Duality of Abelian Groups

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Preface

One of the seminars conducted during the "Lagniappe Semester" at Tulane University in the Summer of 2006 in the Department of Mathematics dealt with abelian groups and their duality with compact abelian groups. The goal was to provide a complete and almost self-contained approach to that portion of the Pontryagin-Van Kampen Duality Theorem that establishes the duality between (discrete) abelin groups and compact abelian groups.
The linear algebra of vector space duality served as a motiation and as an introduction to the general pattern of duality. The rudiments of abelian group theory were discussed, compact groups were introduced, characters and character groups were defined and biduals were exhibited.
The notes present the material of the seminar. The assignments completed by the participants were included with a little bit of editing.

Chapter 1
Introduction: Finite Dimensional Vector Spaces

In the first chapter we introduce the idea of duality through a look at finite dimensional vector spaces.

Definitions and Elementary Duality

We consider finite dimensional vector spaces over a field K which we may take to be $\mathbb{R}$ for the purposes of illustration. If V and W are two vector spaces, we let $\mathrm{Hom}(V, W)$ denote the set of all linear maps $V \to W$. The abbreviation Hom refers to the generic terminology of "homomorphisms" between objects like V and W. We shall assume familiarity with basic facts in linear algebra. and note that $\mathrm{Hom}(V, W)$ is itself a vector space by defining addition $(f + g)(v) = f(v) + g(v)$ and scalar multiplication $(r{\cdot}f)(v) = r{\cdot}f(v)$. If $\dim V = n$ and $\dim W = m$ then upon selecting bases of V and W we attach to each f an $m \times n$-matrix $M(f)$ in such a fashion that if we write $v = r_1{\cdot}e_1 + \cdots + r_n{\cdot}e_n$ as columns

$$M(v) = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix},$$

then the image $f(v)$ in W, written as a column is calculated as matrix product $M(f)M(v)$. The set $M(m, n; K)$ of all $m \times n$- matrices with coefficients from K is a a vector space with respect to matrix addition and scalar multiplication, and the dimension of this vector space is mn. With this notation, the function $f \mapsto M(f) : \mathrm{Hom}(V, W) \to M(m, n; K)$ is an isomorphism of vector spaces, and this shows, in particular

$$\dim \mathrm{Hom}(V, W) = (\dim V)(\dim W).$$

The field K is itself a one-dimensional vector space, so $\widehat{V} \stackrel{\text{def}}{=} \mathrm{Hom}(V, K)$ is an n-dimensional vector space, isomorphic to the space of $1 \times n$-matrices, that is rows of length n.

Definition 1.1. The vector space $\widehat{V}$ is called the *dual* of V Its members are called *linear forms* or, notably when $K = \mathbb{R}$ or $K = \mathbb{C}$, *linear functionals* or simply *functionals*.

For $v \in V$ and $\omega \in \widehat{V}$, the element $\omega(v) \in K$ is often written $\langle \omega, v \rangle$. If we must recall what vector space we refer to we shall also write $\langle \omega, v \rangle_V$. □

We notice that $\dim \widehat{V} = \dim V$. We know that two vector spaces of the same dimension are isomorphic. So $\widehat{V} \cong V$. However, there is no natural way of finding an isomorphism between V and $\widehat{V}$.

If $V = \mathbb{R}^n$, then an element $\omega \in \widehat{V}$ acts on a vector $v = (x_1, \dots, x_n)$ in the form $\langle \omega, v \rangle = a_1x_1 + \cdots + a_nx_n$ for a suitable n-tuple $(a_1, \dots, a_n)$. Solving a system of linear equations:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

now amounts to the following: Let W be the vetor space spanned by

$$\begin{aligned} \omega_1 &= (a_{11}, \dots, a_{1n}), \\ &\vdots \\ \omega_m &= (a_{m1}, \dots, a_{mn}) \end{aligned}$$

in $\widehat{V}$. We must find the space of solutions $W^\perp$ of all vectors $v \in V$ such that $\langle \omega, v \rangle = 0$ for all $\omega \in W$. (A method called the Gauss algorithm yields one way of solving this problem.)

This dual space $\widehat{V}$ captures the entire structure of V and this is seen as follows:

We can form the *bidual* or *double dual* $\widehat{\widehat{V}} = \mathrm{Hom}(\mathrm{Hom}(V, K), K)$, a vector space of dimension $n = \dim \widehat{V} = \dim V$. We find a function

$$\eta_V : V \to \widehat{\widehat{V}}, \quad \eta_V(v)(\omega) = \omega(v).$$

The defining equation for eta_V may be rewritten in the form:

$$\langle \eta_V(v), \omega \rangle_{\widehat{V}} = \langle \omega, v \rangle_V.$$

It is easy to see that η_V is a linear map, that is, it is a member of $\mathrm{Hom}(V, \widehat{\widehat{V}})$.

Definition 1.2. The naturally defined function η_V is called the *evaluation morphism* from V to the bidual $\widehat{\widehat{V}}$. □

The following is crucial for all of finite dimensional linear algebra:

Proposition 1.3. *For every finite dimensional vector space V, the evaluation morphism $\eta_V : V \to \widehat{\widehat{V}}$ is an isomorphism.*

Proof . We begin by showing that η_V is injective, that is, that its kernel $\ker \eta_V$ is zero. Now $v \in \ker \eta_V$ iff $\eta_V(v) = 0$, that is, for all $\omega \in \widehat{V}$ we have $\langle \omega, v \rangle = \eta_V(v)(\omega) = 0$. We claim that this implies that $v = 0$. Suppose that this is not the case. Then v can be completed to a basis $e_1 = v, e_2, \dots, e_n$ of V. Let W be the linear span of $e_2, \ldots, e_n$. Then every $x \in V$ is uniquely written in the form

$$x = \omega(x)\cdot v + w, \quad \omega(x) \in K, \quad w \in W.$$

It is readily checked that $\omega \in \widehat{V} = \operatorname{Hom}(V, K)$ and that $\langle \omega, v \rangle = \omega(v) = 1$. This contradicts the statement that v is annihilated by all linear forms. Thus $\ker \eta_V = \{0\}$ and this shows that η_V is injective.

Next we show that η_V is surjective, that is, that $\eta_V(V) = \widehat{\widehat{V}}$. Since η_V has been shown to be injective, $\dim \eta_V(V) = \dim V$. But we saw that $\dim \widehat{\widehat{V}} = \dim V$ as well. This implies that $\eta_V(V) = \widehat{\widehat{V}}$ as asserted. □

Two comments:

Firstly, the injectivity of η_V means that the linear forms of V separate the points, that is,

$$(\forall v \in V)\ v \neq 0 \Rightarrow (\exists \omega \in \widehat{V})\ \langle \omega, v \rangle \neq 0.$$

If we generalize the duality procedure to infinite dimensional vector spaces, which is easily done, then this remains intact with only a slight generalization, involving, however, the Axiom of Choice. So in general, if we want to show that an evaluation morphism is injective, it is easy if we have enough functions of the relevant type.

Secondly, the surjectivity was a direct consequence of the fact that V, $\widehat{V}$ and $\widehat{\widehat{V}}$ have the same dimension. This argument breaks down for infinite dimensional vector spaces and indeed in that case the evaluation morphism badly fails to be surjective.

Proposition 1.3 could be called the Duality Theorem for finite dimensional vector spaces. It allows us to completely recover V when we know $\widehat{V}$ (and therefore $\widehat{\widehat{V}}$).

Passing to the Duals Reverses Arrows

Let us consider a linear map $f\colon V \to W$. Then we can define a morphism $\widehat{f}\colon \widehat{W} \to \widehat{V}$ by

$$(\forall v \in V, \omega\widehat{W})\ \widehat{f}(\omega)(v) = \omega(f(v)), \text{ equivalently } \langle \widehat{f}(\omega), v \rangle_V = \langle \omega, f(v) \rangle_W.$$

In short, we define

$$\widehat{f}(\omega) = \omega \circ f: \quad \begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \widehat{f}(\omega)}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\ K & \xrightarrow[\text{id}]{\qquad} & K \end{array}$$

Definition 1.4. The map $\widehat{f}\colon \widehat{W} \to \widehat{V}$ is called the *adjoint morphism* of $f\colon V \to W$,but also the *dual morphism* and occasionally the *transposed morphism.*

Proposition 1.5. *The assignment* $f \mapsto \widehat{f} : \mathrm{Hom}(V,W) \to \mathrm{Hom}(\widehat{W},\widehat{V})$ *is an isomorphism of vector spaces.*

Proof . Exercise. □

Exercise E1.1. Prove Proposition 1.5.

[Hint. One way to prove this to explore what the matrix $M(f)' \stackrel{\text{def}}{=} M(\widehat{f}) \in M(n,m;K)$ is after the introduction of bases in V and W. The exercise then amounts to showing that

$$M \mapsto M' : M(m,n;K) \to M(n,m;K)$$

is an isomorphism.

There are other proofs as well; here is one:
Solution. [Patrick Vernon] We show first that the function

$$f \mapsto \widehat{f} : \mathrm{Hom}(V,W) \to \mathrm{Hom}(\widehat{W},\widehat{V})$$

is injective. Since this function is linear, it suffices to show that its kernel is zero. So let $f: V \to W$ be a morphism and assume $\widehat{f} = 0$. Then for every $\omega \in \widehat{W}$ we have $\widehat{f}(\omega) = \omega \circ f = 0$. Thus $\omega\big(f(V)\big) = \{0\}$ for all $\omega \in \widehat{W}$. By Proposition 1.3, the linear forms of W separate the points, and so $f(V) = \{0\}$ follows. Hence $f = 0$ as asserted. Therefore $f \mapsto \widehat{f}$ is injective.

Next we have to show that $f \mapsto \widehat{f}$ is surjective. Since $f \mapsto \widehat{f}$ is injective by he first part of the proof, its image, a vector subspace E of $\mathrm{Hom}(\widehat{W},\widehat{V})$, is isomorphic to $\mathrm{Hom}(V,W)$. We recall from the paragraph preceding 1.1 that $\dim \widehat{V} = \dim V$ and $\dim \widehat{W} = \dim W$. Accordingly, $\dim E = \dim \mathrm{Hom}(V,W) = \dim V \cdot \dim W = \dim W \cdot \dim V = \dim \widehat{W} \cdot \dim \widehat{V} = \dim \mathrm{Hom}(\widehat{W},\widehat{V})$. So $E = \mathrm{Hom}(\widehat{W},\widehat{V})$ and thus $f \mapsto \widehat{f}$ is surjective and therefore is an isomorphism.]

Proposition 1.6. *If* $f: U \to V$ *and* $g: V \to W$ *are linear maps, then* $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$.

Proof . Exercise. □

Exercise E1.2. Prove Proposition 1.6.

[*Solution.* [Patrick Vernon] Let $f: U \to V$, $g: V \to W$ be linear maps and $g \circ f: U \to W$ their composition. According to Definition 1.4 we obtain adjoint maps $\widehat{f}: \widehat{V} \to \widehat{U}$, $\widehat{g}: \widehat{W} \to \widehat{V}$ and $\widehat{g \circ f} : \widehat{W} \to \widehat{U}$. Now let $\omega \in \widehat{W}$. Then $(\widehat{g \circ f})(\omega) = \omega \circ (g \circ f) = (\omega \circ g) \circ f = (\widehat{g}(\omega)) \circ f = \widehat{f}\big(\widehat{g}(\omega)\big) = (\widehat{f} \circ \widehat{g})(\omega)$. So $\widehat{g \circ f} = \widehat{f} \circ \widehat{g}$.]

Exercise E1.3. Prove the following proposition.

Let $f\colon V \to W$ be a linear map. Show that the following diagram of linear maps commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad \eta_V \quad} & \widehat{\widehat{V}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f}}} \\ W & \xrightarrow[\quad \eta_W \quad]{} & \widehat{\widehat{W}}. \end{array}$$

[Hint. You are to show that $\widehat{\widehat{f}} \circ \eta_V = \eta_W \circ f$. So for each $v \in V$ you have to prove that

$$\widehat{\widehat{f}}\big(\eta_V(v)\big) = \eta_W\big(f(v)\big).$$

Recall that on both sides of the equation there are linear forms $\widehat{W}$ whose equality on each $\omega \in \widehat{W}$ has to be verified.]

