to the well-known Tau(τ)-function [33, 34]. In particular, we are able to obtain simple expressions for the $\mathcal{T}$-function, the connection form, and the curvature (see, e.g., Theorem 8.1). As observed in [39] one can calculate in coordinates, but here we have natural embeddings which give the geometry. Using coordinates we can calculate, but we cannot visualize, whereas using the natural embeddings we can both visualize and simplify the final formulas. This means the determination of the Tau-function is reduced purely to analytic questions concerning the existence of determinants of the operator values in the particular subgroup of the algebra which forms the group of the principal bundle. This, along with other related issues, is taken up in [16].

2. Algebraic preliminaries

2.1. The Grassmannian over a semigroup

To commence, let A be a (multiplicative) semigroup with group of units denoted by $G(A)$, if A has an identity. Let

$$P(A) := \{p \in A \ : \ p^2 = p\}, \tag{2.1}$$

that is, $P(A)$ is the set of idempotent elements in A (for suitable A, we can regard elements of $P(A)$ as projections). Recall that the right Green's relation is $p\mathcal{R}q$, if and only if $pA = qA$ for $p, q \in A$.

Let $\mathrm{Gr}(A) = P(A)/\mathcal{R}$ be the set of equivalence classes in $P(A)$ under $\mathcal{R}$. As the set of such equivalence classes, $\mathrm{Gr}(A)$ will be called *the Grassmannian of A*. Note that as the equivalence classes partition A, elements of $\mathrm{Gr}(A)$ are in fact subsets of $P(A)$. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi \ : \ P(A) \longrightarrow \mathrm{Gr}(A). \tag{2.2}$$

In fact if A is a Banach algebra, it follows that $P(A)$ is an analytic submanifold of A, and that $\mathrm{Gr}(A)$ has a unique analytic manifold structure (holomorphic, if A is a complex algebra) such that Π is an analytic open map having local analytic sections passing through each point of $P(A)$ (see [11, § 4], cf. [30]).

Let $h : A \longrightarrow B$ be a semigroup homomorphism. Then it is straightforward to see that the diagram below is commutative:

$$\begin{array}{ccc} P(A) & \xrightarrow{P(h)} & P(B) \\ \Pi\downarrow & & \downarrow\Pi \\ \mathrm{Gr}(A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(B) \end{array} \tag{2.3}$$

Clearly, if A is a semigroup of linear transformations of a vector space E, then we have $\Pi(r) = \Pi(s)$, if and only if $r(E) = s(E)$ as subspaces of E. Notice that $r^{-1}(0)$ is a complement for $r(E)$, so if E is a topological vector space and all members of A are continuous, then $r(E)$ is closed with a closed complement, that is, $r(E)$ is a *splitting subspace.*

If we reverse the multiplication of A, we obtain the opposite semigroup A^{op} and consequently, the right Green's relation in A^{op} is the left Green's relation in A. But $P(A) = P(A^{\mathrm{op}})$, and so this construction gives $\Pi^{\mathrm{op}} : P(A) \longrightarrow \mathrm{Gr}^{\mathrm{op}}(A)$, where by definition $\mathrm{Gr}^{\mathrm{op}}(A) = \mathrm{Gr}(A^{\mathrm{op}})$.

In the case where A is a semigroup of linear transformations of a vector space E, we see immediately that $\Pi^{\mathrm{op}}(r) = \Pi^{\mathrm{op}}(s)$, if and only if $r^{-1}(0) = s^{-1}(0)$ as subspaces of E. Because of this we sometimes denote $\Pi(r) = \mathrm{Im}(r)$, and $\Pi^{\mathrm{op}}(r) = \mathrm{Ker}(r)$, for $r \in P(A)$ with A now taken to be an arbitrary semigroup. Clearly, if $h : A \longrightarrow B$ is a semigroup homomorphism, then so too is $h : A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$. Thus Gr^{op} and Π^{op} produce an analogous commutative diagram to (2.3). We observe that $\Pi(r) = \Pi(s)$ if and only if both $rs = s$ and $sr = r$, so in the dual sense, $\Pi^{\mathrm{op}}(r) = \Pi^{\mathrm{op}}(s)$, if and only if both $rs = r$ and $sr = s$. Consequently, if both $\mathrm{Im}(r) = \mathrm{Im}(s)$ and $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, then $r = s$, and thus the map

$$(\mathrm{Im}, \mathrm{Ker}) : P(A) \longrightarrow \mathrm{Gr}(A) \times \mathrm{Gr}^{\mathrm{op}}(A), \tag{2.4}$$

is an injective map which, in the case A is a Banach algebra, we later show to be an analytic embedding of manifolds whose image is open in the righthand side product.

Remark 2.1. Notice that if A is commutative, then $A^{\mathrm{op}} = A$, so $\mathrm{Im}(r) = \mathrm{Im}(s)$, if and only if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$ and therefore by (2.4), $\Pi = \Pi^{\mathrm{op}}$ is injective and thus bijective.

2.2. The canonical section

As in the case where A is a Banachable algebra, we know that Π is a continuous open map [11]. Then it follows that if A is a commutative Banach algebra, then Π is a homeomorphism. Because of (2.4), we see that if $K \in \mathrm{Gr}^{\mathrm{op}}(A)$, then $\mathrm{Im}|K : K \longrightarrow \mathrm{Im}(K) \subset \mathrm{Gr}(A)$ is a bijection whose inverse, we refer to as the *canonical section* over $\mathrm{Im}(K)$. If $p \in K$, then we denote this canonical section by S_p. We set $U_p = \mathrm{Im}(K) \subset \mathrm{Gr}(A)$ and $W_p = \mathrm{Im}^{-1}(U_p) \subset P(A)$. Thus, we have $S_p : U_p \longrightarrow W_p \subset P(A)$ is a section of $\mathrm{Im} = \Pi$ for $p \in W_p$, and $S_p(\mathrm{Im}(p)) = p$. In this situation we refer to S_p as *the canonical section through* p. In fact, from the results of [11], we know that if A is a Banach algebra, then U_p is open in $\mathrm{Gr}(A)$ and S_p is a local analytic section of $\mathrm{Im} = \Pi$.

2.3. Partial isomorphisms and relative inverses

Definition 2.1. We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$, or equivalently, if $u \in uAu$. If also $vuv = v$, we call v a *relative inverse* (or *pseudoinverse*) for u. In general, such a relative inverse always exists, but it is not unique. Effectively, if $u = uwu$, then $w = wuw$ is a relative inverse for u. We take $W(A)$ to denote the set (or space, if A has a topology) of all partial isomorphisms of A.

Notice that $W(A^{\mathrm{op}}) = W(A)$ and $P(A) \subset W(A)$. If u and v are *mutually* (*relative*) *inverse* partial isomorphisms, then $r = vu$ and $s = uv$ are in $P(A)$. In this latter case, we will find it useful to simply write $u : r \longrightarrow s$ and $v : s \longrightarrow r$. Thus we can say u *maps* r to s, regarding the latter as a specified map of idempotents in $P(A)$. Moreover, v is now uniquely determined by the triple (u, r, s), meaning that if w is also a relative inverse for u and both $wu = r$ and $uw = s$ hold, then it follows that $v = w$. Because of this fact, it is also useful to denote this dependence symbolically as

$$v = u^{-(r,s)}, \tag{2.5}$$

which of course means that $u = v^{-(s,r)}$. If $u, v \in W(A)$ with $u : p \longrightarrow r$ and $v : r \longrightarrow s$, then $vu : p \longrightarrow s$. Thus we have

$$(vu)^{-(p,s)} = u^{-(p,r)}v^{-(r,s)}. \tag{2.6}$$

In particular, the map $u : r \longrightarrow r$ implies that $u \in G(rAr)$ and $u^{-(r,r)}$ is now the inverse of u in this group. Thus $G(rAr) \subset W(A)$, for each $r \in P(A)$. For $u \in G(rAr)$, we write $u^{-r} = u^{-(r,r)}$, for short. It is a trivial, but useful observation that if $r, s \in P(A) \subset W(A)$, and if $\mathrm{Im}(r) = \mathrm{Im}(s)$, then $r : r \longrightarrow s$ and $s : s \longrightarrow r$, are mutually inverse partial isomorphisms. Likewise working in A^{op}, and translating the result to A, we have that if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, then $r : s \longrightarrow r$ and $s : r \longrightarrow s$, are mutually inverse partial isomorphisms. Therefore, if $u : q \longrightarrow r$, if $p, s \in P(A)$ with $\mathrm{Ker}(p) = \mathrm{Ker}(q)$ and $\mathrm{Im}(r) = \mathrm{Im}(s)$, then on applying (2.6), it follows that $u = ruq : p \longrightarrow s$ has a relative inverse

$$u^{-(p,s)} = pu^{-(q,r)}s : s \longrightarrow p. \tag{2.7}$$

Thus the relative inverse is changed (in general) by changing q and r for fixed u, and (2.7) is a useful device for calculating such a change.

Now it is easy to see [11] that the map Π has an extension $\Pi = \mathrm{Im} : W(A) \longrightarrow \mathrm{Gr}(A)$, which is well defined by setting $\Pi(u) = \Pi(s)$, whenever $u \in W(A)$ maps to s. Again, working in A^{op}, we have $\Pi^{\mathrm{op}} = \mathrm{Ker} : W(A) \longrightarrow \mathrm{Gr}^{\mathrm{op}}(A)$, and because $u : r \longrightarrow s$ in A, is the same as $u : s \longrightarrow r$ in A^{op}, this means that $\mathrm{Ker}(u) = \mathrm{Ker}(r)$ if $u : r \longrightarrow s$. More precisely, observe that if $p, q, r, s \in P(A)$, if $u \in W(A)$ satisfies both $u : p \longrightarrow q$ and $u : r \longrightarrow s$, then it follows that $\mathrm{Ker}(p) = \mathrm{Ker}(r)$ and $\mathrm{Im}(q) = \mathrm{Im}(s)$. In fact, if $v = u^{-(p,q)}$ and $w = u^{-(r,s)}$, then we have

$$rp = (wu)(vu) = w(uv)u = wqu = wu = r, \tag{2.8}$$

so $rp = r$ and symmetrically, $pr = p$,which implies $\mathrm{Ker}(p) = \mathrm{Ker}(r)$. Applying this in A^{op}, yields $\mathrm{Im}(q) = \mathrm{Im}(s)$.

Remark 2.2. Of course the commutative diagram (2.3) for Π extends to the same diagram with $W(\ )$ replacing $P(\ )$ and likewise, in the dual sense, for $\Pi^{\mathrm{op}} = \mathrm{Ker}$, on replacing A by A^{op}.

