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RELATIVE INVERSION AND EMBEDDINGS

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Abstract. Commencing from a monoidal semigroup A, we consider the geometry of the space $W(A)$ of pseudoregular elements. When A is a Banachable algebra we show that there exist certain subspaces of $W(A)$ that can be realized as submanifolds of A. The space $W(A)$ contains certain subspaces constituting the Stiefel manifolds of framings for A. We establish several embedding results for such subspaces, where the relevant maps induce embeddings of associated Grassmann manifolds.

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1. Introduction

For a given ring A in an operator-theoretic setting, the geometry of the space of projections $P(A)$ warrants special attention. In the case, where A is a Banach algebra or more generally, a continuous inverse algebra, relative to the similarity class of a fixed projection $p \in P(A)$, it was shown in [6] that there are important subsets $V(p, A)$ of the proper partial isomorphisms $V(A)$ which constitute the Stiefel manifolds of framings (or bases) for A. Taking into account the appropriate order (and category) of differentiability, there are naturally defined principal fiber bundles $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$, where $\mathrm{Gr}(p, A)$ is the associated Grassmanian. The utility of this construction is realized when studying the differentiable structure of families of subalgebras of A (or subspaces in the case of a Banach space) as described in e.g. [6], [7], [8], [9], [10], [11], [19]. Thus the geometry of $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ affords potential applications to subjects such as operational calculus, control theory, besides the Riemann–Hilbert and elliptic transmission problems as studied in [2], [14].

In particular, when A is a C^*-algebra, the geometry of $P(A)$ has been considered by a variety of authors [1], [3], [4], [10], [15], [16], [18] (see also the bibliography of these references). But for a general Banachable algebra A, the geometric structure of the set of *pseudoregular elements* $W(A)$ appears to have received less attention (cf. [6], [10]). For continuous inverse algebras or at least for Banach algebras, one might anticipate that $W(A)$ has a (Banach) submanifold structure as a subspace of A in the usual subspace topology. Thus it is interesting to determine if a submanifold structure ever exists. We note that in [10] it was shown that for continuous inverse algebras, $W(A)$ has the structure of a rational homogeneous space so that $W(A)$ has the structure of a manifold, but there the topology is not the subspace topology, and so it is generally not

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a submanifold of A. Our part aim here is to show that there is a natural class of submanifolds of A which are subsets of $W(A)$, and contain $P(A)$ properly. But keeping in mind applications, several reasons necessitate looking more generally at the case of $W(S)$, where S is a general semigroup. Firstly, if F is a functor on the category of vector spaces and linear maps, in general it will only define a semigroup homomorphism on the algebra level which will be necessary for applications to pseudodifferential operators and control theory. Secondly, in the case of infinite dimensions, the various topological considerations often require restriction to a subsemigroup which in general, is not even a submanifold. Thirdly, regarding e.g. control theory, it is best to be very explicit with the prevailing algebraic concepts and to favor restricting to rational functions whenever possible. Finally, the basic principle amounts to seeing how the underlying theory actually functions at the semigroup level; thus the subject could be best developed as a theory pertaining to general semigroups.

An outline of the paper is as follows. Firstly, the reader will observe that prior to the Banachable assumption, preliminary results are purely algebraic. The spaces $P(S)$, $W(S)$ etc. are definable when S is simply a semigroup. When $S \subset A$, where A is a ring (possibly noncommutative) with identity, we introduce the notion of a *local rational retract* over a rationally open set that plays an instrumental role in the constructions following. This can be viewed as a primitive form of noncommutative algebraic geometry. Then on specializing to the case, where A is a continuous inverse algebra we show that there exist certain interesting subsets $W_L(S) \subset W(S)$ and $V_L(S) \subset W(S)$ that are indeed local rational retracts of S once A is granted a suitable topology. Since a subset of a Banach space is a C^r-submanifold if and only if it is a C^r-local retract (e.g. [5] or [12]), it follows that if A is a Banach algebra, and S is a submanifold of A, then $W_L(S)$ and $V_L(S)$ are in particular analytic submanifolds of A. Further results concern embeddings $\tilde{h} : S \longrightarrow T$ with S, T subsemigroups of Banachable algebras A, B, respectively, with possible induced embeddings $W(S) \longrightarrow W(T)$ restricting to the same on the Stiefel manifolds $V(p, S) \longrightarrow V(\tilde{h}(p), T)$. The latter embeddings descend to the same between the corresponding Grassmannians $\mathrm{Gr}(p, S) \longrightarrow \mathrm{Gr}(\tilde{h}(p), T)$.

2. Algebraic Preliminaries

2.1. **The space of idempotents $P(S)$ and the Grassmannian $\mathrm{Gr}(S)$.** We start with a semigroup S and let $\check{S} = S \cup \{1\}$ denote S or the semigroup obtained by adjoining an identity to S if S does not have one already. We denote by S^{op} the opposite semigroup obtained by simply reversing the multiplication order. The group of units of $\check{S}$ is denoted by $G(S)$. We let Let $P(S)$ denote the space of idempotents in S. Observe that we are somewhat influenced by operator theory, since in the semigroup literature, $P(S)$ is usually denoted by $E(S)$. Recall that the right Green's relation is $p\mathcal{R}q$ if and only if $pS = qS$ for $p, q \in S$. Let $\mathrm{Gr}(S) = P(S)/\mathcal{R}$ be the set of equivalence classes in $P(S)$ under $\mathcal{R}$. Regarding the right Green's relation $\mathcal{R}$, note that there is a natural partial order on $P(S)$, where we say that $p \prec q$ if $qp = p$, and thus "$\prec$ & $\succ$" realizes the equivalence relation $\mathcal{R}$ on $P(S)$. As the set of such equivalence classes, $\mathrm{Gr}(S)$ will be called *the Grassmannian of* S. Relative to a given topology on S, then $\mathrm{Gr}(S)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi : P(S) \longrightarrow \mathrm{Gr}(S). \tag{2.1.1}$$

Let $h : S \longrightarrow T$ be a semigroup homomorphism. Then it is straightforward to see that the diagram below is commutative:

$$\begin{array}{ccc} P(S) & \xrightarrow{P(h)} & P(T) \\ \Pi_S \downarrow & & \downarrow \Pi_T \\ \mathrm{Gr}(S) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(T). \end{array} \tag{2.1.2}$$

Of course, in the case, where S, T have topologies making h continuous, then all the maps in the diagram are continuous. Many of the results of [5] apply verbatim to the semigroup case, as the proofs only used the underlying multiplication. In what follows, where we refer to specific results of [5] in the semigroup setting, it is because the specific result and its proof require no modification to apply to the semigroup setting.

2.2. **The space of pseudoregular elements $W(S)$.**

Definition 2.2.1. We say that $u \in S$ is *pseudoregular* if there exists a $v \in S$ such that $uvu = u$ and $vuv = v$, in which case we call v a *relative inverse* (or *pseudoinverse*) for u. In general such a relative inverse is not unique. We take $W(S)$ to denote the set (or space, if S has a topology) of all pseudoregular elements of S.

Remark 2.2.1. It is useful to keep in mind in what follows that if $w, x \in S$ and $wxw = w$, then $v = xwx$ is a relative inverse for w so $v, w \in W(S)$. Thus we have $W(S) = \{w \in S : w \in wSw\}$. More generally, if also $wyw = w$, then xwy is a relative inverse for w. Clearly, all relative inverses for w are so obtained.

Remark 2.2.2. Of course, both $G(S)$ and $P(S)$ are contained in $W(S)$ and if p is in $P(S)$ then p is a relative inverse for itself. Whereas if g is in $G(S)$, then clearly g^{-1} is its unique relative inverse. More generally, we see that if v and w in $W(S)$ are mutual relative inverses, then gv and wg^{-1} are also mutual relative inverses. Thus $G(S)W(S)$ and $W(S)G(S)$ are both subsets of $W(S)$.

Remark 2.2.3. In the context of operator algebras, a "pseudoregular operator" is often called a "partial isomorphism" and various classes of these have been extensively studied by operator theorists, but relevant to our goal much less has been done. We have specifically made use of e.g. [4], [10].

If $u \in W(S)$ has a relative inverse v, then clearly $v \in W(S)$ with relative inverse u, and it is easy to see that both vu and uv belong to $P(S)$. Even though v is not uniquely determined by u alone, it is uniquely determined once u, vu and uv are all specified [6]. Actually, this fact is contained in the general picture in semigroup theory known as "the eggbox diagram" for the Green's relations.

If $p \in P(S)$, then we take $W(p,S) \subset W(S)$ to denote the subspace of all pseudoregular elements u in S having a relative inverse v satisfying $vu = p$. Likewise, $W(S,q)$ denotes the subspace of all pseudoregular elements u in S having a relative inverse v satisfying $uv = q$, so $W(S,q) = W(q, S^{op})$. Now for $p, q \in P(S)$, we set (making use of remark 2.2.1)

$$\begin{aligned} W(p,S,q) &= W(p,S) \cap W(S,q) \\ &= \{u \in qSp : \exists v \in pSq, \;\; vu = p \;\text{ and }\;\; uv = q\}. \end{aligned} \tag{2.2.1}$$

Consequently, the map Π of (2.1.1) can be shown to extend to a well-defined map $\Pi_S : W(S) \to \mathrm{Gr}(S)$ which is constant on $W(S,p)$ with value $\Pi(p)$ (see [6] Proposition 5.1). It is also useful to note here that for any $p, q, r \in P(S)$, relative inversion induces a bijection of $W(p,S,q)$ onto $W(q,S,p)$. In particular, $G(pSp) = W(p,S,p)$.

As in [6], let us denote the unique relative inverse in $W(q,S,p)$ for x in $W(p,S,q)$ by $x^{-(p,q)}$, and for $g \in G(pSp)$, set $g^{-(p,p)} = g^{-p}$. Then we have

$$W(q,S,r)\, W(p,S,q) \subset W(p,S,r), \tag{2.2.2}$$

together with $(yx)^{-(p,r)} = x^{-(p,q)} y^{-(q,r)}$. Moreover, if instead y is in $G(S)$, then on setting $r = gqg^{-1}$, we have $(yx)^{-(p,r)} = x^{-(p,q)} y^{-1}$.