The Categorical Language

If one has a class of objects such as finite dimensional vector spaces and a class of appropriate functions between them then these are called *morphisms* and the class of objects together with the class of morphisms are often referred to as a *category*. The formal definition is not needed at the moment; it suffices to know the words as a vocabulary. If one has a function F (like $\widehat{\ }$) transforming objects in one category to objects in another (or the same one), and transforming morphisms as well, then F is called a *functor* if it takes identity maps to identity maps and satisfies $F(f \circ g) = F(f) \circ F(g)$; if, however, like the hat it satisfies $F(f \circ g) = F(g) \circ F(f)$, that is if it reversses arrows, then F is called a *contravariant functor*.

Using this terminology we can say that passing to the dual within the category of finite dimensional vector spaces is a contravariant functor *implementing a self-duality* of the category due to the fact that η_V is an isomorphism for all objects V.

In the course of the seminar we want to explore how we can make a duality like this one work for abelian groups. This requires a bit of preparation.

Chapter 2
Some Rudimentary Abelian Group Theory

We study some basic abelian group theory, starting from very elementary examples.

Elementary Examples, Constructions and Results

Definition 2.1. An *abelian group* is a commutative group which we write additively most of the time, but not always. □

Abelian groups are named after the Norvegian Mathematician Niels Henrik Abel (1802–1829), one of the mathematicians having shaped modern algebra during his all too short life.

Exercise E2.1. Look up Niels Henrik Abel in Wikipedia.

Examples 2.2. (i) The group $\mathbb{Z}$ of integers under addition. This is in some sense the root example of all abelian groups, originating from counting. The set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers, the counting numbers proper, do not form a group but only a commutative semigroup. $\mathbb{Z}$ is the smallest group containing $\mathbb{N}$. The group $\mathbb{Z}$ is also called the *free cyclic group*.

(ii) The additive group $\mathbb{Q}$ of *rationals*. It has some interesting subgroups, for instance, the following: Let p be a prime number and consider the set of all rational numbers m/p^n, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. This is a subgroup we denote by $\frac{1}{p^\infty}{\cdot}Z$. Note that

$$\frac{1}{p^\infty}{\cdot}\mathbb{Z} = \mathbb{Z} \cup \frac{1}{p}\mathbb{Z} \cup \frac{\mathbf{1}}{p^2}\mathbb{Z} \cup \cdots$$

with an ascending series of free cyclic subgroups.

(iii) The additive group $\mathbb{R}$ of *reals*, that is of the real number field. We know from anaysis that the set $\mathbb{R}$ has a metric d given by $d(x, y) = |y - x|$; accordingly it has a topology for which a set $U \subseteq \mathbb{R}$ i open iff for every $u \in U$ we find an $r > 0$ such that $d(x, u) < r$ implies $x \in U$. Addition as a function $(x, y) \mapsto x+y : \mathbb{R}^2 \to R$ is continuous, as is inversion $x \mapsto -x : \mathbb{R} \to \mathbb{R}$. So $\mathbb{R}$ is what is called a *topological group*.

We summarize various construction methods of new groups from given ones.

Proposition 2.3. (i) *If A is an abelian group and B is a closed subgroup of A, then the factor group $A/B = \{a + B : a \in A\}$ is an abelian group. The homomorphism $q\colon A \to B$, $q(a) = a + B$ is surjective an has the universal property that every* *morphism* $f\colon A \to C$ *vanishing on* B *factors through* q *with a unique morphism* $f'\colon A/B \to C$ *such that* $f = f' \circ q$.

(ii) Let $\{A_j : j \in J\}$ be a family of abelian groups. Then the the set of all functions $j \mapsto a_j : J \to \bigcup_{j\in J} A_j$ such that $a_j \in A_j$, denoted by $(a_j)_{j\in J}$, form a group $\prod_{j\in J} A_j$ under pointwise addition $(a_j)_{j\in J} + (b_j)_{j\in J} = (a_j + b_j)_{j\in J}$; it is called the *cartesian product* or the *direct product.* of the family.

(iii) The set of all $a = (a_j)_{j\in J} \in \prod_{j\in J} A_j$ for which there is a finite subset $F_a \subseteq J$ such that $a_j = 0$ for $j \in J \setminus F_a$ is a subgroup $\bigoplus_{j\in J} A_j$.

Proof . The assertions are easily verified directly. □

Definition 2.4. For a famliy of abelian groups A_j, the group $\prod_{j\in J} A_j$ is called the *cartesian product* or the *direct product.* of the family. The subgroup $\bigoplus_{j\in J} A_j$ is called the *direct sum* of the family. If the family $\{A_j : j \in J\}$ is constant, that is $A_j = A$ for all j we write $A^J = \prod_{j\in J} A_j$ and $A^{(J)} = \bigoplus_{j\in J} A_j$. □

Now we have further basic examples of abelian groups.

Example 2.5. (i) Let $n \in \mathbb{N}$. The set $n\mathbb{Z}$ of multiples of n is a subgroup of $\mathbb{Z}$, isomorphic to $\mathbb{Z}$ under the map $m \mapsto nm : \mathbb{Z} \to n\mathbb{Z}$. The factor group $\mathbb{Z}/n\mathbb{Z}$ of *integers modulo* n is a finite group also called "the" *cyclic group of order* n.

(ii) For a prime number p, define $\mathbb{Z}(p^\infty) = \frac{1}{p^\infty}\cdot\mathbb{Z}/\mathbb{Z}$. Such a group is called a *Prüfer group.*

(iii) The group $\mathbb{R}/\mathbb{Z}$ of *reals modulo one* is written $\mathbb{T}$ and called the *additive circle group.* □

Note that Example 2.5(iii) has a topology which consists of all images $q(U)$ under the quotient morphism $q\colon \mathbb{R} \to \mathbb{T}$ of open sets U of $\mathbb{R}$ which satisfy $U+\mathbb{Z} = U$.

Exercise E2.2. Prove that we have really defined a topology on $\mathbb{T}$, that is, that the collection of open sets of $\mathbb{T}$ is closed under the formation of arbitrary unions and finite intersections. Show that $q\colon \mathbb{R} \to \mathbb{T}$ is both a continuous and an open function (that is, maps open sets to open sets). Show that addition $(s,t) \mapsto s+t : \mathbb{T}\times\mathbb{T} \to \mathbb{T}$ is continuous and that inversion $t \mapsto -t : \mathbb{T} \to \mathbb{T}$ is continuous. Prove that $\mathbb{T}$ is compact.

[*Solution.* [Jonathan Meddaugh] As a first order of business, we prove that we really have defined a topology on $\mathbb{T}$.

Let U be an open set in $\mathbb{R}$, define $V \subset \mathbb{R}$ to be $U + \mathbb{Z} = q^{-1}\big(q(U)\big)$ and call V the *the saturation of* U. Since translations on $\mathbb{R}$ are homeomorphisms, each of $U + k$ is open for $k \in \mathbb{Z}$. Since $V = \bigcup_{k\in\mathbb{Z}} U + k$, it follows that the saturation V of any open set U is open as well. Moreover, $q(U) = q(V)$.

Now we show that the collection $\{q(U) : U+\mathbb{Z} = U \quad \text{open in } \mathbb{R}\}$ is a topology. Let A be an index set and U_α an open set in $\mathbb{T}$ for each $\alpha \in A$. Then, for each

$\alpha \in A$ there exists $V_\alpha \subset \mathbb{R}$ such that $V_\alpha = V_\alpha + \mathbb{Z}$ and $q(V_\alpha) = U_\alpha$. Then $\bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\in A} q(V_\alpha) = q(\bigcup_{\alpha\in A} V_\alpha)$. Immediately, we have $\bigcup_{\alpha\in A} V_\alpha$ is open.

Also, $\bigcup_{\alpha\in A} V_\alpha + \mathbb{Z} = \bigcup_{\alpha\in A} V_\alpha$; fo a proof let $x \in \bigcup_{\alpha\in A} V_\alpha + \mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that $x + k \in \bigcup_{\alpha\in A} V_\alpha$, and so there exists $\alpha \in A$ with $x + k \in V_\alpha$. But $V_\alpha = V_\alpha + \mathbb{Z}$, so that $x \in V_\alpha$, and so $x \in \bigcup_{\alpha\in A} V_\alpha$. Thus $\bigcup_{\alpha\in A} V_\alpha + \mathbb{Z} \subset \bigcup_{\alpha\in A} V_\alpha$, and it is immediate that the other containment holds, so that we have equality. Thus $\bigcup_{\alpha\in A} U_\alpha \in \{q(U) : U + \mathbb{Z} = U \quad \text{open in } \mathbb{R}\}$.

Similarly, since $V_\alpha = q^{-1}(U_\alpha)$, we have $\bigcap_{\alpha\in A} U_\alpha = q(\bigcap_{\alpha\in A} V_\alpha)$; so that

$$\bigcap_{\alpha\in A} U_\alpha \in \{q(U) : U + \mathbb{Z} = U \text{ open in } \mathbb{R}\}$$

as well, and this applies, in particular, to finite indes sts A. Thus we have a topology on $\mathbb{T}$. We note in passing that functions do not preserve intersections in general: If $f \colon \{0,1\} \to \{0\}$ is the constant function, then $f(\{0\}) \cap f(\{1\}) = \{0\} \neq \emptyset = f(\emptyset) = f(\{0\} \cap \{1\})$.

Continuity of q: If U is open in $\mathbb{T}$, then there is a saturated open st V of $\mathbb{R}$ such that $U = q(V)$; it follows that $V = q^{-1}(U)$ and so q is open.

Openness of q: Let W be open in $\mathbb{R}$. Then the saturation $V = W + \mathbb{Z}$ is open in $\mathbb{R}$ by the first paragraph of the proof, and $q(W) = q(V)$ is open in $\mathbb{T}$ by the definition of the topology of $\mathbb{T}$.

The function q is not closed: Let $n \mapsto r_n$ be a bijection of $\mathbb{N}$ onto the set of rational numbers in $]0,1[$; then $C = \{n - r_n : n \in \mathbb{N}\}$ is closed in $\mathbb{R}$ while $q(C) = q(\mathbb{Q} \cap]0,1[)$ is dense but not closed in $\mathbb{T}$.

A quotient map $q \colon X \to X/R$ of a space modulo an equivalence relation R need not be open.

Example: $X = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1]) \subseteq \mathbb{R}^2$ with the induced topology, R having the equivalence classes $\{(r,0)\}$ for $r \neq 0$ and $\{0\} \times [0,1]$. Now let $V = \{0\} \times]0,1]$, an open subset of X. But $q(V) = q(V \cup \{(0,0)\})$ is not open. Note that q is a closed map.

For a proof of the continuity of addition we observe that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathbb{R} \times \mathbb{R} & \xrightarrow{\quad \text{id} \quad} & \mathbb{R}^2 & \xrightarrow{\quad \text{add} \quad} & \mathbb{R} \\
{\scriptstyle q\times q}\downarrow & & \downarrow{\scriptstyle (x,y)\mapsto(x,y)+\mathbb{Z}^2} & & \downarrow{\scriptstyle q} \\
\mathbb{T} \times \mathbb{T} & \xrightarrow[(x+\mathbb{Z},y+\mathbb{Z})\mapsto(x,y)+\mathbb{Z}^2]{} & \mathbb{R}^2/\mathbb{Z}^2 & \xrightarrow[(x,y)+\mathbb{Z}^2\mapsto x+y+\mathbb{Z}^2]{} & \mathbb{T},
\end{array}$$

where $\text{add} \colon \mathbb{R}^2 \to \mathbb{R}$ is defined by $\text{add}(x,y) = x + y$. The maps q, add, and id are continuous, and $q \times q$ is an open map (since q is), so that for W open in $\mathbb{T}$, we have $\text{add}^{-1}(W) = q \times q(id^{-1} \circ \text{add}^{-1} \circ q^{-1}(W))$ open in $\mathbb{T} \times \mathbb{T}$. The proof of the continuity of $x \mapsto -x$ is similar and much simpler.

For proof of the compactness of $\mathbb{T}$, we observe that $\mathbb{T} = q([0,1])$ and that $[0,1]$ is compact.