2.4. Proper partial isomorphisms

If $p \in P(A)$, then we take $W(p, A) \subset W(A)$ to denote the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $vu = p$. Likewise, $W(A, q)$ denotes the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $uv = q$. So it follows that $W(A, q) = W(q, A^{\mathrm{op}})$. Now for

$p, q \in P(A)$, we set

$$\begin{aligned} W(p,A,q) &= W(p,A) \cap W(A,q) \\ &= \{u \in W(A) : u : p \longrightarrow q\} \\ &= \{\ u \in qAp : \exists\ v \in pAq\ ,\ vu = p \text{ and } uv = q\ \}. \end{aligned} \tag{2.9}$$

Recall that two elements $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A. For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$. Hence it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.2. Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$, where p and q are similar.

We let $V(A)$ denote the space of all proper partial isomorphisms of A. Observe that $G(A)V(A)$ and $V(A)G(A)$ are both subsets of $V(A)$. In the following we set $G(p) = G(pAp)$.

2.5. The spaces $V(p, A)$ and $\mathrm{Gr}(p, A)$

If $p \in P(A)$, then we denote by $V(p, A)$ the space of all proper partial isomorphisms of A having a relative inverse $v \in W(q, A, p)$, for some $q \in \mathrm{Sim}(p, A)$. With reference to (2.9) this condition is expressed by

$$V(p,A) := \bigcup_{q \in \mathrm{Sim}(p,A)} W(p,A,q). \tag{2.10}$$

Observe that $V(p, A) \subset V(A) \cap W(p, A)$, but equality may not hold in general, since for $u \in V(A)$, it may be the case that $\mathrm{Ker}(p) \subset P(A)$ intersects more than one similarity class and that $u \in V(A)$ by virtue of having $u : r \longrightarrow s$ where r and s are similar. But $u : p \longrightarrow q$ only for $q \neq \mathrm{Sim}(p, A)$. However, we shall see that if A is a ring with identity, then each class in $\mathrm{Gr}(A)$ is contained in a similarity class and thus also for $\mathrm{Gr}^{\mathrm{op}}(A)$. Further, as Π and Π^{op} are extended to $W(A)$, this means that as soon as we have $u : p \longrightarrow q$, with p and q belonging to the same similarity class, then $u : r \longrightarrow s$ implies that r and s are in the same similarity class.

Clearly, we have $G(A) \cdot p \subset V(p, A)$ and just as in [11], it can be shown that equality holds if A is a ring. The image of $\mathrm{Sim}(p, A)$ under the map Π defines the space $\mathrm{Gr}(p, A)$ viewed as the Grassmannian naturally associated to $V(p, A)$.

For a given unital semigroup homomorphism $h : A \longrightarrow B$, there is a restriction of (2.3) to a commutative diagram:

$$\begin{array}{ccc} V(p,A) & \xrightarrow{V(p,h)} & V(q,B) \\ {\scriptstyle \Pi_A}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_B} \\ \mathrm{Gr}(p,A) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(q,B) \end{array} \tag{2.11}$$

where for $p \in P(A)$, we have set $q = h(p) \in P(B)$. Observe that in the general semigroup setting, $V(p, A)$ properly contains $G(A)p$. In fact, if $p \in P(A)$, then $V(p, A) = G(A)G(pAp)$ (see [13] Lemma 2.3.1).

Henceforth we shall restrict mainly to the case where A and B are Banach(able) algebras or suitable multiplicative subsemigroups of Banachable algebras. In this case, as shown in [11], the vertical maps of the diagram (2.11) are right principal bundles, the group for $V(p, A)$ being $G(pAp)$. Moreover, $G(A)$ acts $G(pAp)$-equivariantly on the left of $V(p, A)$ simply by left multiplication, the equivariance being nothing more than the associative law.

Let $H(p)$ denote the isotropy subgroup for this left-multiplication. We have then (see [11]) the analytically equivalent coset space representation

$$\mathrm{Gr}(p, A) = G(A)/G(\Pi(p)), \tag{2.12}$$

where $G(\Pi(p))$ denotes the isotropy subgroup of $\Pi(p)$. Then there is the inclusion of subgroups $H(p) \subset G(\Pi(p)) \subset G(A)$, resulting in a fibering $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ given by the exact sequence

$$G(\Pi(p))/H(p) \hookrightarrow V(p, A) = G(A)/H(p) \longrightarrow \mathrm{Gr}(p, A) = G(A)/G(\Pi(p)), \tag{2.13}$$

generalizing the well-known *Stiefel bundle* construction in finite dimensions.

In general, if A is a semigroup, we say that the multiplication is *left trivial* provided that always $xy = x$, whereas we call it *right trivial* if $xy = y$. In either case, we have $P(A) = A$. If the multiplication is right trivial, then obviously $\Pi = \mathrm{Im}$ is constant and $\Pi^{\mathrm{op}} = \mathrm{Ker}$ is bijective. Whereas if the multiplication is left trivial, then Ker is constant and $\mathrm{Im} = \Pi$ is bijective.

Remark 2.3. For the 'restricted algebra' to be considered in § 3.2, we recover the 'restricted Grassmannians' as studied in [29, 32, 34] (cf. [21]). Spaces such as $V(p, A)$ and $\mathrm{Gr}(p, A)$ are infinite-dimensional Banach homogeneous spaces of the type studied in, e.g., [4, 8, 9, 36] in which different methods are employed. Emphasis on the case where A is a C*-algebra, can be found in, e.g., [5, 25, 26, 27, 37], again using different methods. Other approaches involving representations and conditional expectations are treated in [1, 5, 6, 31].

2.6. The role of the canonical section

Suppose that R is any ring with identity. Now for $x \in R$, we define $\hat{x} = 1 - x$. The 'hat' operation is then an involution of R leaving $P(R)$ invariant. Further, it is easy to check that for $r, s \in P(R)$, we have $\mathrm{Im}(\hat{r}) = \mathrm{Im}(\hat{s})$, if and only if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$. This means that there is a natural identification of $\mathrm{Gr}^{\mathrm{op}}(R)$ with $\mathrm{Gr}(R)$ unique such that $\mathrm{Ker}(r) = \mathrm{Im}(\hat{r})$, for all $r \in P(R)$. For instance, if $r \in P(R)$, then $rR\hat{r}$ and $\hat{r}Rr$ are subrings with zero multiplication. On the other hand, $r + \hat{r}Rr$ is a subsemigroup with left trivial multiplication and $r + rR\hat{r}$ is a subsemigroup with right trivial multiplication. Thus $\mathrm{Im}|(r + \hat{r}Rr)$ is injective and $\mathrm{Ker}|(r + \hat{r}Rr)$ is constant, whereas $\mathrm{Im}|(r + rR\hat{r})$ is constant and $\mathrm{Ker}|(r + rR\hat{r})$ is injective. In fact, we can now easily check that (e.g., see [11])

$$\mathrm{Im}^{-1}(\mathrm{Im}(r)) = r + rA\hat{r}, \tag{2.14}$$

and

$$\mathrm{Ker}^{-1}(\mathrm{Ker}(r)) = r + \hat{r}Ar = P(A) \cap V(p, A). \tag{2.15}$$

Thus this section is again none other than the canonical section through r. From (2.15), it now follows immediately that when $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, we have

$$r + \hat{r}Ar = s + \hat{s}As, \tag{2.16}$$

and from the symmetry here, one easily deduces that

$$\hat{r}Ar = \hat{s}As. \tag{2.17}$$

This means that the sub-ring $\hat{s}As$ is constantly the same as $\hat{r}Ar$ along the points of the image of the canonical section through r which is $r+\hat{r}Ar = P(A)\cap V(p, A)$, by (2.15). But this also means that $sA\hat{s}$ is constantly the same as $rA\hat{r}$ at all points of $\hat{r} + rA\hat{r}$. If $s \in r + rA\hat{r}$, then

$$\hat{s} \in \hat{r} - rA\hat{r} = \hat{r} + rA\hat{r}, \tag{2.18}$$

and consequently we obtain again $sA\hat{s} = rA\hat{r}$. Thus $P(A)$ in effect contains a 'flat X-shaped subset' through any $r \in P(A)$, namely

$$\mathsf{X} = (r + \hat{r}Ar) \cup (r + rA\hat{r}). \tag{2.19}$$

This suggests that $P(A)$ is everywhere 'saddle-shaped'.

Now, as in [11], we observe here that if $\mathrm{Im}(r) = \mathrm{Im}(s)$, then r and s are in the same similarity class. For there is $y \in rA\hat{r}$ with $s = r + y$. But the multiplication in $rA\hat{r}$ is zero, so $e^y = 1 + y \in G(A)$ with inverse $e^{-y} = 1 - y$, and

$$s = rs = re^y = e^{-y}re^y. \tag{2.20}$$

As $r : r \longrightarrow s$, this means that $r \in V(r, A, s)$, and so each class in $\mathrm{Gr}(A)$ is contained in a similarity class. In the dual sense then, each class of $\mathrm{Gr}^{\mathrm{op}}(A)$ is also contained in a similarity class, as is easily checked directly by the same technique and (2.15). In particular, we now see that for each $p \in P(A)$, we have $V(p, A) = V(A) \cap W(p, A)$, and if $u : r \longrightarrow s$ belongs to $W(A)$, and also $u \in V(A)$, then r and s belong to the same similarity class.

Recalling the canonical section S_p (through p) let us take $p, r \in P(A)$ with $r \in W_p$, and therefore $\mathrm{Im}(r) = \mathrm{Im}(S_p(\mathrm{Im}(r)))$. We have of course $\mathrm{Ker}(S_p(\mathrm{Im}(r))) = \mathrm{Ker}(p)$, by definition of S_p, and hence r and p are in the same similarity class. Set $r_p = S_p(\mathrm{Im}(r))$. Thus $\mathrm{Im}(r) = \mathrm{Im}(r_p)$ and $\mathrm{Ker}(r_p) = \mathrm{Ker}(p)$. We can find $x \in \hat{p}Ap$ so that $r_p = p + x$, and then we have $pr_p = p = pr_pp$ and $r_ppr_p = r_pp = r_p$. This shows that

$$S_p(\mathrm{Im}(r)) = p^{-(S_p(\mathrm{Im}(r)),p)} \tag{2.21}$$

and

$$(S_p(\mathrm{Im}(r)))^{-(p,S_p(\mathrm{Im}(r))} = p. \tag{2.22}$$

Proposition 2.1.