2.3. **The space of proper pseudoregular elements** $V(S)$**.** Recall that two elements $x, y \in S$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(S)$ on S. For $p \in P(S)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p,S)$, where by it follows that $\mathrm{Sim}(p,S) = G(S) * p$.

Definition 2.3.1. Let $u \in W(S)$. We call u a *proper pseudoregular element* if for some $W(p,S,q)$, we have $u \in W(p,S,q)$, where p and q are similar. We take $V(S)$ to denote the space of all proper pseudoregular elements of S.

Of course, $G(S)V(S)$ and $V(S)G(S)$ are both subsets of $V(S)$. In the following we set $G(p) = G(pSp)$. If $p \in P(S)$, then we take $V(p,S)$ to denote the space of all pseudoregular elements of S having a relative inverse $v \in W(q,S,p)$ for some $q \in \mathrm{Sim}(p,S)$, so we can define $V(p,S,q) = W(p,S,q)$, for p and q similar. With reference to (2.2.1) this condition is expressed by

$$V(p,S) := \bigcup_{q \in \mathrm{Sim}(p,S)} W(p,S,q). \tag{2.3.1}$$

Notice $V(p,S) \subset V(S) \cap W(p,S)$, but equality may not hold. Clearly, we have $G(S) \cdot p \subset V(p,S)$ and just as in [6] it can be shown that equality holds if S is a ring. The image of $\mathrm{Sim}(p,S)$ under the map Π defines the space $\mathrm{Gr}(p,S)$ viewed as the Grassmannian naturally associated to $V(p,S)$. For a given unital semigroup homomorphism $h : S \longrightarrow T$, there is a restriction of (2.1.2) to a commutative diagram:

$$
\begin{array}{ccc}
V(p,S) & \xrightarrow{V(p,h)} & V(q,T) \\
\Pi_S \big\downarrow & & \big\downarrow \Pi_T \\
\mathrm{Gr}(p,S) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(q,T)
\end{array}
\tag{2.3.2}
$$

where for $p \in P(S)$, we have set $q = h(p) \in P(T)$. Observe that in the general semigroup setting, $V(p, S)$ properly contains $G(S)p$.

Lemma 2.3.1. *If $p \in P(S)$, then $V(p, S) = G(S)G(pSp)$.*

Proof. Since $G(S)p \cup G(pSp) \subset V(p, S)$, it follows that $G(S)G(pSp) \subset V(p, S)$. If $v \in V(p, S)$, then there is $w \in V(S, p)$ a proper relative inverse for v, so there is $g \in G(S)$ with $uv = gpg^{-1} = e$. Now, $ev = v$, so $v = gpg^{-1}v = g(pg^{-1}vp)$, but it is easy to see that $pg^{-1}vp$ is in $G(p)$ because both $pg^{-1}vp$ and $pwgp$ belong to pSp, and their product in either order is p. In other words, in $G(p)$ we have the relation $(pg^{-1}vp)^{-p} = pwgp$. □

If $h : S \longrightarrow T$ is a semigroup homomorphism, then we have the following collection of subsets

$$h(W(S)) \subset W(T), \quad h(P(S)) \subset P(T),$$
$$h(W(p,S,q)) \subset W(h(p),T,h(q)), \quad h(W(p,S)) \subset W(h(p),T).$$

If in addition S and T are monoids and h is unital, then

$$h(V(S)) \subset S(T), \quad h(V(p,S,q)) \subset V(h(p),T,h(q)),$$

so consequently, $h(V(p, S)) \subset V(h(p), T)$.

We say that a homomorphism $h : S \longrightarrow T$ of monoids with topology is *proper* if it is unital and $h(G(S))$ is open in $h(S) \subset T$ in its relative topology from T.

Lemma 2.3.2. *If $S \subset T$ is a proper inclusion of topological monoids, then $G(S)$ is open in S and if the retraction $r : S \to Sp$ given by $r(x) = xp$, is an open map, then $V(p, S)$ is open in Sp.*

Proof. Obviously, now $G(S)$ is open in S. By Lemma 2.3.1 we have $V(p, S) = G(S)G(p)$. If r is open, then $G(S)p$ is open in Sp. But, right multiplication by g in Sp defines a self homeomorphism of Sp whose inverse is right multiplication by g^{-p}, hence $G(S)g \subset Sp$ is open in Sp for each $g \in G(p)$. □

Example 2.3.1. Let X be any set and let S be the set of all subsets of $X \times X$. Considering such sets as relations in X, we have the composition operation making S a monoid whose identity is the diagonal. Every idempotent is transitive, but not conversely. Indeed, if x, y, z are three distinct members of X, then for $R = \{(x, y), (y, z), (x, z)\}$ we have R transitive but not idempotent in S.

Example 2.3.2. For the previous example, consider the subsemigroup T in S consisting of relations which are functions. These are often referred to as *partially defined maps of X*. Now, an idempotent is a retraction of one subset of X on another. The group of units is the permutation group of X. Suppose $r \in P(T)$ has domain U and image $Y \subset U$ and $s \in P(T)$ has domain V and image $Z \subset V$. If $f : Y \longrightarrow Z$, then $sf = f$. Also, $F = fr$ extends f to all of U, and $Fr = F = sF$. If f is a bijection and $g : Z \longrightarrow Y$ is an inverse for f, setting $G = gs$ we get $GF = r$ and $FG = s$. Clearly, $F \in W(r, T, s)$ with G as its relative inverse, and every member of $W(r, T, s)$ is so constructed. We also have $\Pi(r) = \Pi(s)$ is equivalent to $Y = Z$. Thus, $\mathrm{Gr}(T)$ can be identified with the power set of T.

Example 2.3.3. In the previous example, take X to be a topological space and U the subsemigroup of T consisting of continuous maps whose domains are open subsets of X. The idempotents are then the neighborhood retracts in X, the members of $W(r, U, s)$ are now identifiable as homeomorphisms of Y on Z, the group of units is of course the homeomorphism group of X, and $V(r, U, s)$ can be identified with the homeomorphisms of Y on Z which have extensions to global homeomorphisms of X composed with self-homeomorphisms of Y as in Lemma 2.3.1. Here, $\mathrm{Gr}(U)$ can be identified with the set of neighborhood retracts in X.

3. Rationally Open Sets and Relative Inversion

3.1. **Noncommutative Rational Functions.** Begin by fixing a commutative integral domain (with identity) of scalars, denoted by R, once and for all, and let $\mathcal{A}$ be a small full subcategory of the category of R-algebras and unital R-algebra homomorphisms. We assume that $\mathcal{A}$ is closed under finite products. For any pair of such algebras, A and B, let $A * B$ denote their amalgamated free product over R, which is therefore a pushout in the category. If X is a set of indeterminates over all algebras in $\mathcal{A}$, then we can form the usual noncommutative polynomial ring $R\langle X\rangle$ over R. The variables do not intercommute, but they all commute with members of R. By a *noncommutative polynomial over A in the indeterminates X*, we mean a member of $B = A*R\langle X\rangle$. Since the amalgamated free product is actually a pushout in the category of such algebras, we can view the members of B as expressions in the variables in X, so that elements of $A\backslash R$ do not commute with the variables in X.

The pushout property together with the universal property of the polynomial algebra make each polynomial in B a function on X^A. Thus as usual, if $f \in B$, then there are $x_1, x_2, x_3, \ldots, x_n \in X$ with $f = f(x_1, x_2, x_3, \ldots, x_n)$, and if C is an A-bialgebra, then f defines a function from C^n to C. Actually, we note that A^n itself is an algebra, with minimal central idempotents $p_1, p_2, p_3, \ldots, p_n$ having sum 1, and if we consider the subalgebra of $A^n * R\langle\{x\}\rangle$ generated by $p_1x, p_2x, p_3x, \ldots, p_nx$, then it is easy to see that it is all of $A^n*R\langle X\rangle$. But the latter is clearly isomorphic to the algebra $(A*R\langle T\rangle)^n$, where $T = \{t_1, t_2, t_3, \ldots, t_n\}$ under the obvious isomorphism fixing elements of A and sending t_k to p_kx for each $k \leq n$. Thus, for most purposes we need only consider the case of a single indeterminate. We now form the small subcategory of the category of sets and functions whose objects are all subsets of objects in $\mathcal{A}$ and whose morphisms are those generated by the restrictions of homomorphisms, restrictions of noncommutative polynomial maps, and the restrictions of inversion maps to subsets of groups of units for objects in $\mathcal{A}$. The mappings in the resulting category will be called *noncommutative rational functions* (see Remark 7.1 of [6]).

3.2. **Rationally open sets.** Observe that $\mathcal{T}_0 = \{\varnothing, A, G(A)\}$ is a topology on the set A. Let us fix a topology $\mathcal{T}_1$ containing $\mathcal{T}_0$ once and for all. For $X \subset A$, we define the topology $(\mathcal{T}_X)_1$ as the topology induced from $\mathcal{T}_1$ by the noncommutative rational functions having domain containing X. Since the composition of two such rational functions is again rational, then given $X, Y \subset A$ and $f : X \longrightarrow Y$ a rational function, the latter is continuous in $(\mathcal{T}_X)_1 \longrightarrow (\mathcal{T}_Y)_1$. The set of inclusions of maximal domains of all rational functions on A together with their $\mathcal{T}_1$-rationally induced open subsets coinduce a topology on A which we denote by $\mathcal{T}_2$.