Alternative Solution [Gail Blaustein]. We prove that $\mathbb{T}$ is a compact topological group by showing that it is algebraically isomorphic and topologically homeomorphic to the multiplicative group $\mathbb{S}^1$ of complex numbers $z \in \mathbb{C}$ of absolute value $|z| = 1$; since complex multiplication and inversion on $\mathbb{C} \setminus \{0\}$ is continuous, and since $\mathbb{S}^1$ is a closed and bounded, hence compact subset of $\mathbb{C}$ we know right away that $\mathbb{S}^1$ is a compact abelian topological group.

We prove the asserted isomorphism by accepting as established the following fact from algebra:

Proposition CD. (Canonical Decomposition, First Isomorphism Theorem) *A continuous homomorphism $f\colon G \to H$ between groups with kernel $N = \ker f$ decomposes canonically in the form*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle j} \\ G/N & \xrightarrow[\quad f' \quad]{} & f(G), \end{array}$$

where $q\colon G \to N$ is the quotient morphism given by $q(g) = gN$, $j\colon f(G) \to H$ is the inclusion morphism given by $j(h) = h$, and $f'\colon G/N \to f(G)$ is the isomorphism of groups given unambiguously by $f'(gN) = f(g)$. □

For a proof of our claim we define $\varepsilon\colon \mathbb{R} \to \mathbb{S}^1$ by $\varepsilon(x) = \exp(2\pi i x)$. By the functional equation of the exponential function, ε is a homomorphism of groups, and it is known to be continuous. We have $x \in \ker\varepsilon$ iff $\varepsilon(x) = 1$ iff $x \in \mathbb{Z}$ whence $\ker\varepsilon = \mathbb{Z}$. The image of ε is $\varepsilon(\mathbb{R}) = \mathbb{S}^1$. If we define the quotient map $p\colon \mathbb{R} \to \mathbb{T} = \mathbb{R}/Z$ by $p(x) = x + \mathbb{Z}$ then by the First Isomorphism Theorem we get an algebraic isomorphism $\varepsilon'\colon \mathbb{T} \to \mathbb{S}^1$ such that he following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad \varepsilon \quad} & \mathbb{S}^1 \\ {\scriptstyle p}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\ \mathbb{T} & \xrightarrow[\quad \varepsilon' \quad]{} & \mathbb{S}^1 \end{array}$$

with the identity function id. From complex function theory we know that there is a continuous (in fact analytic) function $z \mapsto \frac{1}{2\pi}\log z : \mathbb{S}^1 \setminus \{-1\} \to]-\frac{1}{2}, \frac{1}{2}[$ such that $\varepsilon(\frac{1}{2\pi}\log z) = z$. We may conclude that ε is in fact also open.

Therefore a set W is open in $\mathbb{S}^1$ iff $U \stackrel{\text{def}}{=} \varepsilon^{-1}(W)$ is open in $\mathbb{R}$. We note that $U = U + \ker\varepsilon = U + \mathbb{Z}$. Now $q(U) = {\varepsilon'}^{-1}(W)$. Thus the algebraic isomorphism ε' maps the open sets of $\mathbb{S}^1$ bijectively to the members of the set $\{q(U) : U$ is open in $\mathbb{R}$ and satisfies $U = U + \mathbb{Z}\}$, so this set is a topology on $\mathbb{T}$ such that $\varepsilon\colon \mathbb{T} \to \mathbb{S}^1$ is a homeomorphism. As a consequence, $q = {\varepsilon'}^{-1} \circ \varepsilon$ is continuous and open.]

It is clear that we need more formal definitions on topological groups and compact toplogical groups which we will supply soon.

Free Abelian Groups

Definition 2.6. An abelian group A is called *free* if there is a set X such that $A \cong \mathbb{Z}^{(X)}$. We also say that A is *free over* X or *free on* X. □

For each $x \in X$ we define an element $e_x \in \mathbb{Z}^{(X)}$, $e_x = (\delta_{xy})_{y \in X}$, where

$$\delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $\mathbb{Z}^{(X)}$, every element $(n_x)_{x \in X}$ can be correctly written in the form $\sum_{x \in X} n_x \cdot e_x$ since all but finitely many of the n_x ar zero. The function $x \mapsto e_x : X \to \mathbb{Z}^{(X)}$ is injective; in this spirit it is fair to consider X as a subset of an abelian group that is free over X.

Proposition 2.7. *Let $f: X \to B$ be a function from a set to an abelian group B. Then there is a unique homomorphism $F: \mathbb{Z}^{(X)} \to B$ such that $F(e_x) = f(x)$. In other words, if A is a free group over $X \subseteq A$, then any function from X to an abelian group B extends to a unique homomorphism $A \to B$.*

Proof . Exercise. □

Exercise E2.3. Prove Propopsition 2.7.
[Hint. Try $F((n_x)_{x \in X}) = \sum_{x \in X} n_x \cdot f(x)$.]

Corollary 2.8. *Every abelian group is the quotient group of a free abelian group.*

Proof . Exercise. □

Exercise E2.4. Prove Corollary 2.8.
[Hint. Let A be an abelian group and denote by $|A|$ the underlying set. For the set X try $X = |A|$.]

The *Axiom of Choice* may be expressed as follows:

(AC) *For each surjective function $e: A \to B$ there is a function $s: B \to A$ such that $e \circ s = \mathrm{id}_B$, the identity function of B.*

A function like s is called a *cross section* for e. Use this axiom to prove the following:

Proposition 2.9. *For an abelian group P, the following statements are equivalent:*

(i) For each surjective morphism $e: A \to B$ and each morphism $f: P \to B$ there is a morphism $F: P \to A$ such that $f = e \circ F$.
(ii) P is free.

Proof . Exercise. □

Exercise E2.5. Prove Proposition 2.9.

[Hint. (ii)⇒(i): (Easy) Assume that P is free over a subset $X \subseteq P$, define $j: X \to P$ to be the inclusion function ($j(x) = x$), and let $s: B \to A$ be a cross section for e. Extend $s \circ f \circ j: X \to A$ according to Proposition 2.7.

(i)⇒(ii): (Hard) Assume P satisfies property (i). By 2.8 there is a set X and a surjective morphism $e: \mathbb{Z}^{(X)} \to P$. By (i) applied to $f = \mathrm{id}_P$, there is a morphism $F: P \to \mathbb{Z}^{(X)}$ such that $e \circ F = \mathrm{id}_P$. Then $\mathbb{Z}^{(X)} = \ker e \oplus F(P)$ where $F(P) \cong P$. Find a reference for a more difficult theorem saying that a subgroup of a free group is free or devise a simpler proof for the more special fact that a direct summand of a free group is free.]

Abelian groups P satisfying the condition 2.9(i) are called *projective.* This is a condition expressed in terms of morphisms. Frenes is a structural condition. Proposition 2.9 then expresses the fact that an abelian group is free iff it is projective. The fact that a subgroup of a free group is free is remarkably sophisticated, requiring the Well-Ordering Theorem.

It is important that we know the fundamental theorem on the subgroups of finitely generated free groups.

Theorem 2.10. (The Elementary Divisor Theorem) (A) *Let F be a free abelian group over a finite set and G a nonzero subgroup. Then there exist, firstly, a free generating set $X = \{x_1, \ldots, x_n\}$ of F, secondly a natural number $1 \le d \le n$, and thirdly natural numbers $m_1|m_2| \cdots |m_d$ such that $\{m_1{\cdot}x_1, \ldots, m_d{\cdot}x_d\}$ is a free generating set of G.*

(B) If P is a subgroup of G such that F/P has no elements of finite order, then there is a nonnegative integer c, $0 \le c \le d$ such that, in the case that $0 < c$, the elements $x_1, \ldots, x_c$ may be chosen to be a free generating set of P and that $m_1, \ldots, m_c = 1$.

Proof. We consider the set $\mathcal{X}$ of all free generating sets X of F. For $0 \ne g \in G$ and $X \in \mathcal{X}$ we write $g = \sum_{x \in X} n_x{\cdot}x$ in a unique fashion and set $n(g, X) = \min\{|n_x| \mid x \in X,\ 0 \ne n_x\}$. We define

$$\mathbb{N}(G, F) = \{n(g, X) \in \mathbb{N} \mid (g, X) \in G \times \mathcal{X}\}.$$

Then $m = \min \mathbb{N}(G, F)$ is a positive natural number and we find a pair $(g, X) \in G \times \mathcal{X}$ such that $m = n(g, X)$, and since $n(g, X) = n(-g, X)$ we may assume $g = m{\cdot}x + \sum_{y \in X \setminus \{x\}} n_y{\cdot}y$. We claim that $m|n_y$. Proof of the claim: We find integers q_y and r_y such that $n_y = q_y m + r_y$ with $0 \le r_y < m$. Setting $x' = x + \sum_{y \in X \setminus \{x\}} q_y{\cdot}y$ we have $Y = \{x'\} \cup (X \setminus \{x\}) \in \mathcal{X}$, and $g = m{\cdot}x' + \sum_{y \in X \setminus \{x\}} r_y{\cdot}y$. By the minimality of m we conclude $r_y = 0$ for all $y \in X \setminus \{x\}$. This proves the claim. We note that $m{\cdot}x' = g$. The projection $p_{x'}: F \to \mathbb{Z}$, defined by $f = p_{x'}(f){\cdot}x' + \sum_{y \in X \setminus \{x\}} n_y{\cdot}y$, maps G onto a subgroup of $\mathbb{Z}$ with minimal element m, i.e. on $\mathbb{Z}{\cdot}m$. Thus, for $h \in G$ we compute $p_{x'}(h - \dfrac{p_{x'}(h)}{m}{\cdot}g) = 0$ and thus we find $h - \frac{p_{x'}(h)}{m}\cdot g \in G \cap \langle X \setminus \{x\}\rangle$. Therefore, setting $F' = \langle X \setminus \{x\}\rangle$ and $G' = G \cap \langle X \setminus \{x\}\rangle$ we get

$$F = \mathbb{Z}\cdot x' \oplus F',$$
$$G = \mathbb{Z}\cdot g \oplus G'.$$

Moreover, if X' is any free generating set of F' and $g' = \sum_{y'\in X'} n_{y'}\cdot y' \in G'$, then $m|n_{y'}$ for all $n_{y'}$ by the definition of m as we see by the same proof as for the claim above.

The proof of the theorem now follows by induction on $\mathrm{rank}(G)$. □

The (uniquely determined) natural numbers $m_1, \ldots, m_d$ are called the *elementary divisors* of G in F.

Theorem 2.11. (The Fundamental Theorem of Finitely Generated Abelian Groups) *Let A be a finitely generated abelian group.*

(i) Then there is a unique sequence of natural numbers $1 < m_1|m_2\cdots|m_d$ and a natural number m_0 such that

$$A \cong \mathbb{Z}(m_1) \oplus \cdots \oplus \mathbb{Z}(m_d) \oplus \mathbb{Z}^{m_0}. \tag{1}$$

(ii) For each prime p there is a subgroup

$$A_p \cong \mathbb{Z}(p)^{k(p,1)} \oplus \mathbb{Z}(p^2)^{k(p,2)} \oplus \mathbb{Z}(p^3)^{k(p,3)} \oplus \cdots \tag{2}$$

such that

$$A = \mathbb{Z}^{m_0} \oplus \bigoplus A_p \tag{3}$$
$$= \mathbb{Z}^{m_0} \oplus \bigoplus_{\substack{p \text{ prime} \\ n\in\mathbb{N}}} \mathbb{Z}(p^n)^{k(p,n)}, \tag{4}$$

where $k(p,n) = 0$ for all but finitely many pairs (p,n).