(1) *We have the equation*

$$(S_p(\mathrm{Im}(r))^{-(r,p)} = pr : r \longrightarrow p. \tag{2.23}$$

(2) *The canonical section is a local section of* $\Pi|V(p,A) : V(p,A) \longrightarrow \mathrm{Gr}(p,A)$.

Proof. Part (1) follows from (2.7) and (2.22). For part (2), observe that since $\mathrm{Ker}(S_p(\mathrm{Im}(r))) = \mathrm{Ker}(p)$, we have $S_p(\mathrm{Im}(r))$ and p in the same similarity class and thus the canonical section is actually simultaneously a local section of $\Pi|V(p,A) : V(p,A) \longrightarrow \mathrm{Gr}(p,A)$. □

If A is any semigroup and $u : r \longrightarrow s$ is in $W(A)$ and $k \in P(A)$, we say that u *projects along* k provided that $ku = kr$. Thus, if A is a semigroup of linear transformations of a vector space E, then this condition guarantees that $u(h) - h$ belongs to $k^{-1}(0)$, for every $h \in r(E)$.

Remark 2.4. Clearly this last statement has no content unless $k^{-1}(0)$ is close to being complementary to $r(E)$ and $s(E)$, but in applications this is not a problem.

If $m \in P(A)$ with $\mathrm{Ker}(m) = \mathrm{Ker}(k)$, then $mk = m$ and $km = k$, so $u \in W(A)$ projects along k if and only if it projects along m. Thus we can say u projects along $K \in \mathrm{Gr}^{\mathrm{op}}(A)$ provided that it projects along k, for some and hence all $k \in K$. We can now easily check that if $u : r \longrightarrow s$ in $W(A)$ projects along K, then so too does $u^{-(r,s)}$. It will be important to observe this when later we consider the $\mathcal{T}$-function.

If $r, s \in P(A)$ and it happens that $rs : s \longrightarrow r$, then it is the case that rs projects along $\mathrm{Ker}(r)$, and hence $(rs)^{-(s,r)}$ does also. Thus even though $\mathrm{Ker}(rs) = \mathrm{Ker}(s)$, we have rs projecting along $\mathrm{Ker}(r)$. In particular, by (2.23), if $r \in W_p$, then $S_p(\mathrm{Im}(r))$ and its inverse pr both project along $\mathrm{Ker}(p)$, and therefore, if also $p \in W_r$, then $S_r(\mathrm{Im}(p))$ and its inverse rp both project along $\mathrm{Ker}(r)$. If we consider the case of a semigroup of linear transformations of a vector space E, then we see that for rs to be in $W(A)$ requires that $r^{-1}(0)$ has zero intersection with $s(E)$. Thus, if $rs \in W(A)$, then we should think of r as *close to* s. For instance, if A is any ring with identity and $r, p \in P(A)$ with $rp + \hat{r}\hat{p} \in G(A)$, then, for $g = rp + \hat{r}\hat{p}$, we have

$$rg = rp = gp. \tag{2.24}$$

Therefore, $rp = gp$, so $rp : p \longrightarrow r$ must project along $\mathrm{Ker}(r)$. Moreover as $r = gpg^{-1}$, we have $rp : p \longrightarrow r$ is a proper partial isomorphism and $rp \in V(p,A)$ such that $(rp)^{-(p,r)} = pg^{-1} = g^{-1}r$. Note that for A a Banach algebra, the group of units is open in A, and therefore the set of idempotents $r \in P(A)$ for which $rp + \hat{r}\hat{p} \in G(A)$, is itself an open subset of $P(A)$.

2.7. The spatial correspondence

If $\boldsymbol{\mathcal{A}}$ is a given topological algebra and E is some $\boldsymbol{\mathcal{A}}$-module, then $A = \mathcal{L}_{\boldsymbol{\mathcal{A}}}(E)$ may be taken as the ring of $\boldsymbol{\mathcal{A}}$-linear transformations of E. An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E. In order to understand the relationship between spaces such as $\mathrm{Gr}(p,A)$ and the usual Grassmannians of subspaces (of a vector space E), we will describe a 'spatial correspondence'.

Given a topological algebra $\boldsymbol{\mathcal{A}}$, suppose E is an $\boldsymbol{\mathcal{A}}$-module admitting a decomposition

$$E = F \oplus F^c, \qquad F \cap F^c = \{0\}, \tag{2.25}$$

where F, F^c are fixed closed subspaces of E. We have already noted $A = \mathcal{L}(E)$ as the ring of linear transformations of E. Here $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{array}{ccc} V(p,\mathcal{L}(E)) & \xrightarrow{\ \varphi\ } & V(p,E) \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathrm{Gr}(p,\mathcal{L}(E)) & \xrightarrow{\ =\ } & \mathrm{Gr}(F,E) \end{array} \tag{2.26}$$

where $V(p,E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F. If $u \in V(p,\mathcal{L}(E))$, then $\varphi(u) = u|F$ and if $T : F \longrightarrow E$ is a linear homeomorphism onto a closed complemented subspace of E similar to F, then $\varphi^{-1}(T) = Tp : E \longrightarrow E$. In particular, the points of $\mathrm{Gr}(p,\mathcal{L}(E))$ are in a bijective correspondence with those of $\mathrm{Gr}(F,E)$.

Suppose E is a complex Banach space admitting a decomposition of the type (2.25). We will be considering a 'restricted' group of units from a class of Banach Lie groups of the type

$$\widehat{G}(E) \subset \{\begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F),\ T_2 \in \mathrm{Fred}(F^c),\ S_1, S_2 \in \mathcal{K}(E)\}, \tag{2.27}$$

that generates a Banach algebra A acting on E, but with possibly a different norm. Here we mention that both compact and Fredholm operators are well defined in the general category of complex Banach spaces; reference [38] provides the necessary details.

3. The restricted Banach *-algebra A_{res} and the space of polarizations

3.1. Hilbert modules and their polarizations

Let $\boldsymbol{\mathcal{A}}$ be a unital C*-algebra. We may consider the standard (free countable dimensional) Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ over $\boldsymbol{\mathcal{A}}$ as defined by

$$H_{\boldsymbol{\mathcal{A}}} = \{\{\zeta_i\}\ ,\ \zeta_i \in \boldsymbol{\mathcal{A}}\ ,\ i \geq 1\ : \sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \boldsymbol{\mathcal{A}}\ \} \cong \oplus \boldsymbol{\mathcal{A}}_i, \tag{3.1}$$

where each $\boldsymbol{\mathcal{A}}_i$ represents a copy of $\boldsymbol{\mathcal{A}}$. Let H be a separable Hilbert space (separability is henceforth assumed). We can form the algebraic tensor product $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ on which there is an $\boldsymbol{\mathcal{A}}$-valued inner product

$$\langle x \otimes \zeta\ ,\ y \otimes \eta \rangle\ =\ \langle x, y \rangle\ \zeta^* \eta\ , \qquad x, y \in H\ ,\ \zeta, \eta \in \boldsymbol{\mathcal{A}}. \tag{3.2}$$

Thus $H \otimes_{\mathrm{alg}} \boldsymbol{\mathcal{A}}$ becomes an inner product $\boldsymbol{\mathcal{A}}$-module whose completion is denoted by $H \otimes \boldsymbol{\mathcal{A}}$. Given an orthonormal basis for H, we have the following identification (unitary equivalence) given by $H \otimes \boldsymbol{\mathcal{A}} \approx H_{\boldsymbol{\mathcal{A}}}$ (see, e.g., [23]).

3.2. The restricted Banach *-algebra A_{res}

Suppose now that $H_{\boldsymbol{\mathcal{A}}}$ is *polarizable*, meaning that we have a pair of submodules $(\mathsf{H}_+, \mathsf{H}_-)$, such that $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_-$ and $\mathsf{H}_+ \cap \mathsf{H}_- = \{0\}$ (cf., e.g., [24]). Thus we call the pair $(\mathsf{H}_+, \mathsf{H}_-)$ a *polarization of* $H_{\boldsymbol{\mathcal{A}}}$. If we have a unitary $\boldsymbol{\mathcal{A}}$-module map J satisfying $J^2 = 1$, there is an induced eigenspace decomposition $H_{\boldsymbol{\mathcal{A}}} = H_+ \oplus H_-$, for which $H_\pm \cong H_{\boldsymbol{\mathcal{A}}}$. This leads to the Banach algebra $A_{\mathrm{res}} = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ as described in [14] (generalizing that of $\boldsymbol{\mathcal{A}} = \mathbb{C}$ in [32]). Specifically, we define

$$A_{\mathrm{res}} := \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}}) = \{T \in \mathcal{L}_{\boldsymbol{\mathcal{A}}}(H_{\boldsymbol{\mathcal{A}}}) : [J, T] \text{ is Hilbert-Schmidt}\}, \tag{3.3}$$

for which the norm is $\|T\|_J = \|T\| + \|[J, T]\|_2$, for $T \in A_{\mathrm{res}}$.

• Once this restriction is understood, we shall simply write $A = A_{\mathrm{res}} := \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ until otherwise stated, and let $G(A)$ denote its group of units.

Remark 3.1. Note that A is actually a (complex) Banach *-algebra. The spaces $\mathrm{Gr}(p, A)$ are thus generalized 'restricted Grassmannians' [14, 15], which for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, reduce to the usual restricted Grassmannians of [32, 34]. In this case, $V(p, A)$ is regarded as the Stiefel bundle of 'admissible bases' (loosely, those for which a 'determinant' is definable).

The space $\mathrm{Gr}(p, A)$ may be realized more specifically in the following way. Suppose that a fixed $p \in P(A)$ acts as the projection of $H_{\boldsymbol{\mathcal{A}}}$ on H_+ along H_-. Therefore $\mathrm{Gr}(p, A)$ is the Grassmannian consisting of subspaces $W = r(H_{\boldsymbol{\mathcal{A}}})$, for $r \in P(A)$, such that:

(1) the projection $p_+ = pr : W \longrightarrow \mathsf{H}_+$ is in $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$, and
(2) the projection $p_- = (1 - p)r : W \longrightarrow \mathsf{H}_-$ is in $\mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$ (Hilbert-Schmidt operators).

Alternatively, for (2) we may take projections $q \in P(A)$ such that for the fixed $p \in P(A)$, the difference $q - p \in \mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$. Further, there is the *big cell* $C_{\mathrm{b}} = C_{\mathrm{b}}(p_1, A) \subset \mathrm{Gr}(p, A)$ as the collection of all subspaces $W \in \mathrm{Gr}(p, A)$, such that the projection $p_+ \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$ is an isomorphism.