We observe that $\mathcal{T}_1 \subset \mathcal{T}_2$. We then repeat the process with $\mathcal{T}_1$ replaced by $\mathcal{T}_2$ getting a new topology $\mathcal{T}_3$ containing $\mathcal{T}_2$. In this way, we generate a sequential tower of topologies whose union we denote by $\mathcal{B}$. Then $\mathcal{B}$ is closed under finite intersections, and so forms the basis for a topology $\mathcal{T} = \mathcal{Z}$ which we call *the noncommutative Zariski topology on* A. Every open set in $\mathcal{Z}$ is a countable union of sets in $\mathcal{B}$, and if $x \in U \in \mathcal{Z}$, then there is some $n \in \mathbb{N}$ for which there is $V \in \mathcal{T}_n$ with $x \in V \subset U$. It follows from this "finiteness" of the basis, that any rational function is continuous on its domain in this Zariski topology. For if $f(x) \in U \in \mathcal{Z}$, and a choice of $V \in \mathcal{T}_n$ for some n, so that $f(x) \in V \subset U$, then as $\mathcal{T}_{n+1}$ contains the topology induced by all the rational functions, it follows that there exists $W \in \mathcal{T}_{n+1}$ such that $x \in W$ and $f(W) \subset V \subset U$.

Accordingly, sets open in this topology $\mathcal{Z}$ will be said to be *rationally open in* A. For instance, if U is rationally open and $b \in A$, then $b+U$ is rationally open. Moreover, $G(A) \subset A$ is open and inversion is continuous on $G(A)$. If A is a continuous inverse algebra, then we can in particular take $\mathcal{T}_1$ to be the topology A already possesses, and in this case the rationally open sets are just the open sets. On the other hand, we can still take $\mathcal{T}_1 = \mathcal{T}_0$ instead, so obtaining (in general) a smaller Zariski topology on A than its continuous inverse topology.

3.3. **Rational retracts.** Now suppose that A is any (possibly noncommutative) R-algebra with identity 1. For a given $x \in A$, we write $\hat{x} = 1 - x$. In particular, for $p \in P(A)$, we have $\hat{p} = 1 - p \in P(A)$, and the map sending $x \in A$ to $1 - x$, is an affine involution of A which maps $P(A)$ to itself which is clearly a homeomorphism if A has a topology in which the involution is continuous. The preceding sections can be applied to any multiplicative subsemigroup of A; for instance, if p is in $P(A)$ and J is a two-sided ideal of A, then $p + J$ is an example that is not a subring.

Definition 3.3.1. We say that $M \subset A$ is a *local rational retract* of $Y \subset A$, if for every $x \in M$, there exists relative to Y, a rationally open set $U \subset Y$, with $x \in U$ and a rational function $\psi : U \longrightarrow M$ with $\psi(y) = y$, for each $y \in U \cap M$.

Remark 3.3.1. Take for instance the case, where A is a *Banachable algebra*, meaning that A is a topological algebra whose underlying topological vector space structure is a Banach space in which multiplication is continuous. Then in view of [5] Lemma 2.4 (see also below), the local rational retracts in A are analytic Banach (sub)manifolds.

Returning to the general case, if $x, y \in A$ and $1 + xy \in G(A)$, then so too is $1 + yx \in G(A)$, and it is straightforward to see that:

$$x(1+yx)^{-1} = (1+xy)^{-1}x \quad \text{and} \quad (1+yx)^{-1} = 1 - y(1+xy)^{-1}x. \tag{3.3.1}$$

Lemma 3.3.1. *If $p, r \in P(A)$ and $x = p - r$ with both $1 \pm x \in G(A)$, then we have $r = rp(1-x)^{-1}r$.*

Proof. Firstly, we have $\hat{p} + r = 1 - x$ and $\hat{r} + p = 1 + x$, from which it follows that

$$(\hat{r} + p)\hat{p} = \hat{r}\hat{p} = \hat{r}(\hat{p} + r).$$

Then there are the following consequences:

$$\hat{p}(\hat{p}+r)^{-1} = (\hat{r}+p)^{-1}\hat{r} \Rightarrow \hat{p}(\hat{p}+r)^{-1}r = 0,$$
$$\Rightarrow (\hat{p}+r)^{-1}r = p(\hat{p}+r)^{-1}r.$$

Finally we deduce that $r = (\hat{p}+r)p(\hat{p}+r)^{-1}r = rp(\hat{p}+r)^{-1}r$. □

Lemma 3.3.2. *If $p \prec r$ with $1 \pm (p - r) \in G(A)$, then $r \prec p$. Consequently, $p\mathcal{R}r$ and $\Pi(p) = \Pi(r)$ in $\mathrm{Gr}(A)$.*

Proof. Since $p \prec r$ implies $rp = p$, we have from (3.3.1), $r = p(1-x)^{-1}r$. It follows that $pr = r$, and hence $r \prec p$. □

Lemma 3.3.3. *Let $x, w, r \in A$ with $1 \pm x(w - r) \in G(A)$. If $xw \in P(A)$ and $xrx = x$, then $xr\mathcal{R}xw$ in $P(A)$.*

Proof. Firstly, if $x, w \in A$ with $wxw = w$, then xw and wx belong to $P(A)$. Next, if $x, w \in A$ are such that $xw \in P(A)$ and $r \in A$ satisfies $xrx = x$, then we have $xw \prec xr$. Combining these facts with Lemma 3.3.2, the result follows. □

Proposition 3.3.1. *If $x, w, r \in A$ with $wxw = w$ and r is a relative inverse for x so that $1 \pm x(w - r) \in G(A)$, then $xwx = x$. That is, w is also a relative inverse for x.*

Proof. Using Lemma 3.3.3, we have

$$x = (xr)x = (xw)(xr)x = xw(xrx) = xwx,$$

which proves (1). Statement (2) follows essentially from the definitions. □

We proceed to define the map $g : A \times A \longrightarrow A$, given by $g(x, y) = xy + \hat{x}\hat{y}$, and consider the subset

$$\mathcal{U}_p = \{x \in A : g(p, x) \in G(A)\}, \tag{3.3.2}$$

so $\mathcal{U}_p$ is then a rationally open set in A. Fixing $v_0, w_0 \in W(A)$ as mutual relative inverses, we set $p_0 = w_0v_0$ and $s_0 = v_0w_0$, and define

$$\mathcal{U}_0 = \big\{x \in A : \quad \text{both} \quad 1 \pm (x - v_0)w_0 \in G(A)\big\}. \tag{3.3.3}$$

By (3.3.1), it follows that $\mathcal{U}_0 = \{x \in A : 1 \pm w_0(x - v_0) \in G(A)\}$.

Since we will require several maps in the sequel, it is worthwhile to specify them now:

$$\begin{cases} h : \mathcal{U}_0 \longrightarrow G(A), & h(x) = 1 + (x - v_0)w_0, \\ k : \mathcal{U}_0 \longrightarrow G(A), & k(x) = 1 + w_0(x - v_0), \\ w : \mathcal{U}_0 \longrightarrow W(A), & w(x) = w_0h(x)^{-1}, \\ v : \mathcal{U}_0 \longrightarrow W(A), & v(x) = xw(x)x. \end{cases} \tag{3.3.4}$$

For simplicity, we will sometimes use the subscript notation for evaluation of functions, or drop the subscript entirely when it is understood to be simply x. Thus, we write $w(x) = w_x = w$ or $v(y) = v_y$. However, the subscript 0 here signifies evaluation at v_0 and not evaluation at $0 \in A$.

Remark 3.3.2. By (3.3.1) we have $w = w_0h^{-1} = k^{-1}w_0$ observing the relation $v(v_0) = v_0$, and $w(v_0) = w_0$. Also, we have the relations $h(x) = \hat{s}_0 + xw_0$, and $k(x) = \hat{p}_0 + w_0x$.

Lemma 3.3.4 (cf. [10], Lemma 4.1)**.** *The maps w and v satisfy the following:*

(1) *$v(x)$ and $w(x)$ are mutual relative inverses;*
(2) *$w = wxw$.*

Proof. Firstly, we see that (1) is an immediate consequence of (2), in view of our remark following Definition 2.2.1. Since $W(A)G(A)$ is contained in $W(A)$, it follows that w does indeed take values in $W(A)$. Given $wxw = w$, then by the preceding remarks, the alternate expressions $h = \hat{s}_0 + xw_0$, and $k = \hat{p}_0 + w_0x$, immediately yield

$$kw_0 = w_0xw_0 = w_0h.$$

Therefore, $w_0 = k^{-1}w_0xw_0$, and consequently we have $w_0h^{-1} = k^{-1}w_0xw_0h^{-1}$. But by these same remarks we have $k^{-1}w_0 = w = w_0h^{-1}$, and so it follows that $w = wxw$. □

In view of this result, we will set $p = wv = wx$ and $s = wv = xw$, so these two functions take values in $P(A)$ with $svp = v$ and $pws = w$. Moreover, $p(v_0) = p_0$ and $s(v_0) = s_0$, are consistent with our previous notation.

Recall the function $g(x, y) = xy + \hat{x}\hat{y}$. Observe that if $p_1, p_2 \in P(A)$, then

$$p_1g(p_1, p_2) = p_1p_2 = g(p_1, p_2)p_2.$$

So if $p_2 \in \mathcal{U}_{p_1}$, then on setting $g = g(p_1, p_2)$, it follows that $p_1 = gp_2g^{-1}$, and $p_2 = g^{-1}p_1g$. Let us now set $g = g(p_0, p) = g(p_0, wx)$, with $p \in \mathcal{U}_{w_0v_0} = \mathcal{U}_{p_0}$. We proceed to define

$$\mathcal{U}_1 = \{x \in \mathcal{U}_0 : 1 + (x - v_0)pg^{-1}w_0 \in G(A) \text{ and } p_x \in \mathcal{U}_{p_0}\}. \tag{3.3.5}$$

Next we define the map $\lambda : \mathcal{U}_1 \longrightarrow G(A)$ by $\lambda(x) = 1 + (x - v_0)pg^{-1}w_0$.