Proof. (i) Since A is finitely genrated and by the universal property of free abelian groups 2.6 there is a finite set X and a quotient morphism $q\colon F(X) \to A$. Let $G = \ker q$. By the Elementary Divisor Theorem 2.10 there is an ordered sequence of natural numbers $1, \ldots, 1, m_1, \ldots, m_d$, $1 < m_1|\cdots|m_d$ and a free generating set

$$\{e_1, \ldots, e_m, x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+m_0}\}$$

such that

$$\{e_1, \ldots, e_m, m_1\cdot x_1, \ldots, m_d\cdot x_d\}$$

is a free generating set of G. Then

$$A \cong \frac{\mathbb{Z}^{\{e_1,\ldots,e_m,x_1,\ldots,x_d,x_{d+1},x_{d+m_0}\}}}{\mathbb{Z}^{\{e_1,\ldots,e_m,x_1,\ldots,x_d\}}} \cong \mathbb{Z}(m_1) \oplus \cdots \oplus \mathbb{Z}(m_d) \oplus \mathbb{Z}^{m_0}.$$

(ii) For each natural number $m = p_1^{n_1} \cdots p_k^{n_k}$ (p_j prime) we have

$$\mathbb{Z}(m) = \mathbb{Z}(p_1^{n_1}) \oplus \cdots \oplus \mathbb{Z}(p_k^{n_k}). \tag{5}$$

(Exercise.) All we need to do now is to substitute (5) into (1) and rearrange direct summands to get (4). Finally, (2) and (3) are just a consequence of regrouping (4). □

Exercise E2.6. Prove the assertion yielding (5) above.

Divisible Abelian Groups

There is counterpart of free groups, the so called divisible groups. Just in which way they are counterparts we shall see presently.

Definition 2.12. An abelian group A is called *divisible* if for each $a \in A$ and each natural number n there is an $x \in A$ such that $n{\cdot}x = a$. □

Examples of divisible groups are $\mathbb{Q}$ and $\mathbb{R}$.

Exercise E2.7. Prove the following facts.

(i) *Every homomorphic image of a divisible group is divisible.*

(ii) *The circle group* $\mathbb{T}$ *is divisible.*

(iii) For any $g \in \mathbb{Z}/n\mathbb{Z}$ and any $m \in \mathbb{N}$ that is relatively prime to n, there is an $x \in \mathbb{Z}/n\mathbb{Z}$ such that $m{\cdot}x = g$.

(iv) *The Prüfer group* $\mathbb{Z}(p^\infty)$ *is divisible.*

(v) *Direct products and direct sums of divisible groups are divisible.*

[Solutions of (iii) and (iv) [Jonathan Meddaugh, Patrick Vernon]: (iii) Since m and n are erlatively prime, there are integers a and b such that $am + bn = 1$. Therefore $g = 1{\cdot}g = am.g + bn.g = m.(a.g)$ and so $m{\cdot}x = g$ for $x = a{\cdot}g$.

(iv) Let n be a given natual number and $g \in \mathbb{Z}(p^\infty)$; we must show that there is an $x \in \mathbb{Z}(p^\infty)$ such that $n{\cdot}x = g$. Given n, we find a natural number m which is relatively prime to p such that $n = p^k m$ with a nonnegative inter k. We claim

(a) Every element of the Prüfer group is divisible by p^k,

(b) Every element of the Prüfer group is divisible by m.

Assume that claims (a) and (b) are established and let $g \in \mathbb{Z}(p^\infty)$. Then by (a) there is a $h \in \mathbb{Z}(p^\infty)$ such that $p^k{\cdot}h = g$. Now by (b) there is an $x \in \mathbb{Z}(p^\infty)$ such that $m{\cdot}x = h$. Thus $n{\cdot}x = p^k{\cdot}(m{\cdot}x) = p^k{\cdot}h = g$.

It remains to show (a) and (b).

Proof of (a): Let $g \in Z(p^\infty) = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$. Then there is an integer r and a nonnegative integer s such that $g = (r/p^s) + \mathbb{Z}$. Set $h = (r/p^{s+k}) + \mathbb{Z} \in \mathbb{Z}(p^\infty)$; then $p^k{\cdot}h = (r/p^s) + \mathbb{Z} = g$.

Proof of (b): $h \in Z(p^\infty) = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$. Then there is a nonnegative integer u such that $h \in (1/p^u){\cdot}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}(u)$. Then by (iii) there is an $x \in (1/p^u){\cdot}\mathbb{Z}/\mathbb{Z}$ such that $m{\cdot}x = h$.]

In the proof of (2) ⇒(1) of the preceding Proposition 2.14 we implicitly proved the following fact:

In Corollary 2.8 we proved that every abelian group is a quotient of a free group. Now we shall prove

Proposition 2.13. *Every abelian group can be isomorphically embedded into a divisible group.*

Proof . Exercise. □

Exercise E2.8. Prove Proposition 2.13.

[Hint. Find a set X and a surjective homomorphism $p: \mathbb{Z}^{(X)} \to G$. (Reference?) Thus may assume that $G = \mathbb{Z}^{(X)}/K$ with $K = \ker p$. Now $K \subseteq \mathbb{Z}^{(X)} \subseteq \mathbb{Q}^{(X)}$. Then $G = \mathbb{Z}^{(X)}/K \subseteq \mathbb{Q}^{(X)}/K$, and $D = \mathbb{Q}^{(X)}/K$ is divisible.]

The crucial property of a divisible group I is that

for every subgroup A of an abelian group B and a homomorphism $f: A \to I$ there is a homomorphic extension $F: B \to I$ of f,

as we shall argue now.

Definition 2.14. An abelian group I is called *injective* if for every injective morphism $i: A \to B$ and every morphism $j: A \to I$ there is a morphism $f: B \to I$ with $j = f \circ i$.

$$\begin{array}{ccc} I & \xrightarrow{\quad \mathrm{id}_I \quad} & I \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[\quad i \quad]{} & B \end{array}$$

□

One may rephrase injectivity in the following convenient fashion: *An abelian group I is injective if and only if any homomorphism $j: A \to I$ of a subgroup A of a group B extends to a homomorphism $f: B \to I$ on the whole group.*

Proposition 2.15. (AC) *For an abelian group G the following conditions are equivalent:*

(1) G is divisible.
(2) G is injective.

Proof . (1)⇒(2). (AC) Assume that A is a subgroup of B and that a homomorphism $j: A \to G$ is given. We must extend j to a morphism $f: B \to G$. We consider the set of all morphisms $\varphi: C \to G$ with $A \subseteq C \subseteq B$ and $\varphi|A = j$. This set is partially ordered by inclusion of domains and extension of mappings (i.e. $\varphi \le \varphi'$ if $C \subseteq C'$ and $\varphi'|C = \varphi$). One verifies quickly that this set is inductive, hence by Zorn's Lemma contains a maximal element $\mu: M \to G$. We must show $M = B$. Let $b \in B$.

We would like to produce an extension $\mu'\colon M + \mathbb{Z}{\cdot}b \to G$ of μ; the maximality of μ will then show $b \in M$. We consider the morphism $\alpha\colon M \times \mathbb{Z} \to M + \mathbb{Z}{\cdot}b \subseteq B$ defined by $\alpha(m,k) = m - k{\cdot}b$. Then $M + \mathbb{Z}{\cdot}b = \operatorname{im}\alpha \cong (M \times \mathbb{Z})/\ker\alpha$, where $\ker\alpha = \{(m,k) : m = k{\cdot}b\}$. Let n be the nonnegative generator of the subgroup $\{k \in \mathbb{Z} : k{\cdot}b \in M\}$ of $\mathbb{Z}$. Since G is divisible, there is an element $d \in G$ such that $n{\cdot}d = \mu(n{\cdot}b)$. Now we define $\beta\colon M \times \mathbb{Z} \to G$ by $\beta(m,k) = \mu(m) - k{\cdot}d$. We claim that β factors through α, that is, there is a $\mu'\colon M + \mathbb{Z}{\cdot}b \to G$ such that $\beta = \mu' \circ \alpha$; then $\mu'(m) = \mu(m)$ for $m \in M$ and μ' is the desired extension of μ.

Now the factorisation of β exists iff $\ker\alpha \subseteq \ker\beta$, and we claim that this is the case. For a proof of this claim let $(m,k) \in \ker\alpha$, that is, $m = k{\cdot}b \in M \cap \mathbb{Z}{\cdot}b$, and so by the definition of n, there is a $z \in \mathbb{Z}$ such that $k = zn$. Then $\beta(m,k) = \mu(m) - k{\cdot}d = \mu(m) - z{\cdot}(n{\cdot}d) = \mu(m) - z{\cdot}\mu(n{\cdot}b) = \mu(m - zn{\cdot}b) = \mu(m - k{\cdot}b) = \mu(0) = 0$. Hence $(m,k) \in \ker\beta$ and the claim is established, and the proof of $M = B$ is complete.

(2)⇒(1). By 2.13 there is a divisible group D with $G \subseteq D$. Since G is injective there is a morphism $f\colon D \to G$ such that $f|G = \mathrm{id}_G$. Hence G is a homomorphic image of a divisible group and is, therefore, divisible. □

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\ =\ } & \mathbb{T} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \chi} \\ S & \xrightarrow[\text{incl}]{} & A \end{array}$$

For vector spaces we saw that for every K-vector space V, the morphisms of $\mathrm{Hom}(V,K)$ separate the points of V. One might wish to compare this with the following fact on abelian groups.

Lemma 2.16. *Let A be an arbitrary abelian group. Let D be a divisible abelian group containing $\mathbb{Q}/\mathbb{Z}$. Then the morphisms of* $\mathrm{Hom}(A,D)$ *separate the points of A.*

Proof . Assume that $0 \neq a \in A$. We must find a morphism $\chi\colon A \to D$ such that $\chi(a) \neq 0$. Let S be the cyclic subgroup $\mathbb{Z}{\cdot}a$ of A generated by a. If S is infinite, then S is free and for any nonzero element t in $\mathbb{Q}/\mathbb{Z}$ (e.g. $t = \frac{1}{2} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$) and so by Proposition 2.7, there is an $f\colon S \to D$ with $f(a) = t \neq 0$. If S has order n, then S is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$, and thus there is an injection $f\colon S \to D$. If we let $\chi\colon A \to D$ be an extension of f which exists by the injectivity of $\mathbb{T}$, then $\chi(a) = f(a) \neq 0$. □

This applies, in particular to $D = \mathbb{Q}/\mathbb{Z}$ and $D = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. The latter case will play and important role in the next section.

Exercise E2.9. Prove the following fact:

Let D be a divisible subgroup of an abelian group A. Then D is a direct summand of A, that is, $A = D + B$ and $D \cap B = \{0\}$ for a subgroup B of A.

Chapter 3
Compact Abelian Groups and Characters

Some of the basic facts do not at all depend on the commutativity of the group operations, and we therefore formulate them for arbitrary multiplicatively written compact groups.

Definition 3.1. A *compact group* G is a compact Hausdorff space whose underlying set has a group structure such that the function

$$(*) \qquad (x,y) \mapsto xy^{-1}: G \times G \to G$$

is continuous. □

For the concept of compactness and Hausdorff separation of a space see the set of Lecture Notes "Introductction to Topology." Definition 3.1 is a special case of the definition of a *topological group* , which is a topologcial space and a group such that $(*)$ is continuous.

Our principal source of reference for compact groups is

[1] Hofmann, K. H., and S. A. Morris, The Structure of Compact Groups, de Gruyter Verlag, Berlin, 1998, xvii + 834pp.
Second Completely Revised, Corrected and Augmented Edition 2006, xviii + 860pp. To appear July 2006 at de Gruyter Verlag, Berlin.

A copy of [1] is placed on the reserave shelf in the A. H. Clifford Mathematical Research Library.

Exercise E3.1. (i) Let G be a group and a topological space Show that the following conditions are equivalent:
(1) The function $(x,y) \mapsto xy^{-1} : G \times G \to G$ is continuous.
(2) Multiplication and inversion are continuous functions.

Here we recall that multiplication is the function $(x,y) \mapsto xy : G \times G \to G$.

Examples and Constructions of Compact Groups

Exercise E3.2. Prove the following assertion:

Let G be a compact group and H a subgroup. Then the closure $\overline{H}$ of H in G is a compact group with respect to the induced topology.