3.3. The space $\mathfrak{P}$ of polarizations

Let us define $p_\pm \in A$ by

$$p_\pm = \frac{1 \pm J}{2}. \tag{3.4}$$

Then $p_\pm \in P(A)$ can be seen to be the spectral projection of J with eigenvalue ± 1. Clearly $p_- + p_+ = 1$, so $p_- = 1 - p_+ = \hat{p}_+$. Thus,

$$(\mathsf{H}_+, \mathsf{H}_-) = (p_+(H_{\boldsymbol{\mathcal{A}}}), p_-(H_{\boldsymbol{\mathcal{A}}})), \tag{3.5}$$

is a polarization. Notice that if $H_{\boldsymbol{\mathcal{A}}}$ is infinite-dimensional, then members of the group of units $G = G(\mathcal{L}(H_{\boldsymbol{\mathcal{A}}}))$ of the unrestricted algebra, are clearly not Hilbert-Schmidt in general. If $g \in G$ with $g(p_+)g^{-1} = p_-$, then using (3.4), we find

$gJ + Jg = 0$, which means that $[g, J] = 2gJ \in G$. This means that in the restricted algebra $A = A_{\mathrm{res}}$, the projections p_+ and p_- must be in different similarity classes. For this reason, when dealing with the Grassmannian $\mathrm{Gr}(p_+, A)$ and the Stiefel bundle $V(p_+, A)$ over it, the map Ker will take values in $\mathrm{Gr}(p_-, A)$ which is an entirely different space referred to as the *dual Grassmannian of* $\mathrm{Gr}(p_+, A)$. Thus for any $p \in P(A)$, let

$$\mathrm{Gr}^*(p, A) = \mathrm{Gr}(\hat{p}, A) = \mathrm{Gr}^{\mathrm{op}}(p, A). \tag{3.6}$$

We also note that by (3.4), we have $[T, J] = 2[T, p_+]$, for any operator in $\mathcal{L}(H_{\mathcal{A}})$. So the definition of the restricted algebra is equally well given as the set of operators $T \in \mathcal{L}(H_{\mathcal{A}})$ for which $[T, p_+]$ is Hilbert-Schmidt.

Now let $(\mathsf{H}_+, \mathsf{H}_-)$ be the fixed polarization defined by p_+ and $(\mathsf{K}_+, \mathsf{K}_-)$ another polarization, so that $H_{\mathcal{A}} = \mathsf{H}_+ \oplus \mathsf{H}_- = \mathsf{K}_+ \oplus \mathsf{K}_-$, whereby the projections parallel to H_- and K_- are isomorphisms of the spaces H_+ and K_+ respectively. Further, when restricting $\mathsf{K}_\pm$ to be in $\mathrm{Gr}(p_\pm, A)$, then under these specified conditions, the Grassmannian $\mathrm{Gr}(p_-, A)$ is the 'dual Grassmannian' of $\mathrm{Gr}(p_+, A)$. Let us denote this dual Grassmannian by $\mathrm{Gr}^*(p_+, A)$. Then, on setting $p = p_+$, the space $\mathfrak{P}$ of such polarizations can be regarded as a subspace

$$\mathfrak{P} \subset \mathrm{Gr}(p, A) \times \mathrm{Gr}^*(p, A). \tag{3.7}$$

3.4. The case where $\mathcal{A}$ is commutative

Here we address the case where $\mathcal{A}$ is a commutative separable C*-algebra. The Gelfand transform implies there exists a compact metric space Y such that $Y = \mathrm{Spec}(\mathcal{A})$ and $\mathcal{A} \cong C(Y)$. Setting $B = \mathcal{L}_J(H)$, we can now express the Banach *-algebra A in the form

$$A \cong B \otimes \mathcal{A} \cong \{\text{continuous functions } Y \longrightarrow B\}, \tag{3.8}$$

for which the $\| \ \|_2$-trace in the norm of A is regarded as continuous as a function of Y. The Banach algebra $B = \mathcal{L}_J(H)$ corresponds to taking $\mathcal{A} = \mathbb{C}$, and as mentioned in Remark 3.1, with respect to the polarization $H = H_+ \oplus H_-$, we recover the usual restricted Grassmannians $\mathrm{Gr}(H_+, H)$. Given our formulation, and in view of the spatial correspondence, it will sometimes be convenient to set $\mathrm{Gr}(q, B) = \mathrm{Gr}(H_+, H)$, for suitable $q \in P(A)$. In fact, there is a natural inclusion $\mathrm{Gr}(q, B) \subset \mathrm{Gr}(p, A)$ as deduced in [15].

4. Constructions for the submanifold geometry and bundle theory

4.1. Some preliminaries

In this section we will compute in various bundles where the manifolds involved are submanifolds of Banach spaces, and in this context, adopt some notation which will facilitate the calculations. If $\xi = (\pi, B, X)$ denotes a bundle, meaning simply that we start with a map $\pi : B \longrightarrow X$, and denote by $\xi_x = B_x = \pi^{-1}(x)$, the fiber of ξ over $x \in X$. We write $\pi = \pi_\xi$ for the projection of this bundle and $B = B_\xi$ for its total space. When no confusion results, we will simply write B for the bundle

ξ. If $\psi = (h; f) : \xi \longrightarrow \zeta$, meaning that $\pi_\zeta h = f\pi_\xi$, then $\psi_x = h_x$ denotes the restriction of h to a map of ξ_x into $\zeta_{f(x)}$. By the same token we shall simply write h in place of ψ. As usual, by a section of ξ, we simply mean a map $s : X \longrightarrow B$ satisfying $\pi s = \mathrm{id}_X$.

If ξ is a vector bundle over X, then we take z_ξ to denote the zero section of ξ. We denote by $\epsilon(X, F)$ the trivial bundle $X \times F$ over X with fiber F. If M is a manifold (of some order of differentiability), then we will need to distinguish between the tangent bundle $\mathsf{T}(M)$ of M and the total space TM of the former. We let $z_M = z_{\mathsf{T}(M)}$. Thus, z_M is a standard embedding of M into TM.

When ξ is a subbundle of the trivial bundle $\epsilon = \epsilon(X, F)$, then π_ϵ is the first factor projection and the second factor projection, π_2 assigns each $b \in X \times F$ its principal part. Thus we have a subset $F_x = \pi_2(B_x) \subset F$, so that $B_x = \{x\} \times F_x$. Moreover, if s is here a section of $\xi \subset \epsilon$, then we call $\pi_2 s$ *the principal part of* s. Consequently, $s = (\mathrm{id}_X, f)$, where $f = \pi_2 s : X \longrightarrow F$, must have the property that $f(x) \in F_x$ for each $x \in X$, and any $f : X \longrightarrow F$ having this property is the principal part of a section. In particular, if M is a submanifold of a Banach space F, then $\mathsf{T}(M)$ is a vector subbundle of $\epsilon(M, F)$, and we define $T_x M = F_x$. Thus $\mathsf{T}_x(M) = \{x\} \times T_x M$. If H is another Banach space, N a submanifold of H, and $f : M \longrightarrow N$ is smooth, then $T_x f : T_x \longrightarrow T_f(x)$, is the principal part of the tangent map, so that we have $Tf_x = \mathrm{id}_x \times T_x f$.

Locally, we can assume that M is a smooth retract in F which means any smooth map on M can be assumed to have at each point, a local smooth extension to some open set in F containing that point. So if $v \in T_x M$, then $T_x f(v) = D_v f(x) = f'(x)v$, this last term being computed with any local smooth extension. In our applications, the maps will be defined by simple formulas which usually have obvious extensions, as both F and H will be at most products of a fixed Banach algebra A and the formulas defined using operations in A.

4.2. The tangential extension

If $\varphi : M \times N \longrightarrow Q$ is a smooth map, then we have the associated tangent map $T\varphi : TM \times TN \longrightarrow TQ$. If we write $\varphi(a, b) = ab$, then we also have $T\varphi(x, y) = xy$, if $(x, y) \in TM \times TN$. Employing the zero sections, we shall write ay in place of $z_M(a)y$ and xb in place of $xz_N(b)$. Thus it follows that $ab = z_M(a)z_N(b)$ is again identified with $\varphi(a, b)$; that is, we regard $T\varphi$ as an extension of φ which we refer to as *the tangential extension* (of φ).

Since $T(M \times N) = TM \times TN$, which is fiberwise the direct sum of vector spaces, we readily obtain for $(x, y) \in T_a M \times T_b N$, the relation

$$xy = ay + xb = ay +_{ab} xb, \tag{4.1}$$

where for emphasis, we denote by $+_{ab}$ the addition map in the vector space $T_{ab}Q$ (recall that $\varphi(a, b) = ab$).

4.3. Tangential isomorphisms

In the following, we will have to be particularly careful in distinguishing between the *algebraic commutator* '$[\ ,\ ]_{\mathrm{alg}}$' and the *Lie bracket* '$[\ ,\ ]_{\mathfrak{L}}$' (of vector fields), when dealing with functions taking values in a Banach algebra. Specifically, we let $[x,y]_{\mathrm{alg}}$ denote the algebraic commutator which can be taken pointwise if x, y are algebra-valued functions, and $[x,y]_{\mathfrak{L}}$ to denote the Lie bracket of vector fields or principal parts of vector fields which may also be algebra-valued functions.

Relative to the restricted algebra A_{res} in (3.3), let us recall that the space of polarizations is the space $\mathfrak{P}$ of complementary pairs in the product

$$\mathfrak{P} \subset \mathrm{Gr}(p, A_{\mathrm{res}}) \times \mathrm{Gr}^{op}(p, A_{\mathrm{res}}). \tag{4.2}$$

A significant observation, is that as a set, $\mathfrak{P}$ can be identified with the similarity class $\mathrm{Sim}(p, A_{\mathrm{res}})$ of A_{res}. In fact (see below),

$$\mathfrak{P} \cong \mathrm{Sim}(p, A_{\mathrm{res}}) \subset P(A_{\mathrm{res}}). \tag{4.3}$$

Now from [11], we know that $\Pi = \mathrm{Im}$ and $\Pi^{\mathrm{op}} = \mathrm{Ker}$ are analytic open maps. In fact, the calculations are valid in any Banach algebra, *so henceforth, A can be taken to be any Banach algebra with identity.* Thus, we can begin by observing from (2.4) that for any Banach algebra A, the map $\phi = (\Pi, \Pi^{\mathrm{op}}) = (\mathrm{Im}, \mathrm{Ker})$ is an embedding of the space of idempotents $P(A)$ as an open subset of $\mathrm{Gr}(A) \times \mathrm{Gr}(A)$.