Lemma 3.3.5 (cf. [10], Lemma 4.2)**.** *We have $g^{-1}p_0g = p$ and $\lambda v_0 g = v$.*

Proof. Firstly, let us note that since $p_0 = w_0v_0$, it follows that $v_0p_0 = v_0$, and hence $v_0\hat{p}_0 = 0$. Consequently, since we also have $g = p_0p + \hat{p}_0\hat{p}$, it follows that

$$v_0g = v_0p_0g = v_0p_0p = v_0p,$$

and so $v_0(g - p) = 0$. Next, we recall from the above observation that from $g = g(p_0, p)$ we have $g^{-1}p_0g = p$, and consequently for λv_0g, the string of equalities is obtained

$$\begin{aligned}(1 + (x - v_0)pg^{-1}w_0)v_0g &= v_0g + (x - v_0)pg^{-1}p_0g = v_0g + (x - v_0)(p)^2 \\ &= v_0g + (x - v_0)p = v_0g - v_0p + xp \\ &= v_0(g - p) + xp = xp = xwx = v,\end{aligned}$$

which establishes the result. □

Lemma 3.3.6. *Let us recall the map $v : \mathcal{U}_0 \longrightarrow W(A)$ in* (3.3.4)*, as defined by $v(x) = xw(x)x$. If, as above $v_0, w_0 \in V(A)$, then $v(\mathcal{U}_1) \subset V(A)$. Moreover, if n_0 is in $G(A)$ with $s_0 = n_0^{-1}p_0n_0$,* (*where we recall $p_0 = w_0v_0$ and $s_0 = v_0w_0$*)*, and if we set $m_0 = v_0 + \hat{s}_0n_0\hat{p}_0$, then:*

(1) *m_0 belongs to $G(A)$;*
(2) *$m_0^{-1} = w_0 + \hat{p}_0n^{-1}\hat{s}_0$;*
(3) *$v_0 = m_0p_0$ and $w_0 = p_0m_0^{-1}$;*
(4) *$v = (\lambda m_0g)p$ and pointwise, $p(\lambda m_0g)^{-1}$ is a proper relative inverse for the map v.*

Proof. We have $v = \lambda v_0g$, and so

$$v = \lambda v_0g = \lambda m_0p_0g = (\lambda m_0g)(g^{-1}p_0g) = (\lambda m_0g)p,$$

together with a map $\lambda m_0g : \mathcal{U}_1 \longrightarrow G(A)$. On the other hand, there is the map $p : \mathcal{U}_1 \longrightarrow P(A)$, and thus $v(x) \in V(A)$, for all $x \in \mathcal{U}_1$, since $G(A)P(A)$ is contained in $V(A)$. □

Observe that $\mathcal{U}_1 \subset \mathcal{U}_0$ is rationally open, and since $\mathcal{U}_0$ is rationally open in A, we deduce the inclusions of rationally open sets $\mathcal{U}_1 \subset \mathcal{U}_0 \subset A$. Relative to the inner automorphic action, we also observe that $p : \mathcal{U}_1 \longrightarrow G(A) * p_0$.

Next, define the rationally open set

$$\mathcal{U}_{00} = \{x \in \mathcal{U}_0 : v(x) \in \mathcal{U}_0, \quad \text{and} \quad 1 \pm v(x)[w(v(x)) - w(x)] \in G(A)\}. \tag{3.3.6}$$

We observe that $v_0 \in \mathcal{U}_{00}$.

Proposition 3.3.2. *The rational map $v : \mathcal{U}_{00} \longrightarrow W(A)$ satisfies the relation $v(v(x)) = v(x)$, for all $x \in \mathcal{U}_{00}$.*

Proof. If $x \in \mathcal{U}_{00}$, then $v(x) \in \mathcal{U}_0$ and by Lemma 3.3.4, $w(x)$ and $v(x)$ are mutual relative inverses. By Proposition 3.3.1, since $1 \pm v(x)[w(v(x)) - w(x)]$ belong to $G(A)$, it follows that $w(v(x))$ is also a relative inverse for $v(x)$. But then we have $v(x)w(v(x))v(x) = v(x)$. On the other hand, by definition of $v(x)$, we have $v(v(x)) = v(x)w(v(x))v(x)$, so finally, $v(v(x)) = v(x)$, for any $x \in \mathcal{U}_{00}$. □

Note we cannot guarantee that $v(\mathcal{U}_{00})$ contains a rationally open neighborhood of v_0. Let us say that relative inversion is *rationally continuous at* v_0 provided it has some relative inverse w_0 so that $v(\mathcal{U}_{00})$ is a rational neighborhood of v_0 in $W(A)$. We say that A is a $\mathcal{T}_1$*-relative continuous inverse algebra* if relative inversion is rationally continuous at each point of $W(A)$. In this case, we see that $W(A)$ is a local rational retract in A. However, even when A is a Banach algebra, it is not clear that this last hypothesis applies. Calculating as if A is a Banach algebra, we see that if relative inversion is continuous at v_0, then the tangent space of $W(A)$ at v_0 is deduced to be

$$T_{v_0}W(A) = s_0Ap_0 + \hat{s}_0Ap_0 + s_0A\hat{p}_0. \tag{3.3.7}$$

Then as in [10], we obtain a local rational parametrization of $v(\mathcal{U}_{00}) \subset W(A)$ at v_0 by just translating and restricting the retraction v to $(\mathcal{U}_{00} - v_0) \cap T_{v_0}W(A)$. But this still does not make $W(A)$ a submanifold in general, unless A is a Banach algebra which is relatively continuous inverse. Clearly, the problem is that there is no control on the relative inverse. One way out is to consider the mutual relative inverse pairs in $A \times A$.

Let $\widetilde{W}(A) \subset A \times A$ be the set of all pairs (x, y) such that x and y are mutual relative inverses. The functions k, h, w, v previously defined can be thought of as functions of x, t, u on replacing v_0 by t and w_0 by u, respectively. Thus, we now let $w(x, t, u) = k(x, t, u)^{-1}u$ with

$$k(x, t, u) = 1 + u[x - t], \quad h(x, t, u) = 1 + [x - t]u,$$

and $v(x, t, u) = x[w(x, t, u)]x$.

Definition 3.3.2. Let us set

(i) $\mathcal{U}_{(t,u)} = \{x \in A : 1 \pm u[x - t] \in G(A)\}$;
(ii) $\mathcal{W}_{(t,u)} = \{(x, y) \in A \times A : x \in \mathcal{U}_{(t,u)}$ and $1 \pm x[w(x, t, u) - y] \in G(A)\}$.

Further, we define

$$\tilde{w}(x, y, t, u) = w(v(x, t, u), x, y) \quad \text{and} \quad \tilde{v}(x, y, t, u) = v(x, t, u).$$

Finally, let

$$\widetilde{\mathcal{W}}_{(t,u)} = \{(x, y) \in \mathcal{W}_{(t,u)} : 1 \pm \tilde{v}[\tilde{w} - w] \in G(A)\}, \tag{3.3.8}$$

and consider the map

$$\begin{aligned} \tilde{r} : \widetilde{\mathcal{W}}_{(t,u)} &\longrightarrow A \times A, \\ (x, y) &\mapsto (\tilde{v}(x, y, t, u), \tilde{w}(x, y, t, u)). \end{aligned} \tag{3.3.9}$$

Proposition 3.3.3. *If* $(t, u) \in \widetilde{W}(A)$ *and* $(x, y) \in \widetilde{\mathcal{W}}_{(t,u)}$*, then* $\tilde{r}(x, y) \in \widetilde{W}(A)$ *and if* $(x, y) \in \widetilde{W}(A) \cap \widetilde{\mathcal{W}}_{(t,u)}$*, then* $\tilde{r}(x, y) = (x, y)$*.*

Proof. If (x, y) is in $\widetilde{\mathcal{W}}_{(t,u)}$, then by Proposition 3.3.1, $\tilde{w}(x, y, t, u)$ is a relative inverse for $v(x, t, u) = \tilde{v}(x, y, t, u)$, because already $\tilde{w}\tilde{v}\tilde{w} = \tilde{w}$. But if in addition we have $(x, y) \in \widetilde{W}(A)$, then as $x \in \mathcal{W}_{(t,u)}$, we have $\tilde{v}(x, y, t, u) = v(x, t, u) = x$ and therefore $\tilde{w}(x, y, t, u) = w(x, x, y) = y$. □

Corollary 3.3.1. *The set of mutual relative inverse pairs, $\widetilde{W}(A)$, is a local rational retract of $A \times A$.*

Let us consider now a monoid S which is a proper submonoid of A, so $G(S)$ is open in S and $G(S)$ contains the identity of A. In order to find actual local rational retracts inside S, we need to analyze the preceding equations a little more carefully. But first, one application is quick and easy.

Proposition 3.3.4. *For any ring A and any proper submonoid S of A, the subset $P(S)$ of idempotents is a local rational retract of S. In particular, $P(A)$ is a local rational retract in A.*

Proof. The function $g(p_0, x)$ has value 1 when $x = p_0$, but maps $\mathcal{U}_{p_0}$ into $G(A)$ and if $x \in \mathcal{U}_{p_0} \cap P(A)$, then $x = g^{-1}p_0 g$. Since $G(S)$ is open in S and contains $1 = g(p_0, p_0)$, if p_0 is in $P(S)$, then by continuity of g restricted to $S \times S$ at (p_0, p_0), there is $\mathcal{U}'_{p_0}$ open in the relative topology on S resulting in the subset $g(\mathcal{U}'_{p_0} \times \mathcal{U}'_{p_0}) \subset G(S)$. We proceed to define the map $r : \mathcal{U}'_{p_0} \longrightarrow P(S)$ by

$$r(x) = [g(p_0, x)]^{-1} p_0 g(p_0, x).$$

Then r is a local rational retraction of $\mathcal{U}'_{p_0}$ onto a rationally open neighborhood of p_0 in $P(S)$. □

Suppose now that L is a fixed rationally open set containing 0 which is symmetric in the sense that $L = -L$, and such that $1 + L + L \subset G(A)$.