[Solution. [Gail Blaustein and Patrick Vernon] Let $f: G \times G \to G$ be defined by $f(x,y) = xy^{-1}$. Since G is a compact group, f is continuous. Since continuous functions map closures into closures, we have $f(\overline{H \times H}) \subseteq \overline{f(H \times H)} = \overline{HH^{-1}} = \overline{H}$, where in the last equality we have used that H is a subgroup. Now we claim that $\overline{H} \times \overline{H} \subseteq \overline{H \times H}$. For a proof let $(x, y) \in \overline{H} \times \overline{H}$, that is $x, y \in \overline{H}$. Then for each open subsets V_1 and V_2 open in G such that $x \in V_1$ and $y \in V_2$ we have $(V_1 \times V_2) \cap (H \times H) \neq \emptyset$. Thus there are elements $h_1, h_2 \in H$ such that $h_1 \in V_1$ and $h_2 \in V_2$. Thus $(V_1 \times V_2) \cap (H \times H) \neq \emptyset$, and so $(x, y) \in \overline{H \times H}$. Now $\overline{HH}^{-1} = f(\overline{H} \times \overline{H}) \subseteq f(\overline{H \times H}) \subseteq \overline{H}$. Thus $\overline{H}$ is a subgroup.

Since G is a compact group, f is continuous. Then, for every subgroup S of S, the restriction of f to $S \times S$ is also continuous. Furthermore, since $\overline{H}$ is a closed subspace of a compact space, $\overline{H}$ is compact.]

Examples 3.2. (i) All finite groups with the discrete topology are compact groups.

(ii) The multiplicative groups

$$\begin{aligned} \mathbb{S}^0 &= \{r \in \mathbb{R} : |r| = 1\}, \\ \mathbb{S}^1 &= \{z \in \mathbb{C} : |z| = 1\}, \\ \mathbb{S}^3 &= \{q \in \mathbb{H} : |q| = 1\} \end{aligned}$$

are compact groups on the unit spheres of the fields of real and complex numbers and of the skew field of quaternions.

The skew field of *quaternions* is isomorphict to a real 4-dimensional subalgebra of the algebra of 2×2 complex matrices

$$M(u, v) \stackrel{\text{def}}{=} \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \in M_2(\mathbb{C}), \quad u, v \in \mathbb{C}.$$

Using the basis $1, i, j, k$ of $\mathbb{H}$, we find the isomorphism via

$$r \cdot 1 + x \cdot i + y \cdot j + z \cdot k \mapsto M(r + x \cdot i, y + z \cdot i).$$

Accordingly, $\mathbb{S}^3 \cong \mathrm{SU}(2) \stackrel{\text{def}}{=} \{M(u, v) : u\overline{u} + v\overline{v} = 1\}$.

If G is a compact group on an n-sphere $\mathbb{S}^n$, then $n = 0, 1, 3$. But his is a nontrivial result: see [1] 9.59(iv).

(iii) Each of the groups $\mathrm{O}(n)$ of $n \times n$-orthogonal matrices forms a closed and bounded subset in the vector space $M_n(\mathbb{R})$ of all $n \times n$ real matrices and therefore is a compact group, since matric multiplication, being polynomial in each coefficient, is continuous and inversion agrees with transposition and is, therefore, continuous. By a similar argument, each of the groups $\mathrm{U}(n)$ of $n \times n$-unitary matrices is a compact group; alternatively, one may identify $\mathrm{U}(n)$ with a closed subgroup of $\mathrm{O}(2n)$.

(iv) Every *closed subgroup* of a compact group is a compact group.

(v) If $\{G_j : j \in J\}$ is an arbitrary family of compact groups, then their cartesian product $G \stackrel{\text{def}}{=} \prod_{j \in J} G_j$ with componentwise multiplication and the product topology is a compact group. The compactness of G is a consequence of the *Tychonoff product theorem*; the continuity of $(x, y) \mapsto xy^{-1} : G \times G \to G$ follows from the natural homeomorphy of $G \times G$ and $\prod_{j\in J} G_j \times G_j$ and the continuity of

$$(x_j, y_j) \mapsto x_j y_j^{-1} : G_j \times G_j \to G_j$$

for all $j \in J$.

(va) Let $p \in \mathbb{N}$. Define $f\colon \mathbb{Z} \to P \stackrel{\text{def}}{=} \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ by $f(x) = (x+p^n\mathbb{Z})_{n\in\mathbb{N}}$. Then $\mathbb{Z}_p \stackrel{\text{def}}{=} \overline{f(\mathbb{Z})}$ is a compact abelian group. If $\mathrm{pr}_n\colon P \to \mathbb{Z}/p^n\mathbb{Z}$ denotes the projection given by $\mathrm{pr}_n((x_m)_{m\in\mathbb{N}}) = x_n$, then $f_n \stackrel{\text{def}}{=} \mathrm{pr}_n |\mathbb{Z}_p$ is a morphism $\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$. It is an exercise to show that it is surjective. Thus $\ker f_n$ is a subgroup I_n such that $\mathbb{Z}_p/I_n \cong \mathbb{Z}/p^n\mathbb{Z}$. All morphisms in sight also preserve multiplication, so $\mathbb{Z}_p$ is a ring and the I_n are ideals, and I_n turns out to agree exactly with $p^n\mathbb{Z}_p$. One calls $\mathbb{Z}_p$ the *ring of p-adic integers*, and its elements are called *p-adic integers.*

(vb) Every product $\prod_{j\in J} \mathrm{O}(n_j)$ or any product $\prod_{j\in J} \mathrm{U}(n_j)$ of a family of orthogonal, respectively, unitary groups is a compact group, as is any closed subgroup of these. □

It is remarkable that *every compact group is isomorphic as a topological group to a closed subgroup of one of the groups exhibited in* Example 1.2(vb) but we shall not have the time to discuss this this summer. (For a complete proof, see [1], Chapter 2, Corollary 2.29.)

Exercise E3.3. Verify the details of the propositions that a product of an arbitrary family $\{G_j : j \in J\}$ of compact groups is a compact group.

[Solution. [Patrick Vernon] let $G = \Pi_{j\in J} G_j$ be the product of these groups (under the product topology). Let $\pi_j\colon G \to G_j$ be the standard projection map. By the Tychonoff product theorem, G is compact. We let $f\colon G \times G \to G$ be the function defined by $f(x, y) = xy^{-1}$, and let $f_j\colon G_j \times G_j \to G_j$ be the function defined by $f_j(x, y) = xy^{-1}$. We claim that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\quad f \quad} & G \\ {\scriptstyle \pi_j\times\pi_j}\big\downarrow & & \big\downarrow{\scriptstyle \pi_j} \\ G_j \times G_j & \xrightarrow[\quad f_j \quad]{} & G_j \end{array}$$

is commutative. For a proof let $(x, y) \in G \times G$. Then $\pi_j \circ f(x, y) = \pi_j(xy^{-1}) = \pi_j(x)\pi_j(y)^{-1} = f_j(\pi_j(x), \pi_j(y)) = f_j \circ (\pi_j \times \pi_j)(x, y)$. So $\pi_j \circ f = f_j \circ (\pi_j \times \pi_j)$.

Then $\pi_j \circ f$ is continuous for each $j \in J$, so f is continuousby the universal property of the product topology. Thus G is a compact group.]

The following exercise generalizes our earlier one on the circle group $\mathbb{T}$. (See E2.2.)

Exercise E3.4. Prove the following assertion:

If G is a compact group and N a closed normal subgroup, then G/N is a compact group with respect to the quotient topology.

[Solution. [Gail Blaustein, Jonathan Meddaugh, and Patrick Vernon] We know that a subset $V \in G/N$ is open iff $q^{-1}V$ is open in G where $q\colon G \to G/N$ is the quotient map given by $q(g) = gN = Ng$. We consider the functions $f\colon G\times G \to G$ given by $f(x, y) = xy^{-1}$ and $m\colon G/N \times G/N \to G/N$ given by $m(\xi, \eta) = \xi\eta^{-1}$. We claim that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\quad f \quad} & G \\ {\scriptstyle q\times q}\downarrow & & \downarrow{\scriptstyle q} \\ \frac{G}{N} \times \frac{G}{N} & \xrightarrow[\quad m \quad]{} & \frac{G}{N} \end{array}$$

is commutative: Indeed let $x, y \in G$ and set $a = q(x)$ and $b = q(y)$. Then

$$m \circ (q \times q)(x, y) = m(a, b) = ab^{-1} = q(x)q(y)^{-1} = q(xy^{-1}) = q \circ f(x, y),$$

so $m \circ (q \times q) = q \circ f$. Now we show that m is continuous. For a proof, let W be open in G/N. Since f and q are continuous, we know that $m \circ (q \times q))^{-1}(W) = (q \circ f)^{-1}(W)$ is open in G. But since q is an open map, it follows that $(q\times q)$ is also an open map. Furthermore, for any **surjective** map $\varphi\colon X \to Y$, if $M \subset Y$, we have $\varphi\circ\varphi^{-1}(M) = M$. Thus $(q\times q)\circ(m\circ(q\times q))^{-1}(W) = ((q\times q)\circ(q\times q))^{-1})\circ m^{-1}(W) = m^{-1}(W)$ is open in $G/N \times G/N$. Thus m is continuous as claimed and so G/N is a topological group.]

Exercise E3.5. (p-adic integers). Let L denote the subset of all sequences $(x_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \in P \stackrel{\text{def}}{=} \prod_{n\in N} \mathbb{Z}/p^n\mathbb{Z}$, $x_n \in \mathbb{Z}$ such that $x_{n+1} \in x_n + p^n\mathbb{Z}$. Show that L is a compact subring of P and that it contains the subring $\mathbb{Z}' \stackrel{\text{def}}{=} \{(x+p^n\mathbb{Z})_{n\in\mathbb{N}} \in L : x \in \mathbb{Z}\}$. Prove that $\mathbb{Z}' \cong \mathbb{Z}$ and that every open subset of L contains an element of $\mathbb{Z}'$, that is, $\mathbb{Z}'$ is dense in L and $L = \overline{\mathbb{Z}'}$. Conclude that $L = \mathbb{Z}_p$.

Applications to Abelian Groups

An important example arises out of the preceding proposition. For two sets X and Y the set of all functions $f\colon X \to Y$ will be denoted by Y^X.

Definition 3.3. If A is an abelian group (which we now continue to write additively), then the group

$$\mathrm{Hom}(A, \mathbb{T}) \subseteq \mathbb{T}^A$$

of all morphisms of abelian groups into the underlying abelian group of the circle group (no continuity involved!) given the induced group structure and topology of the product group $\mathbb{T}^A$ (that is, pointwise operations and the topology of pointwise convergence) is called *the character group of* A and is written $\widehat{A}$. Its elements are called *characters* of A. □

Proposition 3.4. *The character group $\widehat{A}$ of any abelian group A is a compact abelian group.*

Proof . By Exercise 3.2(v), the product $\mathbb{T}^A$ is a compact abelian group. For any pair $(a, b) \in A \times A$ the set $M(a, b) = \{\chi \in \mathbb{T}^A \mid \chi(a + b) = \chi(a) + \chi(b)\}$ is closed since $\chi \mapsto \chi(c): \mathbb{T}^A \to \mathbb{T}$ is continuous by the definition of the product topology. But then $\widehat{A} = \bigcap_{(a,b)\in A\times A} M(a,b)$ is closed in $\mathbb{T}^A$ and therefore compact. □

Exercise E3.6. Formulate an alternative proof of showing that $M(a,b)$ is closed by verifying that its complement in $\mathbb{T}^A$ is open.

Let us look at a few examples! We recall $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$, the cyclic group of n elements.

Examples 3.5. The following examples are elementary:

$$\widehat{\mathbb{Z}} \cong \mathbb{T}, \tag{1}$$

$$\widehat{\mathbb{Z}(n)} \cong \mathbb{Z}(n). \tag{2}$$

Proof . Exercise. □

Exercise E3.7. Verify the details of 3.5(1,2).