Theorem 4.1. *Let $\phi = (\Pi, \Pi^{\mathrm{op}}) = (\mathrm{Im}, \mathrm{Ker}) : P(A) \longrightarrow \mathrm{Gr}(A) \times \mathrm{Gr}(A)$, be as above and let $r \in P(A)$.*

(1) *We have an isomorphism*

$$T\Pi_r|[\{r\} \times (\hat{r}Ar)] : \{r\} \times (\hat{r}Ar) \xrightarrow{\cong} T_{\Pi(r)}\mathrm{Gr}(A), \tag{4.4}$$

and

$$\mathrm{Ker}\ T\Pi_r = \{r\} \times (rA\hat{r}). \tag{4.5}$$

(2) *In the dual sense, we also have an isomorphism*

$$T\Pi^{\mathrm{op}}_r|[\{r\} \times (rA\hat{r})] : \{r\} \times (rA\hat{r}) \xrightarrow{\cong} T_{\Pi(\hat{r})}\mathrm{Gr}(A), \tag{4.6}$$

and

$$\mathrm{Ker}\ T\Pi^{\mathrm{op}}_r = \{r\} \times (\hat{r}Ar). \tag{4.7}$$

(3) *The map ϕ is an injective open map and an analytic diffeomorphism onto its image $\mathfrak{P}$. Hence $\mathfrak{P}$ is analytically diffeomorphic to* $\mathrm{Sim}(p, A)$.

Proof. As we already know, since the map ϕ is injective, it suffices to apply the Inverse Function Theorem (see, e.g., [20]) when noting that the tangent map $T\phi$ is an isomorphism on fibers of the tangent bundles. To do this, we apply the formulation of [11]. Firstly, from [11], we know that

$$T_rP(A) = \hat{r}Ar + rA\hat{r}. \tag{4.8}$$

If $r \in P(A)$, then we deduce from [11] the canonical section $S_r : U_r \longrightarrow P(A)$ whose image is $P(A) \cap V(r, A) = r + \hat{r}Ar$, which is analytic on its domain $U_r \subset \mathrm{Gr}(A)$. Specifically, we know from [11] that S_r is the inverse of the analytic diffeomorphism $\Pi|(r + \hat{r}Ar)$, which maps onto U_r and that U_r is an open subset of $\mathrm{Gr}(A)$ containing r. This shows that $T_r\Pi|\{r\} \times (\hat{r}Ar)$ is an isomorphism onto $T_{\Pi(r)}\mathrm{Gr}(A)$. On the other hand, Π is constant on $r + rA\hat{r} = \Pi^{-1}(\Pi(r)) \subset P(A)$. Thus, we see that Ker $T_r\Pi = \{r\} \times (rA\hat{r})$. This establishes part (1).

Likewise for part (2), $\mathrm{Ker}|(r + \hat{r}Ar)$ is constant and $\mathrm{Ker}|(r + rA\hat{r})$ is an analytic diffeomorphism onto an open subset of $\mathrm{Gr}(\hat{r}, A)$ which of course is an open subset of $\mathrm{Gr}(A)$ as Π is an open map and $\mathrm{Sim}(q, A)$ is open in $P(A)$. Thus (2) follows.

For part (3), note that since $\hat{r}Ar$ and $rA\hat{r}$ are complementary subspaces of $T_rP(A)$, it follows that $T_r\phi_r = T_r(\Pi, \Pi^{\mathrm{op}})$ is an isomorphism onto $T_{\phi(r)}[\mathrm{Gr}(A) \times \mathrm{Gr}(A)]$. Thus ϕ is indeed an injective open map and an analytic diffeomorphism onto its image $\mathfrak{P}$. Now $\mathrm{Gr}^{\mathrm{op}}(p, A) = \mathrm{Gr}^*(p, A) = \mathrm{Gr}(\hat{p}, A)$, and clearly ϕ carries $\mathrm{Sim}(p, A)$ onto this sub-product, namely the space of polarizations $\mathfrak{P}$. □

5. The space V_Λ and its geometry

5.1. Transversality and the transition map

We now fix any idempotent $p \in P(A)$, and for ease of notation in the following, we set

$$\begin{aligned} &\Lambda = \mathrm{Sim}(p, A),\ \mathrm{Gr}(p) = \mathrm{Gr}(p, A),\ V = V(p, A) \\ &\pi_\Lambda = \Pi|\Lambda,\ \text{and}\ \pi_V = \Pi|V. \end{aligned} \tag{5.1}$$

Note that from Theorem 4.1(3), we have the analytic diffeomorphism $\Lambda \cong \mathfrak{P}$.

From [11, § 7] we know that $(\pi_V, V, \mathrm{Gr}(p))$ is an analytic right principal $G(pAp)$-bundle whose transition map

$$t_V : V \times_\pi V \longrightarrow G(pAp), \tag{5.2}$$

is the analytic map such that if $u, v \in V$, and $r \in \Lambda$, with $\pi_V(u) = \pi_V(v) = \pi_\Lambda(r)$, then (recalling the notation of (2.5)) we have

$$t_V(u, v) = u^{-(p,r)}v. \tag{5.3}$$

Define $V_\Lambda = \pi_\Lambda^*(V)$, so then $V_\Lambda \subset \Lambda \times V$ is an analytic principal right $G(pAp)$-bundle over Λ, and clearly

$$V_\Lambda = \{(r, u) \in \Lambda \times V : \pi_\Lambda(r) = \pi_V(u)\}. \tag{5.4}$$

The fact that V_Λ is an analytic submanifold of $\Lambda \times V$ and hence of $A \times A$, follows from the fact that by (4.4) any smooth map to $\mathrm{Gr}(p)$ is *transversal* over π_Λ.

Likewise, we denote by t_Λ the transition map for V_Λ, as the analytic map given by the formula:

$$t_\Lambda((r, u), (r, v)) = t_V(u, v) = u^{-(p,r)}v. \tag{5.5}$$

We keep in mind that if $(r, u) \in V_\Lambda$, then as $\pi_\Lambda(r) = \pi_V(u)$, it follows that $u : p \longrightarrow r$ and therefore $u^{-(p,r)}$ is defined.

The next step is to uncover the geometry natural to V_Λ coming from the fact that we can calculate $T_{(r,u)}V_\Lambda \subset A \times A$. Since π_Λ and π_V are transversal as maps to $\mathrm{Gr}(p)$, it follows that

$$T_{(r,u)}V_\Lambda = \{(x,y) \in T_r\Lambda \times T_uV_\Lambda : [T\pi_\Lambda]_r(r,x) = [T\pi_V]_u(u,y)\} \subset A \times A. \quad (5.6)$$

Lemma 5.1. *We have $T_uV = Ap$, and rAp is the vertical tangent space of V over $\pi_E(r) = \pi_V(u)$. Further, $\hat{r}Ap$ and rAp are complementary subspaces of $Ap = T_uV$.*

Proof. It is straightforward to see that $T_r\Lambda = \hat{r}Ar + rA\hat{r} \subset A$, and from [11], we know that $V = G(A)p$ is open in Ap. It follows that $T_uV = Ap$. As π_V is a principal bundle projection, we know that Ker $T_u\pi_V = T_u[uG(pAp)]$, the tangent space to the fiber over $u \in V$, is the kernel of $T\pi_V$. As there is a $g \in G(A)$ with $u = gp$, and as left multiplication by g is $G(pAp)$-equivariant (simply by the associative law for multiplication in A), it follows that

$$T_u[uG(pAp)] = gT_pG(pAp) = gpAp = uAp. \quad (5.7)$$

Since $ru = u$, and $uu^{-(p,r)} = r$, it follows that $uAp = rAp$. Thus rAp is the vertical tangent space of V over $\pi_E(r) = \pi_V(u)$, so $\hat{r}Ap$ and rAp are complementary subspaces of $Ap = T_uV$. □

On the other hand, from [11], we know that $\Lambda \cap V = S_p(U_p)$ is the image of the canonical section and both π_Λ, π_V coincide on $\Lambda \cap V$. This means that by (4.4), we know $[T\pi_v]_p$ carries $\{p\} \times \hat{p}Ap$ isomorphically onto $T_{\pi(p)}\mathrm{Gr}(p)$ and agrees with the isomorphism (4.4), so we see easily that

$$T_{(p,p)}V_\Lambda = \{x, y) \in [\hat{p}Ap + pA\hat{p}] : xp = \hat{p}y\}. \quad (5.8)$$

Differentiating the equation $ru = u$, we see that any $(x,y) \in T_{(r,u)}V_\Lambda$ must satisfy $xu + ry = y$ which is equivalent to the equation $xu = \hat{r}y$. Notice this is exactly the equation for the tangent space at (p,p), so we claim

$$T_{(r,u)}V_\Lambda = \{(x,y) \in T_r\Lambda \times Ap : xu = \hat{r}y\}. \quad (5.9)$$

Effectively, a straightforward calculation using (5.8) and the fact that $G(A)$ acts $G(pAp)$-equivariantly on V on the left by ordinary multiplication to translate the result in (5.8) over to the point (r,u), establishes (5.9).

5.2. The connection map $\mathcal{V}$

Now the projection $\pi^* = \pi_{V_\Lambda}$ of V_Λ is a restriction of the first factor projection of $A \times A$ onto A which is linear. Thus $T_{(r,u)}\pi^*(x,y) = x$, and therefore the vertical subspace of $T_{(r,u)}V_\Lambda$ is the set $\{0\} \times rAp$. The projection of the tangent bundle TV_Λ onto this vertical subbundle is clear, and we define

$$\begin{aligned} \mathcal{V} : TV_\Lambda &\longrightarrow TV_\Lambda \\ \mathcal{V}((r,u),(x,y)) &= ((r,u),(0,ry)), \end{aligned} \quad (5.10)$$

for any $(x,y) \in T_{(r,u)}V_\Lambda$, and for any $(r,u) \in V_\Lambda$. For convenience, let $\mathcal{V}_{(r,u)}$ be the action of $\mathcal{V}$ on principal parts of tangent vectors, so that we obtain

$$\mathcal{V}_{(r,u)}(x,y) = (0, ry). \quad (5.11)$$

It is obvious that $\mathcal{V}$ is a vector bundle map covering the identity on V_Λ and that $\mathcal{V} \circ \mathcal{V} = \mathcal{V}$. Thus we call $\mathcal{V}$ *the connection map.*

Since the right action of $G(pAp)$ on V is defined by just restricting the multiplication map on $A \times A$, it follows that the tangential extension of the action of $G(pAp)$ to act on TV is also just multiplication on the right, that is, yg is just the ordinary product in A. This means that in TV_Λ we have $(x, y)g = (x, yg)$ as the tangential extension of the right action of $G(pAp)$ on $T_{(r,u)}V_\Lambda$. From this, the fact that $\mathcal{V}$ is $G(pAp)$-equivariant, is clear. Thus the map $\mathcal{V}$ defines a connection on V_Λ.