Definition 3.3.3.

$$W_L(S) := \{u \in W(S) : u \quad \text{has a relative inverse} \quad t \quad \text{satisfying} \quad tu - u \in L\}.$$

Definition 3.3.4.

$$V_L(S) := \{u \in S : u \quad \text{has a proper relative inverse} \quad t \quad \text{satisfying} \quad tu - u \in L\}.$$

Observe then the inclusions $P(S) \subset V_L(S) \subset W_L(S) \subset W(S)$. Next we define

$$\mathcal{U}_2 = \{x \in \mathcal{U}_1 : p(x) - x \in L\}, \tag{3.3.10}$$

so $\mathcal{U}_2$ is seen to be rationally open (but may be empty). We further define

$$\mathcal{U}_3 = \{x \in \mathcal{U}_2 : v(x) - p(x) \in L\}. \tag{3.3.11}$$

Notice that if v_0 is in $W_L(A)$ and w_0 is chosen to satisfy $v_0 - p_0 \in L$, then $v_0 \in \mathcal{U}_3$ and $\mathcal{U}_3$ is rationally open. Moreover, v on $\mathcal{U}_3$ takes its values in $W_L(A)$ because $w(x)$ is a relative inverse for $v(x)$, and $p = wv$. In fact, if v_0 is in $V_L(A)$, then we can choose w_0 so as to satisfy $v_0 - p_0 \in L$ with w_0 a proper relative inverse, and then by Lemma 3.3.6, v on $\mathcal{U}_3$ takes its values in $V_L(A)$.

Proposition 3.3.5. *If $x \in \mathcal{U}_3 \cap W_L(A)$, then $v(x) = x$. Thus $V_L(S)$ and $W_L(S)$ are local rational retracts in S if S is a proper submonoid of A.*

Proof. If $x \in \mathcal{U}_3 \cap W_L(A)$ or $x \in \mathcal{U}_3 \cap V_L(A)$, as the case may be, we can find a relative inverse u for x so that $ux - x \in L$, and this is proper if v_0 is in $V_L(A)$. Then

$$wx - ux = (wx - x) + (x - ux) \in L + L,$$

and likewise as $L = -L$, we have $ux - wx \in L + L$. Thus $1 \pm (w - u)x$ and $1 \pm x(w - u)$ all belong to $G(A)$. Hence, by Proposition 3.3.1, we find that w is also a relative inverse for x and $xwx = x$. But, $xwx = v(x)$, and consequently $v(x) = x$. In case that we have to deal with a proper submonoid S of A, as the maps h, k take values in $G(A)$ and have the value 1 when $x = v_0$, it follows that all the maps are still defined on a rationally open subset of S in the relative topology on S, hence the same argument applies to S. □

We note that even though the Zariski topology is defined with an infinite tower of inducing and coinducing constructions, if $1 \in B \in \mathcal{T}_n$, with $n \geq 3$ and $B \subset L$, then it follows that

$$\mathcal{U}_0, \mathcal{U}_p \in \mathcal{T}_2, \quad \mathcal{U}_1, \mathcal{U}_{00} \in \mathcal{T}_3, \quad \mathcal{U}_3 \in \mathcal{T}_{n+1}.$$

This means that all the open sets we need are finitely constructible with rational functions.

Definition 3.3.5. For any $X \subset W(A)$ we say $(v_1, v_2, v_3, ..., v_n)$ is a *chain* in X provided v_k has a relative inverse w_k so that both v_k and w_k belong to S for each $k \leq n$, and such that with $p_k = w_k v_k$ and $s_k = v_k w_k$ for each $k \leq n$ we have $p_{k+1} = s_k$ for each $k < n$. We can also say that $(v_1, v_2, v_3, \ldots, v_n)$ is a chain in X from p_1 to s_n, in this case.

Now, $(w_{n-1}, w_{n-2}, w_{n-3}, \ldots, w_1)$ is a chain in X from s_n to p_1. Consequently, the products $w_1 w_2 w_3 \cdots w_n$ and $v_n v_{n-1} v_{n-2} \cdots v_1$, now belong to $W(S)$ by (2.2.2).

Remark 3.3.3. Clearly, if $(v_1, v_2, v_3, \ldots, v_m)$ is a chain in X from p to e and $(v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{m+n})$ is a chain in X from e to s, then

$$(v_1, ..., v_m, v_{m+1}, \ldots, v_{m+n}),$$

is a chain in X from to p to s. Moreover, if $1 \leq k \leq l \leq n$, then $(v_k, v_{k+1}, \ldots, v_l)$ is a chain in X from p_k to s_l.

Now for each $e \in P(S)$, using $g = g(e, x)$ on $\mathcal{W}_e$ open in S and containing e, and mapping into $G(S)$, we define the rationally open set

$$\mathcal{V}_L(e) = \{x \in \mathcal{W}_e : ge - e \in L \quad \text{and} \quad eg - g^{-1}eg \in L\}. \tag{3.3.12}$$

Lemma 3.3.7. *If $p_1, s_2 \in \mathcal{V}_L(e)$ with $e \in P(A)$, then (v_1, v_2) is a chain in $V_L(A)$ from p_1 to s_2, where $v_1 = eg(e, p_1)$ and $v_2 = g(e, s_2)^{-1}e$.*

Proof. We know $w_1 = g(e, p_1)^{-1}e$ is a relative inverse for v_1 and

$$w_1 v_1 = g(e, p_1)^{-1} e g(e, p_1) = p_1,$$

so $v_1 - p_1 \in L$ as $p_1 \in \mathcal{V}_L(e)$, and thus $v_1 \in V_L(S)$. Similarly, with $s_1 = v_1 w_1 = e$, we have $w_1 - s_1 \in L$ as $s_1 \in \mathcal{V}_L(e)$, and this means w_1 also belongs to $V_L(S)$.

We also know that $w_2 = eg(e, s_2)$ is a relative inverse for v_2, so we have now $p_2 = w_2 v_2 = e = s_1$ and $s_2 = g(e, s_2)^{-1} eg(e, s_2) = v_2 w_2$, hence $v_2 - p_2 \in L$ and $w_2 - s_2 \in L$. Since $s_2 \in \mathcal{V}_L(e)$, this now forces v_2 and w_2 to belong to $V_L(S)$. $\square$

The next proposition is aimed at showing that $V_L(S)$ is a substantial part of $V(S)$. Since $V_L(S) \subset W_L(S)$, it follows that $W_L(S)$ is likewise a substantial part of $W(S)$.

Proposition 3.3.6. *If $p, s \in P(S)$ belong to the same connected component of $P(S)$, then there is a chain in $V_L(S)$ from p to s. In particular, this means that there is $v \in V(S)$ with relative inverse $w \in V(S)$ such that v and w are products of members of $V_L(S)$ with $wv = p$ and $vw = s$.*

Proof. It is clear from the above lemma and remarks that fixing $p \in P(S)$, the set of $s \in P(S)$ for which there is a chain in $V_L(S)$ from p to s, is both open and closed, contains p, and therefore contains the entire connected component of $P(S)$ containing p. $\square$

3.4. **Topological inverse algebras.** Following [17], consider the case, where A is a (continuous) *topological inverse algebra* meaning that A is a topological algebra whose group of units is open and inversion is continuous in A. In this instance it is easily seen that every rationally open set is open. Moreover, the essential properties of (2.1.1) follow straight away from the algebraic arguments used in establishing [6] Propositions 4.1, 4.2 as reasoned in [6] Remark 7.1; these properties are summarized in the following:

Proposition 3.4.1 ([6] cf. [10])**.** *The map $\Pi : P(A) \longrightarrow \mathrm{Gr}(A)$ is a surjective rational equivariant open map which admits local rational sections. For a given element $p \in P(A)$, the fiber over $\Pi(p)$ is the linear flat $p + pA\hat{p}$.*

As shown in [6] the restriction of Π defines a principal $G(p)$-bundle

$$G(p) \hookrightarrow V(p, A) \longrightarrow \mathrm{Gr}(p, A), \tag{3.4.1}$$

which is shown to be an infinite dimensional generalization of the well-known *Stiefel bundle* concept in finite dimensions. We will have more to say about this construction at a later stage.

Now, taking the topology $\mathcal{T}_1$ to be the topology on A making it a continuous inverse algebra, we can find a neighborhood basis $\mathcal{L}$ at $0 \in A$ consisting of symmetric open sets each having the property that $1 + L + L \subset G(A)$.

The following result follows from the above considerations.

Theorem 3.4.1. *Let $S \subset A$ be any proper submonoid of A. The subset $P(S) \subset S$ and for each $L \in \mathcal{L}$ the subsets $W_L(S)$ and $V_L(S)$ are all local rational retracts in S. Moreover, $\widetilde{W}(A)$ is a local rational retract in $A \times A$.*

4. Submanifolds of Pseudoregular Elements and Embeddings

4.1. **The submanifold $V_\epsilon(S)$.** Henceforth we specialize to the case that A is a Banachable algebra and consider the following two subspaces. Fix ϵ with $0 \leq \epsilon \leq 1/2$. Let B_ϵ denote the open unit ball of radius ϵ in A, so that we have the containment $1 + B_\epsilon + B_\epsilon \subset G(A)$. Again we fix S a proper submonoid of A. Next, consider the subset $W_\epsilon(S) = W_{B_\epsilon}(S) \subset W(S)$ defined by

$$W_\epsilon(A) = \{z \in W(S) : \ \ z \text{ has a relative inverse } u \text{ with } \ \|z - uz\| < \epsilon\}.$$

Likewise, we define $V_\epsilon(S) = V_{B_\epsilon}(S)$. Notice that $V_0(S) = P(S) = W_0(S)$.

For the next lemma, we say that maps f and g are C^k-conjugate provided there is a diffeomorphism H such that $Hg = fH$.