[Solution. [Gail Blaustein] (1) We define a function $f \mapsto f(1): \mathrm{Hom}(\mathbb{Z},\mathbb{T}) \to \mathbb{T}$ by $f(\chi) = \chi(1)$ and a function $g: \mathbb{T} \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$ by $g(t)(n) = n{\cdot}t$. We show (a) that f is a morphisms of topological groups, and that (b) f and g are inverses of each other. Regarding (a), for any $\chi_1, \chi_2 \in \widehat{\mathbb{Z}}$ we have $f(\chi_1+\chi_2) = (\chi_1 + \chi_2)(1) = \chi_1(1) + \chi_2(1) = f(\chi_1) + f(\chi_2)$. This says that f is a homomorphism. Since $\mathrm{Hom}(\mathbb{Z},\mathbb{T})$ has the topology of pointwise convergence, that is, the topology induced by the inclusion $\mathrm{Hom}(\mathbb{Z},\mathbb{T}) \subseteq \mathbb{T}^{\mathbb{Z}}$, the function f is evaluation at 1, that is the projection $\mathrm{pr}_1: \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}$ restricted to $\mathrm{Hom}(\mathbb{T},\mathbb{Z})$ and is, therefore continuous. Thus f is a morphism of compact groups.

Now (b): First we show $g \circ f = \mathrm{id}_{\mathrm{Hom}(\mathbb{Z},\mathbb{T})}$: Let $\chi \in \widehat{\mathbb{Z}}$. Then $g\big(f(\chi)\big)(n) = n{\cdot}f(\chi) = n{\cdot}\chi(1) = \chi(n)$, whence $(g\circ f)(\chi) = \chi$ and this implies the claim. Next we show that $f \circ g = \mathrm{id}_{\mathbb{T}}$. So ;et $t \in \mathbb{T}$. then $f\big(g(t)\big) = g(t)(1) = 1{\cdot}t = t$ which proves the claim. Thus g is an inverse function of f and is therefore algebraically an isomorphism. Since for all $n \in \mathbb{Z}$ the function $t \mapsto g(t)(n) = n{\cdot}t : \mathbb{T} \to \mathbb{T}$ is continuous, the function $g: \mathbb{T} \to \mathrm{Hom}(\mathbb{Z},\mathbb{T})$ is continuous by the definition of the topology of pointwise convergence. Thus g is an isomorphism of topological groups as is f and assertion (1) is proved.

(2) Every morphism $\mathbb{Z}(n) \to \mathbb{T}$ maps the cyclic group of order n into the unique cyclic subgroup $\frac{1}{n}{\cdot}\mathbb{Z}/\mathbb{Z}$ of $\mathbb{T}$. Therefore the inclusion $i: \frac{1}{n}{\cdot}\mathbb{Z}/\mathbb{Z} \to \mathbb{T}$ induces an isomorphism $\mathrm{Hom}(Z(n), i) : \mathrm{Hom}(\mathbb{Z}(n), \frac{1}{n}{\cdot}\mathbb{Z}/\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}(n),\mathbb{T}) = \widehat{\mathbb{Z}(n)}$. The isomorphism $z + n\mathbb{Z} \mapsto \frac{1}{n}z + \mathbb{Z} : \mathbb{Z}(n) \to \frac{1}{n}{\cdot}\mathbb{Z}/\mathbb{Z}$ gives us an isomorphism $\mathrm{Hom}\big(\mathbb{Z}(n),\mathbb{Z}(n)\big) \to \mathrm{Hom}(\mathbb{Z}(n), \frac{1}{n}{\cdot}\mathbb{Z}/\mathbb{Z})$. Thus $\widehat{\mathbb{Z}}(n)$ is isomorphic to $\mathrm{Hom}\big(\mathbb{Z}(n),\mathbb{Z}(n)\big)$. The group $\mathbb{Z}(n)$ is a ring and a $\mathbb{Z}(n)$-module. For each element $v \in \mathbb{Z}(n)$, the map $z \mapsto z{\cdot}v$ is a morphism $\mathbb{Z}(n) \to \mathbb{Z}(n)$, and if $\varphi: \mathbb{Z}(n) \to \mathbb{Z}(n)$ is a morphism, then we set $v = \varphi(1 + n\mathbb{Z})$. Then $z{\cdot}v = z{\cdot}\varphi(1) = \varphi(z)$. Therefore $\varphi \mapsto \varphi(1 + n\mathbb{Z})$ is an isomorphism $\mathrm{Hom}\big(\mathbb{Z}(n),\mathbb{Z}(n)\big) \to \mathbb{Z}(N)$. All isomorphism properly superposed yield an isomorphism $\widehat{\mathbb{Z}(n)} \to \mathbb{Z}(n)$. We have shown (2).]

In order to get more involved examples we consider is a set X , and a family $\{A_x \mid x \in X\}$ of abelian groups. We denote by $\bigoplus_{x\in X} A_x$ the direct sum of the A_x, that is, the subgroup of the cartesian product $\prod_{x\in X} A_x$ consisting of all elements $(a_x)_{x\in X}$ with $a_x = 0$ for all x outside some finite subset of X. A special case is the free abelian group $\mathbb{Z}^{(X)} = \bigoplus_{x\in X} A_x$ with $A_x = \mathbb{Z}$ for all $x \in X$.

Proposition 3.6. *Let $\{A_x : x \in X\}$ be a family of topological abelian groups and T a topological abelian group. The function*

$$\Phi\colon \prod_{x\in X} \mathrm{Hom}(A_x, T) \to \mathrm{Hom}(\bigoplus_{x\in X} A_x, T)$$

which associates with a family $(f_x)_{x\in X}$ of morphisms $f_x\colon A_x \to T$ the morphism

$$(a_x)_{x\in X} \mapsto \sum_{x\in X} f_x(a_x)\colon \bigoplus_{x\in X} A_x \to T$$

is an isomorphism of compact groups. Notably,

$$(\bigoplus_{x\in X} A_x)\widehat{\ } \cong \prod_{x\in X} \widehat{A_x}. \tag{3}$$

In particular

$$\mathbb{Z}^{(X)}\widehat{\ } \cong \widehat{\mathbb{Z}}^X \cong \mathbb{T}^X. \tag{4}$$

Proof. Abbreviate $\bigoplus_{x\in X} A_x$ by A and $\prod_{x\in X} \mathrm{Hom}(A_x, T)$ by P. For $y \in X$, we define where $\mathrm{copr}_y\colon A_y \to A$ to be the natural inclusion given by

$$\mathrm{copr}_y(a_y) = (a_{xy})_{x\in X}, \quad a_{xy} = \begin{cases} a_y & \text{if } x = y, \\ = 0 & \text{otherwise.} \end{cases}$$

We notice that Φ is well defined, since the $f_x(a_x)$ vanish with only finitely many exceptions for $(a_x)_{x\in X}$. Clearly Φ is a morphism of abelian groups.

We define $\Psi\colon \mathrm{Hom}(A, T) \to P$ as follows: If $f\colon A \to T$ is a morphism, set $\Psi(f) = (f \circ \mathrm{copr}_x)_{x\in X}$. If one identifies A_x with the subgroup $\mathrm{copr}_x(A_x)$, then $\Psi(f) = (f|A_x)_{x\in X}$.

Again, Ψ is a morphism of abelian groups. If $(f_x)_{x\in X} \in P$ then

$$\Phi((f_x)_{x\in X})|A_y = f_y \text{ and so } \Psi\big(\Phi((f_x)_{x\in X})\big) = (f_x)_{x\in X}.$$

Conversely, if $f \in \mathrm{Hom}(A, T)$ then $\Phi\big(\Psi(f)\big)(a_x)_{x\in X} = \Phi\big((f|A_x)_{x\in X}\big)(a_x)_{x\in X} = \sum_{x\in X} f(a_x) = f\big((a_x)_{x\in X}$ and so Φ and Ψ are inverses of each other.

It remains to show that both Φ and Ψ are continuous.

First we show that Φ is continuous. By the definition of the topology on $\mathrm{Hom}(A, T) \subseteq T^A$, it suffices to show that for each $(a_x)_{x\in X} \in A$, the function $(f_x)_{x\in X} \mapsto \Phi\big((f_x)_{x\in X}\big)\big((a_x)_{x\in X}\big) = \sum_{x\in X} f_x(a_x) : \mathrm{Hom}(A_x, T)^X \to T$ is continuous. Since only finitely many a_x are nonzero, this is the case if $(f_x)_{x\in X} \mapsto f_y(a_y)$ is continuous for each fixed y, and this holds if $f_y \mapsto f_y(a_y)\colon \mathrm{Hom}(A_y, T) \to T$ is continuous. However, by definition of the topology of pointwise convergence, this is indeed the case.

Next we prove the continuity of Ψ. Now Ψ is continuous iff for each $x \in X$ the function $f \mapsto f \circ \text{copr}_x : \text{Hom}(A,T) \to \text{Hom}(A_x,T)$ is continuous which, once more is continuous iff for each $a_x \in A_x$, the function $f \mapsto f(a_x) : \text{Hom}(A,T) \to T$ is continuous, where we have again identified A_x with the subgroup $\text{copr}(A_x)$ of A. But this continuity is secured by the definition of pointwise convergence on $\text{Hom}(A,T)$. This shows that Ψ is continuous.

We have completed the proof that Φ is n isomorphism of topological groups.

The last assertion of the proposition is a special case which we obtain by taking $T = \mathbb{T}$. This remark concludes the proof of the proposition. □

We shall use only the cae that T is a compact group. Then, since the domain of Φ is compact by the theorem of Tychonoff and the range is Hausdorff, this suffices for Φ to be a homeomorphism. Thus in this case we could pass up the proof of the continuity of Ψ.

The compact abelian groups $\mathbb{T}^X$ are called *torus groups.* The finite dimensional tori $\mathbb{T}^n$ are special cases.

The Fundamental Theorem of Finitely Generated Abelian Groups and (1), (2) and (3) above imply the following result:

Proposition 3.7. *If E is a finite abelian group, then $\widehat{E}$ is isomorphic to E (although not necessarily in any natural fashion!). If F is a finitely generated abelian group of rank n, that is, $F = E \oplus \mathbb{Z}^n$ with a finite abelian group E, then $\widehat{F} \cong \widehat{E} \times \mathbb{T}^n$.* □

In particular, the character groups of finitely generated abelian groups are compact manifolds. (We shall not make any use of this fact right now.)

There are examples of compact abelian groups whose topological nature is quite different.

Example 3.8. Let $\{G_j \mid j \in J\}$ be any family of finite discrete nonsingleton groups. Then $G = \prod_{j\in J} G_j$ is a compact group. All connected components are singletons, and G is discrete if and only if J is finite. □

A topological space in which all connected components are singletons is called *totally disconnected.* Arbitrary products of totally disconnected spaces are totally disconnected, and all discrete spaces are totally disconnected. The standard Cantor middle third set C is a compact metric totally disconnected space. In fact it may be realized as the set of all real numbers r in the closed unit interval, whose expansion $r = \sum_{n=1}^{\infty} a_n 3^{-n}$ with respect to base 3 has all coefficients a_n in the set $\{0,2\}$. Then the map $f\colon \{-1,1\}^{\mathbb{N}} \to C$ given by $f\big((r_n)_{n\in\mathbb{N}}\big) = \sum_{n=1}^{\infty}(r_n+1)3^{-n}$ is a homeomorphism. The set $\mathbb{S}^0 = \{-1,1\}$ is a finite group, and thus, by Exercise 1.2(v), the domain of f is a compact group.

Hence the Cantor set can be given the structure of a compact abelian group. In this group, every element has order 2, so that in fact, algebraically, it is a vector space over the field GF(2) of 2 elements, and by (2) and (3) above, it is the character group of $\mathbb{Z}(2)^{(\mathbb{N})}$.

One can show that all compact metric totally disconnected spaces without isolated points are homeomorphic to C. In particular, all metric compact totally disconnected infinite groups are homeomorphic to C.

Aiming for Duality

We have defined characters of (discrete) abelian groups, and we concluded that the character group of an abelian group had a natural compact group topology. Now we turn around and define characters of compact abelian groups.

Definitions 3.9. For a compact abelian group G a morphism of compact groups $\chi: G \to \mathbb{T}$ is called a *character of* G. The set $\mathrm{Hom}(G, \mathbb{T})$ of all characters is an abelian group under pointwise addition, called the *character group of* G and written $\widehat{G}$. Notice that we do not consider any topology on $\widehat{G}$. □

Of course we can define characters of any topological group G. However, the concept of a character group would require some thought how it might be topologized. The topology normaly taken is the so-called topology of uniform convergence on compact sets (also called the compact open topoloy). We do not use it here, however; for a reference see for instance [1], Chapter 7.