Let $\mathcal{H} = (\mathrm{id}_{TV} - \mathcal{V})$, so $\mathcal{H}$ is the resulting horizontal projection in each fiber. Then clearly for $(x, y) \in T_{(r,u)}V_\Lambda$, we have on principal parts of tangent vectors

$$\mathcal{H}_{(r,u)}(x, y) = (x, y) - (0, ry) = (x, \hat{r}y) = (x, xu). \tag{5.12}$$

Moreover, this clarifies that $(x, xu) \in \mathcal{H}(T_{(r,u)}V_\Lambda)$ is (the principal part of) the horizontal lift of $x \in T_r\Lambda$.

If σ is any smooth local section of V_Λ, then for a vector field χ on Λ it follows that the covariant derivative is just the composition

$$\nabla_\chi \sigma = \mathcal{V}[T\sigma]\chi, \tag{5.13}$$

which is a map of Λ to $\mathcal{V}(TV_\Lambda)$ lifting σ. Because the differentiation here is essentially applied to the principal part of the vector field, if f is the principal part of σ and w is the principal part of χ, then for the purpose of calculations, we can also write $\nabla_w f = \mathcal{V}[f'w] = \mathcal{V}D_w f$, where the meaning is clear.

6. The connection form and its curvature

6.1. The connection form ω_Λ

The right action of $G(pAp)$ on V_Λ in (5.4), when tangentially extended, gives $(r, u)y \in T_{(r,u)}V_\Lambda$ when $y \in T_pG(pAp) = pAp$. As the right action of $G(pAp)$ on V_Λ is defined by $(r, u)g = (r, ug)$, it follows that $(r, u)y = (0, uw)$, for any $w \in T_pG(pAp) = pAp$. The connection 1-form $\omega = \omega_\Lambda$ can then be determined because it is the unique 1-form such that, in terms of the connection map $\mathcal{V}$, we have

$$(r, u)\omega_{(r,u)}(x, y) = \mathcal{V}_{(r,u)}(x, y). \tag{6.1}$$

Notice that if $(x, y) \in T_{(r,u)}V_\Lambda$, then we have $y \in Ap$, and so $u^{-(p,r)y} \in pAp = T_pG(pAp)$. We therefore have both

$$(r, u)\omega_{(r,u)}(x, y) = (0, ry) \text{ and } (r, u)u^{-(p,r)}y = (0, ry), \tag{6.2}$$

which by comparison expresses the connection form as

$$\omega_{(r,u)}(x, y) = u^{-(p,r)}y \in T_pG(pAp) = pAp. \tag{6.3}$$

6.2. The curvature form Ω_Λ

To find the curvature 2-form Ω_Λ of ω_Λ, we simply take the covariant exterior derivative of ω_Λ:

$$\Omega_\Lambda = \nabla\omega_\Lambda = \mathcal{H}^* d\omega_\Lambda. \tag{6.4}$$

Notice that by (5.12), as $r\hat{r} = 0$, we have $\omega_\Lambda(\mathcal{H}v) = 0$, for any $v \in TV_\Lambda$, as should be the case, and therefore if w_1 and w_2 are local smooth tangent vector fields on V_Λ, then, on setting $\Omega = \Omega_\Lambda$ for ease of notation, we have

$$\Omega(w_1, w_2) = -\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}). \tag{6.5}$$

This means that the curvature calculation is reduced to calculating the Lie bracket of two vector fields on V_Λ. Since $V_\Lambda \subset A \times A$ is an analytic submanifold, it is a local smooth retract in $A \times A$.

In order to facilitate the calculation, let

$$(\tilde{r}, \tilde{u}) : W \longrightarrow W \cap V_\Lambda, \qquad (W \subset A \times A), \tag{6.6}$$

be an analytic local retraction of an open set W in $A \times A$, onto the open subset $W \cap V_\Lambda$ of V_Λ. We can then use $(\tilde{r}, \tilde{u})$ to extend all functions on $W \cap V_\Lambda$ to be functions on W. As w_1 and w_2 are tangent vector fields, assumed analytic on $W \cap V_\Lambda$, their principal parts can be expressed in the form $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$, and we can therefore assume that as functions, they all are defined on W. We then have pointwise on $W \cap V_\Lambda$,

$$x_i\tilde{u} = \hat{\tilde{r}}y_i = (1 - \tilde{r})y_i, \qquad \text{for } i = 1, 2. \tag{6.7}$$

But then $\mathcal{H}_{(r,u)}(x_i, y_i) = (x_i, x_i u)$ on $W \cap V_\Lambda$, meaning that the principal part of $[\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}$ is just $[(x_1, x_1\tilde{u}), (x_2, x_2\tilde{u})]_{\mathfrak{L}}|(W \cap V_\Lambda)$.

The next simplification is to notice that on $W \cap V_\Lambda$, the function $\tilde{u}$ is just the same as the second factor projection $A \times A \longrightarrow A$. On differentiating, this simplifies the application of the product rule. The result is that the principal part of $[\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}$ evaluated at $(r, u) \in V_\Lambda$, has the form

$$(c, cu + [x_2, x_1]_{\mathrm{alg}}u), \tag{6.8}$$

for suitable c, and where x_i is now just the value of the preceding function of the same symbol at (r, u).

Proposition 6.1. *For $w_1, w_2 \in (TV_\Lambda)_{(r,u)}$ having principal parts (x_1, y_1) and (x_2, y_2) respectively, we have the curvature formula*

$$\Omega_\Lambda(w_1, w_2) = u^{-(p,r)}[x_1, x_2]_{\mathrm{alg}}u. \tag{6.9}$$

Proof. As the Lie bracket of a pair of vector fields tangent to a submanifold, again remains tangent to that submanifold, this means that $(c, cu + [x_2, x_1]_{\mathrm{alg}}u)$ in (6.8), is tangent to V_Λ. Hence, we must also have

$$cu = \hat{r}(cu + [x_2, x_1]_{alg}u), \tag{6.10}$$

and therefore,

$$rcu = \hat{r}[x_2, x_1]_{\mathrm{alg}}. \tag{6.11}$$

Applying (6.3) and (6.5), we now obtain

$$\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}})_{(r,u)} = u^{-(p,r)}(cu + [x_2, x_1]_{\text{alg}}u). \tag{6.12}$$

In view of the fact that $u^{-(p,r)}r = u^{-(p,r)}$ and (6.12) above, we deduce that

$$\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}})_{(r,u)} = u^{-(p,r)}[x_2, x_1]_{\text{alg}}u. \tag{6.13}$$

Thus by (6.5), we finally arrive at

$$\Omega(w_1, w_2) = u^{-(p,r)}[x_1, x_2]_{\text{alg}}u, \tag{6.14}$$

where now $w_1, w_2 \in (TV_\Lambda)_{(r,u)}$ have principal parts (x_1, y_1) and (x_2, y_2) respectively. □

This of course means that $x_1, x_2 \in T_r\Lambda = \hat{r}Ar + rA\hat{r}$, that $y_1, y_2 \in T_uV_\Lambda = Ap$, and thus $x_iu = \hat{r}y_i$, for $i = 1, 2$. But, $V_\Lambda = G(A)u$, so there is $g \in G(A)$ with $u = gp$. It then follows that $u^{-(p,r)} = pg^{-1}$, and therefore we can also write, when $u = gp$,

$$\Omega(w_1, w_2) = [g^{-1}x_1g, g^{-1}x_2g]_{\text{alg}}. \tag{6.15}$$

In this way we can simply transfer the computation to the Lie algebra of $G(pAp)$. We make the following observations:

(1) Because $ru = u$ and $u^{-(p,r)}r = u^{-(p,r)}$, when $(r, u) \in V_\Lambda$, it follows that (6.14) can also be written as

$$\Omega(w_1, w_2) = u^{-(p,r)}r[x_1, x_2]_{\text{alg}}ru, \tag{6.16}$$

and the factor $r[x_1, x_2]_{\text{alg}}r$ simplifies greatly because $x_1, x_2 \in rA\hat{r} + \hat{r}Ar$.

(2) If x_1 and x_2 both belong to $rA\hat{r}$, or both belong to $\hat{r}Ar$, then the result is $\Omega(w_1, w_2) = 0$.

(3) If $x_1 \in rA\hat{r}$ and $x_2 \in \hat{r}Ar$, the result is

$$\Omega(w_1, w_2) = u^{-(p,r)}x_1x_2u. \tag{6.17}$$

Whereas if the reverse is the case, that is $x_1 \in \hat{r}Ar$ and $x_2 \in rA\hat{r}$, the result is

$$\Omega(w_1, w_2) = -u^{-(p,r)}x_2x_1u. \tag{6.18}$$

Remark 6.1. Again, by Theorem 4.1(3), since $\Lambda \cong \mathfrak{P}$, the construction of the principal bundle with connection $(V_\Lambda, \omega_\Lambda) \longrightarrow \Lambda$, may be seen to recover that of the principal bundle with connection $(V_{\mathfrak{P}}, \omega_{\mathfrak{P}}) \longrightarrow \mathfrak{P}$ as in [39, § 3]. We will elaborate on matters when we come to describe the $\mathcal{T}$-function in § 8.1. This principal bundle has for its associated vector bundle (with connection) the universal bundle $(\gamma_{\mathfrak{P}}, \nabla_{\mathfrak{P}}) \longrightarrow \mathfrak{P}$. In the following section, the latter will be recovered when we construct the universal bundle (with connection) $(\gamma_\Lambda, \nabla_\Lambda) \longrightarrow \Lambda$ associated to $(V_\Lambda, \omega_\Lambda) \longrightarrow \Lambda$.

7. The universal bundle over Λ

7.1. The Koszul connection

Next we relate the geometry of V_Λ to the geometrical context of [39] (cf. [28]). First we must show that V_Λ is the principal bundle of the universal bundle in an appropriate sense. In fact, if E is a Banach A-module, then we can form an obvious universal vector bundle, denoted γ_Λ over Λ, as defined by

$$\gamma_\Lambda = \{(r, m) \in \Lambda \times E : rm = m\}, \tag{7.1}$$

and whose projection π_γ is just the restriction of first factor projection. Thus the principal part of a section is here simply a map $f : \Lambda \longrightarrow E$ with the property that $f(r) \in rE$, for every $r \in \Lambda$.