Lemma 4.1.1 (cf. [5], [12]). *Suppose that M is a C^k-Banach manifold which is modeled on a Banach space E and let $X \subseteq M$. Let ψ be a local C^k-retraction at $x \in X$, and let $p = T_x\psi$. Then p is a continuous linear retraction and ψ is locally C^k-conjugate to p at x. Thus X is a C^k-Banach submanifold if and only if X has the property that for each $x \in X$, there is an open set U in M with $x \in U$ and a C^k-map $\psi : U \to X$ such that $\psi|U \cap X$ is the identity. Moreover, ψ is an open map onto its image and $T_x\psi$ is a continuous linear retraction of T_xM onto T_xX.*

Proof. Let $V = \psi^{-1}(U \cap X)$ and $\varphi = \psi|V$. Then the map $\varphi : V \longrightarrow V \cap X$ is a C^k-retraction, $V \cap X = U \cap X$, the subset V is open in X, and $x \in V$. The result then follows by applying [5] (Lemma 2.4), where we note that the proof given there actually shows that ψ is C^k-conjugate to ψ at x. Specifically, after locally conjugating by a local chart at x, which transforms x to $0\ inE$, setting $p = T_0\psi : E \longrightarrow E$, the fact that ψ is a local retraction makes p a continuous linear retraction so that on setting $q = \hat{p}$ we can define H by $H(x) = p(\psi(x)) + q(x - \psi(x))$. Then the derivative of H at 0 is the identity map of E, so H itself is a local diffeomorphism at 0. But notice that $H\psi = p\psi = pH$ as $pq = 0$. Now any continuous linear retraction is conjugate to a coordinate projection and is therefore open, hence ψ is open as a map onto its image. □

We now arrive at our main theorem which is mostly a consequence of the results just above.

Theorem 4.1.1. *The subsets $V_\epsilon(A)$ and $W_\epsilon(A)$ together with $P(A)$, are Banach analytic submanifolds of A, and $\widetilde{W}(A)$ is a Banach analytic submanifold of $A \times A$. If $S \subset A$ is a proper submonoid of A which is a Banach submanifold of A, then $V_\epsilon(S)$ and $W_\epsilon(S)$ together with $P(S)$, are Banach analytic submanifolds of A. Moreover, for each $p \in P(S)$, $V(p, S)$ is a submanifold of A which is open in Sp.*

Proof. We deal only with the last statement since the rest is clear from the preceding results. If $p \in P(S)$, then defining the retraction $r : S \longrightarrow Sp$ by $r(x) = xp$, we see that r is the restriction of right multiplication by p on all of A, so r is a rational retraction. By Lemma 4.1.1, r is then an open map onto its image Sp. So by Lemma 2.3.2, $V(p, S)$ is open in Sp. But as Sp is the image of r, it is a submanifold of S and hence a submanifold of A. □

4.2. **Retractions and sectional Maps.** Since we have dealt so much with C^k-retractions, for $0 \leq k \leq \omega$, it is useful to consider them in slightly more detail. Suppose that M and N are C^k-Banach manifolds and we restrict our attention to subsets of such manifolds. If X and Y are such subsets of M and N, respectively, and $f : X \longrightarrow Y$, we say that f is C^k at $x \in X$ provided that there is a C^k-map $g : U \longrightarrow N$ with $x \in U$ and so that f and g agree on $U \cap X$. Then clearly id_X is C^k, and the composition of C^k-maps is again C^k. We then have a category of C^k-subsets of C^k-manifolds and C^k-maps, that is, any subset of M has a C^k structure provided by its inclusion in M as in [5]. Thus, it makes sense to speak of a C^k-diffeomorphism of one subset onto another.

Now we have been only requiring our local retraction $\psi : U \longrightarrow X$ to have the property $\psi(x) = x$ for $x \in U \cap X$. But that is because we can always restrict to $V = \psi^{-1}(U \cap X)$, which is again open, and obtain an actual retraction, $\psi : V \longrightarrow V$. For any subset X of M, the retraction $r : X \longrightarrow X$ can be viewed as a bundle over $r(X)$, and as such, the inclusion map $i : r(X) \subset X$ is a distinguished section. Thus the category of retractions and maps preserving the inclusions is equivalent to the category of bundles with distinguished section. In particular, vector bundles are in this category, the distinguished section being the zero section in each case. Obviously, a C^k-conjugation of a C^k-retraction yields the same.

Thinking in bundle terms, it is useful to consider what triviality means. Suppose that $r : Y \longrightarrow Y$, is a retraction and $X = r(Y)$. Let $\pi_1 : X \times Y \longrightarrow X$, and $\pi_2 : X \times Y \longrightarrow Y$, be coordinate projections.

Definition 4.2.1. We call $H : X \times Y \longrightarrow Y$ a *coretraction operator for* r if the following three conditions are satisfied:
(1) $rH = \pi_1$; (2) $H(r, \mathrm{id}_Y) = \mathrm{id}_Y$; (3) $H(r\pi_2, H) = \pi_2$.

For any $(x, y) \in X \times Y$, the first condition implies $rH(x, y) = x$, the second $H(r(y), y) = y$, and the third $H(r(y), H(x, y)) = y$. Equivalently, the first two conditions together imply that for any $x \in X$, and for any $y \in Y$, we have $H(x, y) \in r^{-1}(x)$. Furthermore, for each fixed $x \in X$, we see that the operator $H(x, -) : Y \longrightarrow Y$ is a retraction of Y onto $r^{-1}(x)$.

The coretraction operator is like a parallel translation operator. The third condition says that translating, and then translating back to, where you began, creates no change. A coretraction operator provides a trivialization on choosing a fixed point $x_0 \in X$. We set $F = r^{-1}(x_0)$ and define $h : Y \longrightarrow X \times F$ by $h(y) = (r(y), H(x_0, y))$. Notice the inverse of h is just the restriction of H to $X \times F$. We have the retraction $s : X \times F \longrightarrow X \times F$ given by $s(x, y) = (x, x_0)$, and the coretraction operator for this trivial retraction is K, given simply by $K((x, x_0), (x', y')) = (x, y')$.

If $r' : X' \longrightarrow X'$ is another retraction and f is a C^k-diffeomorphism of X onto X', so that $r'f = fr$, and if H' is a coretraction for r', then we can pull it back to a coretraction for r on defining $H(x, y) = f^{-1}H'(f(x), f(y))$. We therefore see that a retraction has a coretraction operator if and only if it is C^k-conjugate to a trivial retraction. For a topological vector space E and any continuous linear retraction $p \in \mathcal{L}(E)$, we define the coretraction operator by $H(x, v) = x + \hat{p}(v)$ for any $x \in p(E)$ and any $v \in E$.

Proposition 4.2.1. *If U is an open subset of the C^k-manifold M and there exists a C^k-retraction $r : U \longrightarrow U$, then for each $x \in r(U)$, there exists an open subset $V \subset M$ so that r restricted to V is a retraction with a C^k-coretraction operator.*

Proof. By Lemma 4.1.1, such a V can be chosen so that once so restricted, r is C^k-conjugate to a continuous linear retraction. □

Example 4.2.1. Recall Example 2.3.3 and take $M = X$, where M is a C^k-manifold and let D_k be the subsemigroup of partially defined C^k-maps on open subsets of M. Then $P(D_k)$ can no be identified with the C^k-submanifolds of M which are C^k- neighborhood retracts in M. Here, $V(p, D_k)$ can be identified with the set of all embeddings of $\mathrm{Im}(p)$ in M as C^k- neighborhood retracts in M. Consequently, we can view $\mathrm{Gr}(D_k)$ as the set of all such submanifolds of M. For instance in the Hilbert manifold case, following the tubular neighborhood theorem, this would include all separable manifolds modeled on Hilbert space.

Example 4.2.2. Let E be a Banach space, and let B_k be the set of all partially defined maps on $M \times E$ which are C^k-bundle maps whose domain and codomain bundles are C^k-Banach locally trivial subbundles of the trivial bundle over M with fiber E. Now the members of $P(B_k)$ can be identified with C^k-locally trivial Banach bundles over C^k-submanifolds of M whose base spaces are C^k-neighborhood retracts and where the bundles are direct summands of $M \times E$. If we take $M = E$, then identifying the tangent bundle of a submanifold of E with a subbundle of the trivial bundle in the natural way, we get the tangent functor is a homomorphism of semigroups $T : D_k \longrightarrow B_k$.

In the following we will need to consider embeddings of certain manifolds that are defined via diagram (2.1.2). We will also need to impose conditions which ensure that the horizontal maps in the diagram are initially topological embeddings. Suppose that

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow{f} & Y \end{array} \tag{4.2.1}$$

is a commutative diagram of topological spaces and continuous maps. We say the diagram is *sectional* or that the map h is *sectional over* f provided that for any $b \in E$, and any open subset U of E containing b, there exists W open in F with $h(b) \in W$ such that

$$\pi_F(W) = \pi_F(W \cap h(E)) \quad \text{and} \quad W \cap h(E) \subset h(U). \tag{4.2.2}$$

For instance, if $X \subset Y$ and E is the image of a continuous section of π_F with domain X, the horizontal maps being merely inclusions, then the diagram is sectional. Notice in this case, the horizontal maps are homeomorphisms onto their images.

Proposition 4.2.2. *Suppose the diagram* (4.2.1) *is sectional, the vertical maps are surjective,* π_F *is open and* f *is injective. Then both* $h : E \longrightarrow h(E)$ *and* $f : X \longrightarrow f(X)$ *define open maps* (*onto their images*)*.*

Proof. In the case of h, it is immediate from the definition, whereas for f, the result follows from a little routine chasing of the diagram. □

If $h : S \longrightarrow T$ is a continuous homomorphism of topological semigroups, we say that it is sectional provided that the induced diagram (2.1.2) is sectional, that is to say, if $P(h)$ is sectional over $\mathrm{Gr}(h)$. In this case, then both these maps are open onto their images, and if h is injective, then as $\mathrm{Gr}(h)$ is automatically injective when h is, it follows that these maps define topological embeddings.

4.3. **Differentiable Semigroups.** If $S \subset M$ is a semigroup whose multiplication is C^k we call S a *C^k-semigroup.* If $e \in P(S)$, then let us consider the maps $p_e, r_e, l_e : S \longrightarrow S$ defined by

$$p_e(x) = exe, \quad r_e(x) = xe, \quad l_e(x) = ex,$$

which are then C^k-retractions. Hence if S is a submanifold of M, then so too are eSe, Se and Se. Collectively, these subsets of S are subsemigroups and hence are C^k-subsemigroups of S. Notice that eSe has e as an identity. Recall that, for convenience, we denote the inverse of $x \in G(eSe)$ by x^{-e}. The following proposition is most likely well known to semigroup theorists.