After the definition of character groups of compact abelian groups, we can of course iterate the formation of character groups and oscillate between abelian groups and compact abelian groups. This deserves some inspection; the formalism is quite general and is familiar from the duality of finite-dimensional vector spaces in our introduction.

The result proved in Lemma 2.16 implies at once the following result:

Proposition 3.10. *The characters of an abelian group A separate the points. That is, for every nonzero element $a \in A$ there is a character $\chi \in \widehat{A}$ such that $\chi(a) \neq 0$.* □

We shall use in the sequel a result which we shall not prove here because it requires an excursion into the functional analysis of Hilbert space and integration theory of compact spaces. The result is an exact parallel of Proposition 3.10 and is therefore easily understood.

Theorem 3.11. *The characters of a compact abelilan group G separate the points. That is, for every nonzero element $g \in G$ there is a character $\chi \in \widehat{G}$ such that $\chi(g) \neq 0$.*

Proof . For a proof see for instance [1], Corollary 2.31. □

The proof is deeper than that of 3.10. It is a consequence of the basic theorem for compact groups saying that every compact group has enough finite dimensional continuous linear representations to separate the points.

Lemma 3.12. (i) *If A is an abelian group, then the function*

$$\eta_A\colon A \to \widehat{\widehat{A}}, \quad \eta_A(a)(\chi) = \chi(a)$$

is an injective morphism of abelian groups.

(ii) If G is a compact abelian group, then the function

$$\eta_G\colon G \to \widehat{\widehat{G}}, \quad \eta_G(g)(\chi) = \chi(g)$$

is an injective morphism of compact abelian groups.

Proof . (i) The morphism property follows readily from the definition of pointwise addition in $\widehat{A}$. An element a is in the kernel of η_A if $\chi(a) = \eta_A(a)(\chi) = 0$ for all characters $\chi \in \widehat{A}$. Since these separate the points by Proposition 3.10, we conclude $a = 0$. Hence η_A is injective.

(ii) Again it is straightforward that η_G is a morphism of abelian groups. The claim that it is injective follows from Theorem 3.11 in exactly the same way as the injectivity of η_A followed from Proposition 3.9. We still must prove the continuity of η_G: The function $g \mapsto \chi(g)\colon G \to \mathbb{T}$ is continuous for every character χ by the continuity of characters. Hence the function $g \mapsto \big(\chi(g)\big)_{\chi\in\widehat{G}}\colon G \to \mathbb{T}^{\widehat{G}}$ is continuous by the definition of the product topology. Since $\widehat{\widehat{G}} = \mathrm{Hom}(\widehat{G}, \mathbb{T}) \subseteq \mathbb{T}^{\widehat{G}}$ inherits its structure from the product, η_G is continuous. □

Exercise E3.8. For a discrete group A and a compact group G the members of $\widehat{\widehat{A}}$ and $\widehat{\widehat{G}}G$ separate the points of $\widehat{A}$, respectively, $\widehat{G}$. Equivalently, the evaluation morphisms $\eta_{\widehat{A}}\colon \widehat{A} \to \widehat{\widehat{\widehat{A}}}$ and $\eta_{\widehat{G}}\colon \widehat{G} \to \widehat{\widehat{\widehat{G}}}$ are injective.

[Solution. [Jonathan Meddaugh] Since $\widehat{A}$ and $\widehat{G}$ are groups, and the elements of $\widehat{\widehat{A}}$ and $\widehat{\widehat{G}}$ are homomorphisms, we need only show that for each $\chi \in \widehat{A}$, $\chi \neq 0$, there exists $\varphi \in \widehat{\widehat{A}}$ with $\varphi(\chi) \neq 0$.

In particular, let $\chi \neq 0 \in \widehat{A}$. Choose $a \in A$ such that $\chi(a) \neq 0$. Then we have $\eta_A(a)(\chi) = \chi(a) \neq 0$ and $\eta_A(a) \in \widehat{\widehat{A}}$. Thus $\eta_A(A) \subset \widehat{\widehat{A}}$ separates the points of $\widehat{A}$. Note that the above argument holds for G as well.

Now we show that this is equivalent to the injectivity of the evaluation morphisms.

Assume that the points of $\widehat{A}$ are separated by $\widehat{\widehat{A}}$. Suppose that $\chi \in \widehat{A}$ such that $\eta_{\widehat{A}}(\chi) = 0$. Then for all $\varphi \in \widehat{\widehat{A}}$, we have $0 = \eta_{\widehat{A}}(\chi)(\varphi) = \varphi(\chi)$, so that $\chi = 0$ since $\widehat{\widehat{A}}$ separates the points of $\widehat{A}$.

Conversely, if the points of $\widehat{A}$ are not separated by $\widehat{\widehat{A}}$, then there exists a non-zero element $\chi \in \widehat{A}$ such that $\varphi(\chi) = 0$ for all $\varphi \in \widehat{\widehat{A}}$. Thus $\eta_{\widehat{A}}(\chi) = 0$, so that $\eta_{\widehat{A}}$ is not injective.

Again, note that this argument only uses the characteristics of abelian groups, so that it does not matter whether we are using compact or discrete abelian groups. Thus this applies to G as well.] □

Let us look at our basic examples: If A is a finite abelian group, then $\widehat{A}$ is isomorphic to A by Proposition 3.7. Hence $\widehat{\widehat{A}}$ is isomorphic to A and $\eta_A\colon A \to \widehat{\widehat{A}}$ is injective by Lemma 3.12. Hence η_A is an isomorphism.

Every character $\chi\colon \mathbb{T} \to \mathbb{T}$ yields a morphism of topological groups $f\colon \mathbb{R} \to \mathbb{T}$ via $f(r) = \chi(r + \mathbb{Z})$. Let $q\colon \mathbb{R} \to \mathbb{T}$ be the quotient homomorphism. We set $V =]-\frac{1}{3}, \frac{1}{3}[\subseteq \mathbb{R}$ and $W = q(V)$. Then $q|V\colon V \to W$ is a homeomorphism. Assume that x and y are elements of W such that $x + y \in W$, too. Then $r = (q|V)^{-1}(x)$, $s = (q|V)^{-1}(y)$ and $t = (q|V)^{-1}(x + y)$ are elements of V such that $q(r + s - t) = q(t) + q(s) - q(t) = x + y - (x + y) = 0$ in $\mathbb{T}$. Hence $r + s - t \in \ker q = \mathbb{Z}$. But also $|r + s - t| \le |r| + |s| + |t| < 3\cdot\frac{1}{3} = 1$. Hence $r + s - t = 0$ and $(q|V)^{-1}(x) + (q|V)^{-1}(y) = r + s = t = (q|V)^{-1}(x + y)$. Now let U denote an open interval around 0 in $\mathbb{R}$ such that $f(U) \subseteq W$. If we set $\varphi = (q|V)^{-1} \circ f|U\colon U \to \mathbb{R}$ then for all $x, y, x + y \in U$ we have $\varphi(x + y) = \varphi(x) + \varphi(y)$. Under these circumstances φ extends uniquely to a morphism $F\colon \mathbb{R} \to \mathbb{R}$ of abelian groups (see Exercise E3.9 below). Now $q \circ F = f = \chi \circ q$ since F extends φ and U generates the abelian group $\mathbb{R}$. Then $\mathbb{Z} = \ker q \subseteq \ker(q \circ F)$, that is, $F(\mathbb{Z}) \subseteq \ker q = \mathbb{Z}$. Thus if we set $n = F(1)$, then $n \in \mathbb{Z}$. Since φ is continuous, then F is continuous at 0. As a morphism, F is continuous everywhere (see Exercise E3.10 below). As a morphism of abelian groups, F is quickly seen to be $\mathbb{Q}$-linear, and from its continuity it follows that it is $\mathbb{R}$-linear. Thus $F(t) = nt$ and $\chi(t + \mathbb{Z}) = nt + \mathbb{Z}$ follows. Thus the characters of $\mathbb{T}$ are exactly the endomorphisms $\mu_n = (g \mapsto ng)$ and $n \mapsto \mu_n\colon \mathbb{Z} \to \widehat{\mathbb{T}}$ is an isomorphism.

Exercise E3.9. Prove the following proposition:

The Extension Lemma. *Let U be an arbitrary interval in $\mathbb{R}$ containing 0 and assume that $\varphi\colon U \to G$ is a function into a group such that x, y, $x + y \in U$ implies $\varphi(x + y) = \varphi(x)\varphi(y)$. Then there is a morphism $F\colon \mathbb{R} \to G$ of groups extending φ. If U contains more than one point then F is unique.*

[Solution. [JONATHAN MEDDAUGH] Induction quickly gives us that $\varphi(ku) = \varphi(u)^k$ for $u \in U$ and $k \in \mathbb{N}$ such that $ku \in U$; observe that is u and $nu \in U$, then $(n-1)u \in U$ since U is an interval. Assume that the statement is true for $k = n-1$, then $\varphi(nu) = \varphi((n-1)u + u) = \varphi((n-1)u)\varphi(u) = \varphi(u)^{n-1}\varphi(u) = \varphi(u)^n$.

It is also immediately apparent that if $u, mu \in U$, we have

$$\varphi(mu)^n = (\varphi(u)^m)^n = \varphi(u)^{nm} \text{ for any } n \in \mathbb{N}.$$

Thus, for $n, m \geq 1$ and $r \in \mathbb{R}$ such that $r/m, r/n \in U$ we have

$$\varphi(r/m)^m = \varphi(n(r/nm))^m = \varphi(r/mn)^{mn} = \varphi(m(r/mn))^n = \varphi(r/n)^n.$$

For $n, m < 0$, take $r' = -r$ and if $r'/n, r'/m \in U$, we see that $\varphi(r/m) = \varphi(r'/-m)$ (and $\varphi(r/n) = \varphi(r'/-n)$) so that

$$\varphi(r/m)^m = (\varphi(r'/-m)^{-m})^{-1} = (\varphi(r'/-n)^{-n})^{-1} = \varphi(r/n)^n.$$

Lastly, we have the situation where $n > 0$, $m < 0$. In this case, provided that $r'/m, r'/n \in U$, we have

$$\varphi(r/m)^m = \varphi(r'/-m)^m = (\varphi(r'/-m)^{-m})^{-1} = (\varphi(r'/n)^n)^{-1}.$$

Since $\varphi(0) = \varphi(0+0) = \varphi(0)\varphi(0)$ as $0 = 0 + 0 \in U$, we have $\varphi(0) = 1$, the identity of G. Moreover, if $r, -r \in U$, then $u + (-u) = 0 \in U$ and we therefore have $\varphi(r)\varphi(-r) = \varphi\big(r + (-r)\big) = \varphi(0) = 1$ and so $\varphi(-r) = \varphi(r)^{-1}$. Therefore, by induction again, $(\varphi(r'/n)^n)^{-1} = \varphi(r/n)^n$ since φ is a morphism, so we have for $n, m \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathbb{R}$ with $r/n, r/m \in U$; $\varphi(r/n)^n = \varphi(r/m)^m$.

Finally, given $r \in \mathbb{R}$ we have $\{n \in \mathbb{Z} | r/n \in U\} \neq \emptyset$ unless $U = \{0\}$. If this is the case, then $F : \mathbb{R} \to G$ given by $F(r) = \mathrm{id}_G$ is an extension of φ.

Now assume that U has nonempty interior. Define $F(r) = \varphi(r/n)^n$ with $n \in \{n \in \mathbb{Z} | r/n \in U\}$. By the above comments, this is nonempty, and it matters not which element of $\{n \in \mathbb{Z} | r/n \in U\}$ we choose.

We claim that F as defined above is the unique extension of gp. Observe first that, for $u \in U$, $F(u) = \varphi(u/1)^1 = \varphi(u)$, so that F is indeed an extension of φ.

Now, let $r, s \in \mathbb{R}$. Then $F(r+s) = \varphi((r+s)/k)^k$ for some $k \in \mathbb{Z}$. In particular, we can choose k large so that $r/k, s/k, (r+s)/k \in U$. Then $\varphi((r+s)/k)^k = (\varphi(r/k)\varphi(s/k))^k$. Since $r/s, s/k$ commute in U and $\varphi\big((r/k) + (s/k)\big) = \varphi(r/k)\varphi(s/k)$, their images commute in G. Thus

$$\varphi((r+s)/k)^k = (\varphi(r/k)\varphi(s/k))^k = \varphi(r/k)^k\varphi(s/k)^k = F(r)F(s).$$

Thus F is indeed a morphism.] □

Exercise E3.10. Prove the following assertion:

A homomorphism between topological groups is continuous iff it is continuous at the identity.