In this case, a natural Koszul connection ∇_Λ arises. Effectively, we have a covariant differentiation operator, given by its operation on principal parts of sections of γ_Λ, via the formula

$$\nabla_x f(r) = rD_x f(r) = rT_r f(x), \; x \in T_r\Lambda. \tag{7.2}$$

If x is the principal part of a tangent vector field on Λ, then it follows that

$$\nabla_x f = \mathrm{id}_\Lambda D_x f = \mathrm{id}_\Lambda T_{\mathrm{id}_\Lambda} f(x). \tag{7.3}$$

If $(r, m) \in \gamma_\Lambda$, then the principal part of the tangent space to γ_Λ at the point (r, m) is just

$$T_{(r,m)}\gamma_\Lambda = \{(x, w) \in T_r\Lambda \times E : \; rw + xm = w\}, \tag{7.4}$$

which can also be written as

$$T_{(r,m)}\gamma_\Lambda = \{(x, w) \in T_r\Lambda \times E : xm = \hat{r}w\}. \tag{7.5}$$

Since π_γ is simply the restriction of first factor projection which is linear, it follows that the vertical subspace is

$$VT_{(r,m)}\gamma_\Lambda = \mathrm{Ker}\; T_{(r,m)}\pi_\gamma = \{(0, w) \in T_r\Lambda \times E : rw = w\}, \tag{7.6}$$

so the vertical projection

$$\mathcal{V}_\gamma : \mathsf{T}\gamma_\Lambda \longrightarrow \mathsf{T}\gamma_\Lambda, \tag{7.7}$$

as a vector bundle map covering $\mathrm{id}_{\gamma_\Lambda}$, is given by

$$\mathcal{V}_\gamma((r, m), (x, w)) = ((r, m), (0, rw)). \tag{7.8}$$

This of course means that the horizontal projection $\mathcal{H}_\gamma$ is given by

$$\mathcal{H}_\gamma((r, m), (x, w)) = ((r, m), (x, \hat{r}w)) = ((r, m), (x, xm)), \tag{7.9}$$

which makes it clear that the horizontal lift to $(r, m) \in \gamma_\Lambda$ of $(r, x) \in T\Lambda$ is just $((r, x), (x, xm))$.

Thus, the geometry of the universal bundle γ_Λ turns out to be very natural and straightforward. In order to see that γ_Λ is the associated vector bundle to the principal bundle V_Λ, we first note that the principal part of the fiber of γ_Λ over $p \in \Lambda$ is pE and we can define the *principal map*

$$Q : V_\Lambda \times pE \longrightarrow \gamma_\Lambda, \tag{7.10}$$

by

$$Q((r,u),m) = (r,um), \ \ ((r,u),m) \in V_\Lambda \times pE. \tag{7.11}$$

Proposition 7.1. *The map Q in* (7.11) *is the analytic principal bundle map for which the universal bundle* $(\gamma_\Lambda = V_\Lambda[pE], \nabla_\Lambda)$ *is an analytic vector bundle with connection associated to the principal bundle with connection* $(V_\Lambda, \omega_\Lambda)$.

Proof. Clearly $V_\Lambda \times pE$ has a principal right $G(pAp)$-action given by

$$((r,u),m))g = ((r,u)g, g^{-p}m) = ((r,ug), g^{-p}m), \tag{7.12}$$

with transition map

$$t(((r,u),m),((r,v),n)) = t_\Lambda((r,u),(r,v)), \tag{7.13}$$

and Q establishes a bijection with the orbit space of this action. To conclude that Q is the actual principal map making $\gamma_\Lambda = V_\Lambda[pE]$ the associated bundle to V_Λ with fiber pE, it suffices to show that Q has analytic local sections, because Q itself is clearly analytic.

To that end, observe that if σ is a local section of V_Λ over the open subset $U \subset \Lambda$, then $\sigma = (\mathrm{id}_\Lambda, u)$ where $u : U \longrightarrow V = V(p,A)$, such that for every $r \in U$, we have $u(r) : p \longrightarrow r$ is a proper partial isomorphism. We then define λ, the corresponding local analytic cross section of Q by

$$\lambda(r,m) = ((r,u(r)), u(r)^{-(p,r)}m). \tag{7.14}$$

Following [11] we know that $u^{-(p,r)}$ as a function of $r \in U$, is analytic as a map to $V(A)$. Indeed, Q is the principal map and $\gamma_\Lambda = V_\Lambda[pM]$. It is now a routine calculation to see that the connection on γ_Λ defined above is the same as the connection derived from the connection ω_Λ already defined on V_Λ. □

For instance, if $f : V_\Lambda \longrightarrow pE$ is an equivariant smooth map, and x is any section of $T\Lambda$, then f defines a smooth section s of γ_Λ whose covariant derivative $\nabla_x s$ is the same as the section defined by the derivative of f in the direction of the horizontal lift of x. As Q is the principal map, it is the projection of a principal bundle and therefore TQ is vector bundle map covering Q which is surjective on the fibers. We have

$$TQ(((r,u),m),((x,y),w)) = ((r,um),(x, ym+uw)), \tag{7.15}$$

and

$$\mathcal{V}_\gamma TQ((r,u),m),((x,y),w)) = ((r,um),(0, r[ym+uw])), \tag{7.16}$$

along with

$$\begin{aligned} TQ(\mathcal{V}^*(((r,u),m),((x,y),w)) &= TQ(((r,u),m),((0,ry),w)) \\ &= ((r,um),(0,rym+uw)). \end{aligned} \tag{7.17}$$

But $ru = u$ for $(r,u) \in V_\Lambda$. Hence from (7.16) and (7.17), we have $\mathcal{V}_\gamma TQ = TQ\mathcal{V}^*$, where $\mathcal{V}^*$ denotes the connection map of the vertical projection on $V_\Lambda \times pE$ pulled back from V_Λ by the first factor projection map of $V_\Lambda \times pE \longrightarrow V_\Lambda$, which being equivariant, defines a pullback square. This shows that the vertical projection on

γ_Λ is that defined by the vertical projection on V_Λ. Thus we have constructed V_Λ to be the principal bundle for any universal bundle defined by any left Banach A-module such as E. In particular, we could take $E = A$ for the existence of one, but for the $\mathcal{T}$-function construction we would take $E = H_{\boldsymbol{\mathcal{A}}}$. In other words, we would take E to be the underlying Banach space of $H_{\boldsymbol{\mathcal{A}}}$ so $\boldsymbol{\mathcal{A}}$ would act as a subalgebra of the commutant of A in the algebra of bounded operators.

8. The $\mathcal{T}$-function

8.1. Definition of the $\mathcal{T}$-function

From our constructions so far, even though they are quite general, it should be clear that we have all the ingredients for the construction of a function, denoted by $\mathcal{T}$, that generalizes the function, denoted by $\mathfrak{T}$ and defined via cross-ratio in [28, 39] as a pre-determinant, thus providing the Tau (τ)-function studied in [28, 34]. Similar to [39], we will define two local sections α_p and β_p over W_p^0, the latter taken to be an open neighborhood of $p \in P(A)$, which is our reference projection. For W_p^0 we take the set of $r \in W_p = \pi_\Lambda^{-1}(p + pA\hat{p})$ such that $\phi_p(r) = rp + \hat{r}\hat{p} \in G(A)$. As $G(A)$ is open in A, and as $\phi_p(p) = 1 \in G(A)$, it follows that W_p^0 is indeed open in Λ and contains p.

Next we describe the sections α_p and β_p:

(1) For α_p we take the restriction of the pullback by π_Λ of the canonical section S_p which is defined over $\pi_\Lambda(W_p) \subset \mathrm{Gr}(p, A)$. Thus, as in the pullback, α_p becomes a composition with π_Λ. It follows from (4.5) that if $w = (r, x) \in T\Lambda$ with $x \in rA\hat{r}$, then $\nabla_w \alpha_p = 0$.

(2) For β_p, with $g = \phi_p(r)$ and $r \in W_p^0$, we have $g \in G(A)$ and $rp : p \longrightarrow r$ is a proper partial isomorphism which projects along $\mathrm{Ker}(r)$, so we define $\beta_p(r) = (r, rp)$.

As $S_p(\mathrm{Im}(r))$ projects along $\mathrm{Ker}(p)$, we generalize the $\mathfrak{T}$-function of [39] by the function $\mathcal{T}$ by recalling the transition map t_Λ in (5.5), and then defining

$$\mathcal{T}(r) = t_\Lambda(\alpha_p(r), \beta_p(r)). \tag{8.1}$$

Hence we may express the latter by $\mathcal{T} = t_\Lambda(\alpha_p, \beta_p)$.

In [39], the function $\mathfrak{T}$ constructed via cross-ratio is used to define the connection form $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}} \longrightarrow \mathfrak{P}$, where the corresponding curvature 2-form $\Omega_{\mathfrak{P}}$ can be computed in coordinates on the product of Grassmannians. In order to see that the geometry here is essentially the same as that of [39], we show that under certain conditions, α_p and β_p are parallel (covariantly constant) sections. Specifically, it suffices to show that $\nabla_w \alpha_p = 0$, if $w = (r, x)$ with $x \in rA\hat{r}$, and that $\nabla_w \beta_p = 0$ if $w = (r, x)$ with $x \in \hat{r}Ar$. The first of these has already been observed in (1) above. As for the second, since $\beta_p(r) = (r, rp)$, it follows that $T_r\beta_p(x) = (x, xp)$, for any $x \in T_r\Lambda$, and therefore

$$\nabla_w \beta_p = \mathcal{V}((r, rp), (x, xp)) = ((r, rp), (0, rxp)). \tag{8.2}$$

As $x \in \hat{r}Ar$ implies $rxp = 0$, we also have $\nabla_w \beta_p = 0$, for $w = (r, x)$ with $x \in \hat{r}Ar$. We therefore know that the geometry is the same as in [39] and we can now apply our formulas to calculate $\mathcal{T}$. But, we know from the definition of the transition function t_Λ in (5.5), that we have

$$t_\Lambda((r,u),(r,v)) = u^{-(p,r)}v, \tag{8.3}$$

and we know that the relative inverse for the canonical section is p itself, independent of r. Hence, we finally have $\mathcal{T}(r) = prp$.