Proposition 4.3.1. *If* S *is a* C^k*-semigroup and a* C^k*-Banach manifold, and if* $e \in P(S)$*, then the group of units,* $G(eSe)$ *is open in eSe, and inversion in* $G(eSe)$ *is* C^k*, thus making* $G(eSe)$ *a* C^k*-Banach Lie group.*

Proof. Since p_e is a C^k-retraction of S onto eSe, it follows that eSe is a submanifold of S. Let m denote the multiplication of S. We have that $m(x, e) = x = m(e, x)$ for $x \in eSe$. Thus adding partial derivatives, we see that $T_{(e,e)}m$ is the addition map $T_e(eSe) \times T_e(eSe) \longrightarrow T_e(eSe)$ which is then surjective. Thus, by the implicit function theorem, there is an open subset U of eSe containing e and a C^k-function f on U into eSe, satisfying $m(f(x), x) = e$. Further, there is an open subset V of eSe containing e and we have a C^k-map g from V into eSe satisfying $m(x, g(x)) = e$. So if $x \in U \cap V = W$, then $f(x)$ is a left inverse for x, and $g(x)$ is a right inverse for x. But then $f(x) = x^{-e} = g(x)$, hence $e \in W \subset G(eSe)$ and inversion in $G(eSe)$ is C^k on W. Now left (or right) multiplication by $g \in G(eSe)$ is a C^k-diffeomorphism of eSe onto itself, hence gW is open in eSe and contains g. Thus $G(eSe)$ is open in eSe, and is therefore a C^k-submanifold. Moreover, if $x \in gW$, then $x^{-e} = g^{-e}gx^{-e} = g(xg^{-e})^{-e}$, so inversion is C^k on gW. Thus $G(eSe)$ is a Banach Lie group. □

Remark 4.3.1. In fact, it is shown in [13] that if S is a C^k-semigroup and a C^k-manifold, then $P(S)$ is a submanifold of S.

In view of the preceding proposition, we will say that S is an *inverse continuous semigroup* if S is a topological semigroup with identity for which the group of units is open and for which inversion is continuous. Thus, if S is a proper submonoid of a continuous inverse algebra A, then S is an inverse continuous semigroup.

Recalling that for S a semigroup with identity, 1, we use $g * x = gxg^{-1}$ to denote the inner automorphic action of $G(S)$ on S, suppose $X \subset S$ is invariant under all inner automorphisms and $\mathcal{R}$ is a relation on X. We say that $\mathcal{R}$ is *invariant under inner automorphisms* if $x\mathcal{R}y$ implies $g * x\mathcal{R}g * y$. On the other hand, we say that $\mathcal{R}$ is continuous if U open in X implies that the set defined by $\mathcal{R}U = \{y \in X : y\mathcal{R}x \text{ for some } x \in U\}$, is again open in X.

Proposition 4.3.2. *Suppose that S is an inverse continuous semigroup and that for each $e \in P(S)$ the map $v_e : G(S) \longrightarrow P(S)$ defined by $v_e(g) = g*e$ is an open map. Then, any relation on $P(S)$ invariant under all inner automorphisms is continuous. In particular, $\Pi : P(S) \longrightarrow \mathrm{Gr}(S)$ is an open map.*

Moreover, there is a continuous action of $G(S)$ induced on $\mathrm{Gr}(S)$, which is unique with the property that Π is equivariant. Additionally, the orbit spaces $P(S)/G(S)$ and $\mathrm{Gr}(S)/G(S)$ are discrete spaces, the map $G(S) \longrightarrow \Pi(G(S)*p)$ is open onto the orbit of $\Pi(p)$, and both $P(S)$ and $\mathrm{Gr}(S)$ are discrete unions of topological homogeneous spaces of the topological group $G(S)$.

Proof. Suppose that U is open in $P(S)$ and $e \in \mathcal{R}U$. Then there is $c \in U$ with $e\mathcal{R}c$. Choose V open in $G(S)$ such that $1 \in V$ and $V * c \subset U$, using continuity of the inner automorphic action at $1 \in G(S)$. Then by hypothesis, $V * e$ is an open set in $P(S)$ containing e, but then as $\mathcal{R}$ is invariant under all inner automorphisms, $V * e \subset \mathcal{R}U$. Applying this to the case where $\mathcal{R}$ is the equivalence relation on $P(S)$ defining $\mathrm{Gr}(S)$, this implies immediately that $\Pi : P(S) \longrightarrow \mathrm{Gr}(S)$ is an open map. Since the product of open maps is again open, the inner automorphic action on $P(S)$ induces a continuous action on $\mathrm{Gr}(S)$ which is obviously the unique action making Π equivariant. Again, since the map $v_e : G(S) \longrightarrow P(S)$ is open, it follows that in particular, $G(S) * e$ is an open subset of $P(S)$, hence orbits are open and the orbit space $P(S)/G(S)$ is discrete. But since Π is open and equivariant, it then follows that the same is true for the induced action on $\mathrm{Gr}(S)$, so $\mathrm{Gr}(S)/G(S)$ is also discrete. The remainder of the proof is straightforward. □

Proposition 4.3.3. *Suppose that A is a continuous inverse algebra and $S \subset A$ is a proper submonoid of A. Then S is an inverse continuous semigroup,and for each $e \in P(S)$ the map $v_e : G(S) \longrightarrow P(S)$ defined by $v_e(g) = g * e$ is an open map and has rational (relative to A) local continuous sections through each point of $G(S)$. In particular, $P(S)$ is a local rational retract of S, each orbit $G(S) * e$ is a topological homogeneous space, and $v_e : G(S) \longrightarrow G(S) * e$ is a topologically locally trivial principal bundle. Moreover, if $e \in P(S)$, then Π restricted to $V(e, S) \cap P(S)$ is injective.*

Proof. Recall the map $g : A \times A \longrightarrow A$ defined by $g(x, y) = xy + \hat{x}\hat{y}$. It is continuous on $S \times S$ and if $e \in P(S)$, then $g(e, e) = 1$. Since $G(S)$ is open in S, as before, it follows that there is U_e open in S with the property that $g(U_e \times U_e) \subset G(S)$. But, if $x \in U_e \cap P(S)$ then $g(x, e) * e = x$. Thus $g(x, e)$

viewed as a function of $x \in U_e \cap P(S)$ is a local section for the inner automorphic action of $G(S)$ on $P(S)$. Thus, $v_e : G(S) \longrightarrow G(S) * e$ is a topologically locally trivial principal bundle. But as $g(x,e) * e = x$ for $x \in U_e \cap P(S)$, this also gives a local (A-rational) retraction of U_e on $U_e \cap P(S)$. Since $V(e,S) \subset V(e,A)$ and $P(S) \subset P(A)$, it follows that

$$V(e,S) \cap P(S) \subset V(e,A) \cap P(A).$$

By observing the obvious subscript meaning, it is seen from Lemma 7.1 and Remark 7.1 of [6] that Π_A is injective on $V(e,A) \cap P(A)$. Also, as an immediate consequence of the definition of the equivalence relation defining Π, we see that Π_S will be injective on $V(e,S) \cap P(S)$. □

In view of this result, we will in the following refer to $V(e,S) \cap P(S)$ as *the canonical section of* Π_S, as in the cases of interest, it will be the image of a continuous section. Of course, in case of $S = A$, the canonical section is analytic and in fact, rational following Remark 7.1 of [6].

Corollary 4.3.1. *If A is a Banach algebra and S is a subsemigroup of A which is a C^k-submanifold, then $P(S)$ is a C^k-submanifold of A which is a discrete union of C^k-homogeneous spaces of $G(S)$. That is, for each $p \in P(S)$, letting $G(S)_p$ denote the isotropy subgroup of the inner automorphic action we have $v_p : G(S) \longrightarrow G(S) * p$ is a C^k-locally trivial principal bundle isomorphic to $G(S) \longrightarrow G(S)/G(S)_p$ and the orbit space $P(S)/G(S)$ is discrete.*

Proof. If e is the identity of S, then S is a C^k-submanifold of eAe which is a Banach algebra with identity e. Applying the preceding proposition we have $P(S)$ is a local C^k-retract of S and is therefore a C^k-submanifold of S. If $c \in P(S)$, let L_c and R_c denote left multiplication by c and right multiplication by c, respectively. These are restrictions of linear maps on A to A, so $T_a L_p(x) = px$ and $T_a R_p(x) = xp$ for any $x \in T_a S$, and any $a \in S$.

On the other hand, if J denotes the inversion operation on $G(S)$, then as J is the restriction of the inversion operation on $G(eAe)$, we have $T_e J(x) = -x$. Since $v_p = R_p L_p J$, it follows that the image of $T_p v_p$ is just $p(T_e S)p$, which we need to be complemented in $T_p S$. Now, defining $r_p : S \longrightarrow pSp$ by $r_p(x) = pxp$ gives a retraction of S onto pSp which is the restriction of a continuous linear retraction on A, hence by Lemma 4.1.1 we know that the image of $T_p r_p$ is $T_p(pSp)$, which is therefore a closed complemented subspace of $T_p S$. But, $T_e r_p(x) = pxp$, so we conclude that $T_p(pSp) = p(T_p S)p$. We can now apply Corollary 5.6 of [5] to make the final conclusion. □

Lemma 4.3.1. *Suppose that S is a semigroup and $p \in P(S)$ and consider the set $K = \{g \in G(S) : gp = pgp\}$. Then the isotropy subgroup of the inner automorphic induced action of $G(S)$ on $Gr(S)$ at the point p is given by $G(S)_{\Pi(p)} = K \cap K^{-1}$.*

As noted previously, from [6], $\mathrm{Gr}(A)$ has an intrinsic structure as a rational manifold due to the principal bundle $V(p,A) \longrightarrow \mathrm{Gr}(p,A)$ for each $p \in P(A)$

which is analytically equivalent to that as a discrete union of the homogeneous spaces from the inner automorphic action of $G(A)$ on $\mathrm{Gr}(A)$.