8.2. Curvature formulas

Returning to the universal bundle (with connection) $(\gamma_\Lambda, \nabla_\Lambda) \longrightarrow \Lambda$, we can easily calculate the curvature form using the Koszul connection of the connection ∇_Λ operating on principal parts of sections of γ_Λ. If x and y are principal parts of local smooth tangent vector fields to Λ, and if f is an E-valued smooth function on the same domain, then we can consider that ordinary differentiation D acting on functions, is the Koszul connection of the flat connection on $\epsilon(\Lambda, E)$. So the curvature operator $\mathcal{R}_\nabla$ can be computed keeping in mind that $\mathcal{R}_D = 0$. Thus, letting $L : \Lambda \longrightarrow \mathcal{L}(E, E)$ be the action of left multiplication of Λ on E, noting that $L(r)m = em$, we then have

$$\mathcal{R}_\nabla(x,y)f = [\nabla_x, \nabla_y]f - \nabla_{[x,y]_\mathfrak{L}} f. \tag{8.4}$$

Theorem 8.1. *With respect to the above action $L : \Lambda \longrightarrow \mathcal{L}(E, E)$ of left multiplication of Λ on E, we have the following formulas for the curvature operator $\mathcal{R}_\nabla$, for $x, y \in T_r\Lambda$:*

(1)
$$\mathcal{R}_\nabla(x,y) = L[(D_xL)D_y - (D_yL)D_x]. \tag{8.5}$$

(2)
$$\mathcal{R}_\nabla(x,y) = L[x,y]_{\mathrm{alg}}. \tag{8.6}$$

Proof. Firstly, observe that notationally $\nabla_x f = LD_x f$. Since the pointwise product is $LL = L$, it follows that

$$\nabla_x\nabla_y f = LD_x(LD_y f) = L[D_xL][D_y f] + LD_xD_y f, \tag{8.7}$$

and therefore (8.4) becomes

$$\mathcal{R}_\nabla(x,y)f = L[D_xL][D_yf] + LD_xD_yf - (L[D_yL][D_xf] + LD_yD_xf) - LD_{[x,y]_\mathfrak{L}}f. \tag{8.8}$$

Consequently, we have

$$\mathcal{R}_\nabla(x,y)f = (L[D_xL]D_y - [D_yL]D_x)f + L\mathcal{R}_D(x,y)f, \tag{8.9}$$

and therefore, as $\mathcal{R}_D = 0$, it follows that

$$\mathcal{R}_\nabla(x,y)f = L[(D_xL)D_y - (D_yL)D_x]f. \tag{8.10}$$

Thus we may write

$$\mathcal{R}_\nabla(x,y) = L[(D_xL)D_y - (D_yL)D_x], \tag{8.11}$$

which establishes (1).

On the other hand, we note that L is the restriction of the linear map defined by the left regular representation L_A of A on E, defined by the module action of A on M. So we have $D_x L = L_A(x)$, the composition of L_A with x. This means that

$$[(D_x L)(r)]m = L_A(x(r))m = [x(r)]m = (xm)(r), \tag{8.12}$$

for $r \in \Lambda$ and $m \in rE$. Therefore, we have for f, that

$$\mathcal{R}_\nabla(x,y)f = L[(L_A(x))D_y - (L_A(y))D_x]f = L[xD_y - yD_x]f. \tag{8.13}$$

For the curvature operator at a specific point, we can take any $m \in E$, and define $f_m = Lm$, so that we have $f_m(r) = L(r)m = rm$. Then f is given by the module action of A on E which is linear, for fixed $m \in E$. Thus, $D_x f = L_A(x)m = xm$ and (8.13) becomes

$$\mathcal{R}_\nabla(x,y)f = L[x,y]_{\mathrm{alg}} m, \tag{8.14}$$

which means that we finally arrive at (2):

$$\mathcal{R}_\nabla(x,y) = L[x,y]_{\mathrm{alg}}. \tag{8.15}$$

□

8.3. Remarks on the operator cross ratio

Returning to the case $A = \mathcal{L}_J(H_{\mathcal{A}})$, let us now mention some examples (to be further developed in [16]). Firstly, we recall the $\mathfrak{T}$ function of [39] defined via cross-ratio. Consider a pair of polarizations $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in \mathfrak{P}$. Let $\mathsf{H}_\pm$ and $\mathsf{K}_\pm$ be 'coordinatized' via maps $P_\pm : \mathsf{H}_\pm \longrightarrow \mathsf{H}_\mp$, and $Q_\mp : \mathsf{K}_\pm \longrightarrow \mathsf{K}_\mp$, respectively. Following [39] (Proposition 2), we can consider the composite map

$$\mathsf{H}_+ \xrightarrow{\mathsf{K}_-} \mathsf{K}_+ \xrightarrow{\mathsf{H}_-} \mathsf{H}_+, \tag{8.16}$$

as represented by the operator cross-ratio (cf. [39]):

$$\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = (P_- P_+ - 1)^{-1}(P_- Q_+ - 1)(Q_- Q_+ - 1)^{-1}(Q_- P_+ - 1). \tag{8.17}$$

For this construction there is no essential algebraic change in generalizing from polarized Hilbert spaces to polarized Hilbert modules. The principle here is that the transition between charts define endomorphisms of $W \in \mathrm{Gr}(p, A)$ that will become the transition functions of the universal bundle $\gamma_{\mathfrak{P}} \longrightarrow \mathfrak{P}$. These transition functions are defined via the cross ratio as above and thus lead to $\mathrm{End}(\gamma_{\mathfrak{P}})$-valued 1-cocyles, in other words, elements of the cohomology group $H^1(\mathrm{Gr}(p,A), \mathrm{End}(\gamma_{\mathfrak{P}}))$.

Regarding the universal bundle $\gamma_\Lambda \longrightarrow \Lambda$, the transition between charts is already achieved by means of the $\mathcal{T}$-function on Λ. From Theorem 4.1 (3) we have an analytic diffeomorphism $\tilde{\phi} : \mathfrak{P} \longrightarrow \Lambda$ (where $\tilde{\phi} = \phi^{-1}$), and effectively, $\tilde{\phi}^* \mathcal{T} = \mathfrak{T}$ in this case.

8.4. The connection and curvature forms on $V_{\mathfrak{P}}$

In view of § 8.1, we will exemplify the construction of [39, § 3] for the connection form $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}} \longrightarrow \mathfrak{P}$, and the curvature form $\Omega_{\mathfrak{P}}$. We start by fixing a point $\mathcal{P} = (\mathsf{H}_+, \mathsf{H}_-) \in \mathfrak{P}$, and consider a pair of local sections α, β of $V_{\mathfrak{P}}$, which are related as follows:

$$\alpha = \beta\mathfrak{T} \ , \qquad \beta = \alpha\mathfrak{T}^{-1}. \tag{8.18}$$

Next let $\nabla_{\pm}$ denote covariant differentiation with respect to the direction $\mathsf{H}_{\pm}$. The local sections α, β have the property that:

(a) α is covariantly constant along $\{\mathsf{H}_+\} \times \mathrm{Gr}^*(p, A)$, with respect to fixed H_+.

(b) β is covariantly constant along $\mathrm{Gr}(p, A) \times \{\mathsf{H}_-\}$ with respect to fixed H_-.

(c) Properties (a) and (b) imply the equations $\nabla_-\alpha = 0, \nabla_+\beta = 0$, along with $\nabla_+\alpha = \beta\nabla_+\mathfrak{T} = \alpha\mathfrak{T}^{-1}\nabla_+\mathfrak{T}$.

We obtain the connection $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}}$ by setting $\omega_{\mathfrak{P}} = \omega_+ = \mathfrak{T}^{-1}\nabla_+\mathfrak{T}$. We have the exterior covariant derivative $d = \partial_+ + \partial_-$, where $\partial_{\pm}$ denotes the covariant derivative along $\mathsf{H}_{\pm}$. Straightforward calculations as in [39, § 3] yield the following:

$$\partial_+\omega_+ = 0 \ , \tag{8.19}$$

$$\partial_-\omega_+ = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+.$$

The curvature form $\Omega_{\mathfrak{P}}$ relative to $\omega_{\mathfrak{P}}$ is then given by

$$\Omega_{\mathfrak{P}} = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+. \tag{8.20}$$

8.5. Trace class operators and the determinant

An alternative, but equivalent, operator description leading to $\mathfrak{T}$ above can be obtained following [28]. Suppose $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in \mathfrak{P}$ are such that H_+ is the graph of a linear map $S : \mathsf{K}_+ \longrightarrow \mathsf{K}_-$ and H_- is the graph of a linear map $T : \mathsf{K}_- \longrightarrow \mathsf{K}_+$. Then on $H_{\boldsymbol{\mathcal{A}}}$ we consider the identity map $\mathsf{H}_+ \oplus \mathsf{H}_- \longrightarrow \mathsf{K}_+ \oplus \mathsf{K}_-$, as represented in the block form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{8.21}$$

where $a : \mathsf{H}_+ \longrightarrow \mathsf{K}_+,\ d : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are zero-index Fredholm operators, and $b : \mathsf{H}_+ \longrightarrow \mathsf{K}_+,\ c : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are in $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ (the compact operators), such that $S = ca^{-1}$ and $T = bd^{-1}$.

The next thing is to consider the operator $1 - ST = 1 - ca^{-1}bd^{-1}$. In particular, with a view to defining a generalized determinant leading to *an operator-valued Tau-function*, we need to consider cases where ST is assuredly of *trace class*.

(a) When $\boldsymbol{\mathcal{A}} = \mathbb{C}$ as in [28, 34, 39], we take b, c to be Hilbert-Schmidt operators. Then ST is of trace-class, the operator $(1 - ST)$ is essentially

$$\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-)$$

above, and the Tau (τ)-function is defined as

$$\tau(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = \text{Det } \mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = \text{Det}(1 - ca^{-1}bd^{-1}). \qquad (8.22)$$

Starting from the universal bundle $\gamma_{\mathcal{E}} \longrightarrow \text{Gr}(p, A)$, then with respect to an admissible basis in $V(p, A)$, the Tau function in (8.22) is equivalently derived from the canonical section of $\text{Det}(\gamma_{\mathcal{E}})^* \longrightarrow \text{Gr}(p, A)$.

(b) The case where $\boldsymbol{\mathcal{A}}$ is a commutative C*-algebra is relevant to von Neumann algebras (see, e.g., [7]), and we may deal with a continuous trace algebra. In particular, for Hilbert *-algebras in general, we have the nested sequence of Schatten ideals in the compact operators [35]. Thus if we take the operators b, c as belonging to the Hilbert-Schmidt class, then ST is of trace class [35], and $\tau(\mathsf{H}_+, \mathsf{K}_-, \mathsf{K}_+, \mathsf{K}_-)$ is definable when the operator $(1 - ST)$ admits a determinant in a suitable sense.

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