Proposition 4.3.4. *Suppose S is a proper submonoid of A which is a C^k-submanifold of A such that for each $p \in P(S)$, the group $G(S)_{\Pi(p)}$ is a Banach Lie subgroup of $G(S)$. Further, suppose that the inclusion $S \subset A$ is sectional. Then $\mathrm{Gr}(S)$ is a C^k-submanifold of $\mathrm{Gr}(A)$ which is diffeomorphic to a discrete union of C^k-principal homogeneous spaces.*

Proof. By Proposition 4.3.3, if s denotes the inverse of the restriction of Π to $\Pi(P(S) \cap V(p, S))$, then it is a local section of $\Pi : P(S) \longrightarrow \mathrm{Gr}(S)$ which we call the canonical local section. But following this section with a local section of the inner automorphic action of $G(S)$ on $P(S)$ provides a local continuous section of the inner automorphic action of $G(S)$ on $\mathrm{Gr}(S)$. Thus, $\mathrm{Gr}(S)$ is the discrete union of spaces each of which is homeomorphic to a C^k-principal homogeneous space of $G(S)$. Now, as the inclusion of S in A is sectional, it follows that $\mathrm{Gr}(S)$ can be identified with a subspace of $\mathrm{Gr}(A)$, and note that $\mathrm{Gr}(A)$ is already an analytic manifold. Moreover, now the canonical section shows that the evaluation map defined by the inner automorphic action of the Banach Lie group $G(S)$ on $\mathrm{Gr}(S) \subset \mathrm{Gr}(A)$ is an open map. Thus by Corollary 5.6 of [5], it follows that $\mathrm{Gr}(S)$ is a C^k-submanifold of $\mathrm{Gr}(A)$ which is C^k-diffeomorphic to a discrete union of $G(S)$-homogeneous manifolds. □

4.4. **Induced embeddings.** Next we describe how suitable embedding of Banachable algebras induce embeddings at the level of the corresponding subspaces such as $P(A)$, $W(A)$, etc. For the remainder, we fix Banachable algebras A and B, and multiplicative subsemigroups S and T of A and B, respectively. Let $\phi : S \longrightarrow T$ be a semigroup homomorphism.

Remark 4.4.1. If $\tilde{\phi} : A \longrightarrow B$ is a continuous linear algebra homomorphism such that $\phi(S) \subset T$, then $\phi = \tilde{\phi}|S$ is an analytic homomorphism. Moreover, if $\tilde{\phi}$ is injective with image a submanifold of B, which here merely means that the image is a closed and complemented linear subspace of B, then ϕ embeds S into T.

Suppose that e_A is an identity for S and e_B is an identity for T. Given a homomorphism $\phi : S \longrightarrow T$ satisfying $\phi(e_A) = e$, then eBe is a complemented subalgebra of B and ϕ defines a unital homomorphism into $T \cap eBe$. Thus, since $S \subset e_A A e_A$, we can often restrict attention to the case where e_A is the identity of A, e_B is the identity of B, and ϕ is unital. The following lemma provides further justification.

Lemma 4.4.1. *Let $\phi : S \longrightarrow T$ be a homomorphism with $e = \phi(e_A)$. If $a, b \in S$ are similar in S, it follows that $\phi(a)$ and $\phi(b)$ are similar in B. If ϕ is injective and $G(eBe) \subset \phi(S)$ with $\phi(a)$ and $\phi(b)$ similar in $eBe + \hat{e}B\hat{e}$, then a and b are similar in S.*

Proof. Let $e = h(e_A)$. If $a, b \in S$ are similar, then there exists $g \in G(S)$ such that $ga = bg$. Thus we have $\phi(g)\phi(a) = \phi(b)\phi(g)$ in eBe and $\phi(g) \in G(eBe)$.

Since $\phi(a)$ and $\phi(b)$ are in eBe, we further have $k = \hat{e} + \phi(g) \in G(B)$. Then it follows that $k\phi(a) = \phi(b)k$, which shows that $\phi(a)$ and $\phi(b)$ are similar in B. If $\phi(a)$ and $\phi(b)$ are similar in $eBe + \hat{e}B\hat{e}$, then we can find $\tilde{g} \in G(B)$ commuting with e such that $\tilde{g}\phi(a) = \phi(b)\tilde{g}$. Since $\phi(a), \phi(b)$ both belong to eBe, it follows that on noting $e\tilde{g} \in G(eBe) \subset \phi(S)$, we can find $g \in S$ with $\phi(g) = e\tilde{g}$, and obtain $\phi(ga) = \phi(bg)$. □

In view of the preceding results we now arrive at the main results for induced embdeddings.

Proposition 4.4.1. *Suppose that $V(A) \subset S$ and let $h : S \longrightarrow T$ be an analytic sectional monoid homomorphism. Suppose that for each $p \in P(A)$, the restriction $h|P(A) \cap V(p, A)$ is an analytic embedding. Suppose also that* $\mathrm{Gr}(h)$ *is injective. Then the induced map* $\mathrm{Gr}(h) : \mathrm{Gr}(A) \longrightarrow \mathrm{Gr}(B)$ *is an analytic embedding of Banach manifolds.*

Proof. Following [6] §7 we see that $P(A) \cap V(p, A)$ is the image of an analytic section of the map $\Pi : P(A) \longrightarrow \mathrm{Gr}(A)$ defined on an open set in $\mathrm{Gr}(A)$ containing $\Pi(p) \in \mathrm{Gr}(A)$. This shows that $\mathrm{Gr}(h)$ is locally an embedding which suffices to prove the result. □

Theorem 4.4.1. *Suppose that S and T are submanifolds of A and B respectively. Let $h : S \longrightarrow T$ be an analytic proper semigroup homomorphism which is an embedding of Banach manifolds. Assume that $G(h(S))_{\Pi(p)}$ is a Banach Lie subgroup of $G(B)$ and that h is sectional. Then in the* (*commutative*) *diagram below, the horizontal maps are analytic embeddings of Banach manifolds:*

$$\begin{array}{ccc} P(S) & \xrightarrow{P(h)} & P(T) \\ \Pi_S \downarrow & & \downarrow \Pi_T \\ \mathrm{Gr}(S) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(T). \end{array} \tag{4.4.1}$$

Proof. Let T be the image of h. The hypothesis guarantees that T is a proper C^k-submonoid of B which is a C^k-submanifold. Now, we can simply apply the results of the preceding section to T and conclude that $V(h(p), T)$, $P(T)$, $W_\epsilon(T)$, $V_\epsilon(T)$ are all submanifolds of B and that $\mathrm{Gr}(T)$ is a submanifold of $\mathrm{Gr}(B)$. But, h is a C^k-diffeomorphism of S onto T which is a semigroup isomorphism. Consequently, it defines diffeomorphisms of all these subsets for S onto the corresponding subsets for T. □

Remark 4.4.2. We remark straight away that Theorem 4.4.1 also implies induced embeddings at the level of the spaces $W(S)$ and $V(S)$, etc.

Note that conversely, if $h : A \longrightarrow B$ is a linear homomorphism and induces an embedding of $G(A) \to G(B)$, then as the derivative $h'(1) = h$, it follows that $h(A)$ is a closed complemented subspace of B. Since $G(A) \subset V(A)$, it follows that if h induces an embedding $V(h) : V(A) \to V(B)$, then h itself is an embedding.

In certain applications, we may just want to commence with the Stiefel manifolds $V(p,A)$ and $V(h(p),B)$ and the analytic map $V(p,h)$ induced by h.

Proposition 4.4.2. *Suppose that* $V(p,A) \cup \mathrm{Sim}(p,A) \cup G(A) \subset S$, *and suppose that* $h : S \longrightarrow B$ *is a sectional monoid homomorphism. With respect to the commutative diagram*

$$\begin{array}{ccc} V(p,A) & \xrightarrow{V(p,h)} & V(h(p),B) \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ \mathrm{Gr}(p,A) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(h(p),B) \end{array} \tag{4.4.2}$$

if $V(p,h)$ *is an analytic embedding, then so too is* $\mathrm{Gr}(p,h)$.

Proof. This might appear now an immediate consequence of Proposition 4.4.1, but we observe that the map $\Pi_A : V(p,A) \longrightarrow \mathrm{Gr}(p,A)$ does not have canonical sections passing through each point. However, $G(A)$ acts transitively on the left of $V(p,A)$ by left multiplication, and the map $\Pi_A : V(p,A) \longrightarrow \mathrm{Gr}(p,A)$ is equivariant with respect to this action and the inner automorphic action of $G(A)$ on $\mathrm{Gr}(p,A)$, by Theorem 6.1 of [6]. It follows from the homomorphism property of h and the transitivity of the left multiplication action on $V(p,A)$, that $\mathrm{Gr}(p,h)$ is a local embedding. But since h is sectional, the map $\mathrm{Gr}(h)$ is a homeomorphism onto its image, and thus an embedding of analytic manifolds. □

Proposition 4.4.3. *Let* $\Phi : G(A) \to G(B)$ *be an analytic embedding of Banach Lie groups and* $p \in P(A)$ *and* $\tilde{p} \in P(B)$. *Suppose the map* Φ *is equivariant with respect to a homomorphism* $\phi_p : G(\Pi(p)) \longrightarrow G(\Pi(\tilde{p}))$, *and that* Φ *is transversal to* $G(B)_{\tilde{p}}$. *If the map* $g \longrightarrow \Pi_B(\Phi(g)\tilde{p})$ *is an open map onto its image in* $\mathrm{Gr}(\tilde{p},B)$ *then there exists an induced analytic embedding of Grassmannians* $\mathrm{Gr}(p,h) : \mathrm{Gr}(p,A) \to \mathrm{Gr}(\tilde{p},B)$.

Proof. We just apply Corollary 5.6 of [5]. The transversality condition insures that the isotropy subgroup is a Banach Lie subgroup of $G(A)$. □

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