Now that we have determined $\widehat{\mathbb{T}}$ we look at $\eta_{\mathbb{Z}}$. We have $\eta_{\mathbb{Z}}(n)(\chi) = \chi(n) = n\chi(1) = \mu_n\big(\chi(1)\big)$ for any character χ of $\mathbb{Z}$. Since $\chi \mapsto \chi(1): \widehat{\mathbb{Z}} \to \mathbb{T}$ is an isomorphism by (1) above and since every character of $\mathbb{T}$ is of the form μ_n, this shows that $\eta_{\mathbb{Z}}$ is an isomorphism.

Now we show that $\eta_{\mathbb{T}}$ is an isomorphism, too. We recall that $\widehat{\mathbb{T}}$ is infinite cyclic and is generated by the identity map $\varepsilon: \mathbb{T} \to \mathbb{T}$. In other words, any character $\chi: \mathbb{T} \to \mathbb{T}$ of $\widehat{\mathbb{T}}$ is of the form $\chi = n{\cdot}\varepsilon = \mu_n(\varepsilon)$. Now we observe $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon) = n{\cdot}\varepsilon(g) = n{\cdot}g$ for all $n \in \mathbb{Z}$. Taking $n = 1$ we note that the kernel of $\eta_{\mathbb{T}}$ is singleton and thus $\eta_{\mathbb{T}}$ is injective. In order to show surjectivity we assume that $\Omega: \widehat{\mathbb{T}} \to \mathbb{T}$ is a character of $\widehat{\mathbb{T}} \cong \mathbb{Z}$. Then $\Omega(\varepsilon)$ is an element $g \in \mathbb{T}$ and we see $\eta_{\mathbb{T}}(g)(n{\cdot}\varepsilon) =$

$n{\cdot}g = n{\cdot}\Omega(\varepsilon) = \Omega(n{\cdot}\varepsilon)$. Thus $\eta_{\mathbb{T}}(g) = \Omega$. This shows that $\eta_{\mathbb{T}}$ is surjective, too. Thus $\eta_{\mathbb{T}}$ is an isomorphism.

Remark 3.13. (i) Assume that A and B are abelian groups such that η_A and η_B are isomorphisms. Then $\eta_{A\oplus B}$ is an isomorphism.

(ii) If G and H are compact abelian groups and η_G and η_H are isomorphisms, then $\eta_{G\times H}$ is an isomorphism.

(iii) For any finitely generated abelian group A, the map $\eta_A\colon A \to \widehat{\widehat{A}}$ is an isomorphism.

(iv) If $G \cong \mathbb{T}^n \times E$ for a natural number n and a finite abelian group E then $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism.

(v) Every torus group $\mathbb{T}^n$ contains an element such that the subgroup generated by it is dense.

Proof . Exercise E3.11. □

Exercise E3.11. Prove Remarks 3.13(i)–(v).

[Solution. [Jonathan Meddaugh] (i) Lemma 3.12 gives us injectivity for free. We must show surjectivity. Let $\varphi \in \widehat{\widehat{A \oplus B}}$, and $\chi \in \widehat{A \oplus B}$. Observe that $\chi(a, b) = \chi(a, 0) + \chi(0, b)$, let us define $\chi_A = \chi(a, 0)$ and $\chi_B = \chi(b, 0)$ so that $\chi = \chi_A + \chi_B$ and hence $\varphi(\chi) = \varphi(\chi_A) + \varphi(\chi_B)$. Furthermore, $\chi(\cdot, 0) \in \widehat{A}$ and $\chi(0, \cdot) \in \widehat{B}$; by surjectivity of η_A and η_B, we find $a \in A$ and $b \in B$ such that $\varphi(\chi_A) = \chi_A(a)$ and $\varphi(\chi_B) = \chi_B(b)$ Thus

$$\varphi(\chi) = \chi_A(a) + \chi_B(b) = \chi(a, 0) + \chi(0, b) = \chi(a, b) = \eta_{A\oplus B}(\chi),$$

and so $\varphi \in \eta_{A\oplus B}(A \oplus B)$, i.e. $\eta_{A\oplus B}$ is surjective.

(ii) Lemma 3.12 once again gives us injectivity, we are left to show surjectivity. This argument is identical to the one presented above.

(iii) Observe that a finite abelian group is a finite direct sum of cyclic groups, and we know that $\eta_{\mathbb{Z}(n)}$ and $\eta_{\mathbb{Z}}$ are isomorphisms for all n. The result follows from (i) by induction on the number of summands.

(iv) Similarly, we have a finite product of compact abelian groups, and hence the product and the direct sum are isomorphic. It follows that since $\eta_{\mathbb{T}}$ and η_E are isomorphisms, that η_G is an isomorphism as well.] □

[Hint for (v). Set $T = \mathbb{T}^n$. Every quotient group of T modulo some closed subgroup is a compact group which is a quotient group of $\mathbb{R}^n$ and is, therefore, a torus by [1] Appendix 1, Theorem 1.12(ii). Now let $x \in T$; then $T/\overline{\langle \mathbb{Z}{\cdot}x\rangle}$ is a torus, and by (iv) above, its characters separate the points. Thus, $\mathbb{Z}{\cdot}x$ is dense in T iff all characters of T vanish on $\mathbb{Z}{\cdot}x$, i.e. on x, iff

$$(\forall \chi \in \widehat{T}) \quad [\chi(\mathbb{Z}{\cdot}x) = \{0\}] \Rightarrow [\chi = 0]$$

iff the map $\chi \mapsto \big(n \mapsto \chi(n{\cdot}x)\big)\colon \widehat{T} \to \widehat{\mathbb{Z}}$ is injective iff the map $\chi \mapsto \chi(x)\colon \widehat{T} \to \mathbb{T}$ is injective (via the natural isomorphism $\widehat{Z} \cong \mathbb{T}$). But since $\eta\colon T \to \widehat{\widehat{T}}$ is an isomorphism by (iv) above, any homomorphism $\alpha\colon \widehat{T} \to \mathbb{T}$ is an evaluation, i.e. there is a unique $x \in T$ such that for any $\chi \in \widehat{T}$ we have $\chi(x) = \alpha(\chi)$. Thus, in conclusion, we have an element $x \in T$ such that $\mathbb{Z}{\cdot}x$ is dense in T iff we have an injective morphism $\mathbb{Z}^n \cong \widehat{T} \to \mathbb{T}$. But the injective morphisms $\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z}$ abound (cf. [1] Appendix A1.43).

Provide a direct proof of Remark 3.13(v) as follows: *Let $r_j \in \mathbb{R}$, $j = 1, \dots, n$, be n real numbers such that $\{1, r_1, \dots, r_n\}$ is a set of linearly independent elements of the $\mathbb{Q}$-vector space $\mathbb{R}$. Then the element $x + \mathbb{Z} \in \mathbb{R}^n/\mathbb{Z}^n$, $x = (r_1, \dots, r_n)$ has the property that $\mathbb{Z}{\cdot}(x + \mathbb{Z})$ is dense.*] □

Exercise E3.12. Prove the following universal property of the evaluation morphism:

Lemma A. *For every morphism $f\colon A \to \widehat{G}$ from an abelian group A to the character group of a compact abelian group G there is a unique morphism $f'\colon G \to \widehat{A}$ such that $f = \widehat{f'} \circ \eta_A$.*

Proof . [Jonathan Meddaugh] We first handle existence. Define $f' : G \to \widehat{A}$ by $[f'(g)](a) = [f(a)](g)$ for $a \in A$ and $g \in G$. Then $\widehat{f'} : \widehat{\widehat{A}} \to G$ given by $[\widehat{f'}(\alpha)](g) = \alpha(f'(g))$ for all $\alpha \in \widehat{\widehat{A}}$ and $g \in G$.

Then for $g \in G$ and $a \in A$ we have

$$[\widehat{f'}(\eta_A(a))](g) = [\eta_A(a)](f'(g)) = [f'(g)](a) = [f(a)](g),$$

i.e. $\widehat{f'} \circ \eta_A = f$.

Now we handle uniqueness. Suppose that $h : G \to \widehat{A}$ such that $\widehat{h} \circ \eta_A = f$. Then for $a \in A$ and $g \in G$, we have $[f'(g)](a) = [f(a)](g) = [\widehat{h}(\eta_A(a))](g) = [\eta_A(a)](h(g)) = [h(g)](a)$. Thus $f'(g) = h(g)$ for all g, and so $f' = h$. □

Lemma B. *For every morphism $f\colon G \to \widehat{A}$ from a compact abelian group G to the character group of an abelian group A there is a unique morphism $f'\colon A \to \widehat{G}$ such that $f = \widehat{f'} \circ \eta_G$.*

Proof . [Jonathan Meddaugh] Lemma B is proved in the same way as Lemma A. □

Apply this to show

Lemma C. *For each abelian group A we have $\widehat{\eta_A} \circ \eta_{\widehat{A}} = \mathrm{id}_{\widehat{A}}$ and for each compact abelian group wie have $\widehat{\eta_G} \circ \eta_{\widehat{G}} = \mathrm{id}_{\widehat{G}}$.*

Proof . [Jonathan Meddaugh] Let A be an abelian group. Then $\eta_A : A \to \widehat{\widehat{A}}$, $\eta_{\widehat{A}} : \widehat{A} \to \widehat{\widehat{\widehat{A}}}$, and $\widehat{\eta_A} : \widehat{\widehat{\widehat{A}}} \to \widehat{A}$.

Let $\chi \in \widehat{A}$ and $a \in A$. Then

$$[\widehat{\eta_A} \circ \eta_{\widehat{A}}(\chi)](a) = [\widehat{\eta_A}(\eta_{\widehat{A}}(\chi))](a) = [\eta_{\widehat{A}}(\chi)](\eta_A(a)) = [\eta_A(a)](\chi) = \chi(a);$$

i.e. $\widehat{\eta_A} \circ \eta_{\widehat{A}} = \mathrm{id}_{\widehat{A}}$.

Observe that this proof holds replacing A with a compact abelian group G, as the properties of the evaluation morphisms $\eta_\cdot$ are the same in both categories. □

We have seen in Remark 3.13(iii, iv) that η_A and η_G are isomorphisms if A is a finitely generated (discrete) abelian groups and $G \cong \mathbb{T}^n \times E$ for a finite abelian group E. It will be the general goal of this seminar to show that these maps are isomorphisms for all abelian groups A and all compact groups G. We prepare for a proof with introducing a structural tool of the structure theory of compact groups that is intrumental in many situations.

Chapter 4
Projective Limits

Definition 4.1. Let J be a directed set, that is, a set with a reflexive, transitive and antisymmetric relation $\leq$ such that every finite nonempty subset has an upper bound. A *projective system of topological groups over* J is a family of morphisms $\{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\, j \leq k\}$, where G_j, $j \in J$ are topological groups, satisfying the following conditions:
(i) $f_{jj} = \mathrm{id}_{G_j}$ for all $j \in J$
(ii) $f_{jk} \circ f_{kl} = f_{jl}$ for all $j,k,l \in J$ with $j \leq k \leq l$. □

Lemma 4.2. (i) *For a projective system of topological groups, define the topological group P by $P = \prod_{j \in J} G_j$. Set*

$$G = \{(g_j)_{j \in J} \in P \mid (\forall j,k \in J)\, j \leq k \Rightarrow f_{jk}(g_k) = g_j\}.$$

Then G is a closed subgroup of P. If $\mathrm{incl}: G \to P$ denotes the inclusion and $\mathrm{pr}_j: P \to G_j$ the projection, then the function $f_j = \mathrm{pr}_j \circ \mathrm{incl}: G \to G_j$ is a morphism of topological groups for all $j \in J$, and for $j \leq k$ in J the relation $f_j = f_{jk} \circ f_k$ is satisfied.

(ii) *If all groups G_j in the projective system are compact, then P and G are compact groups.*

Proof . (i) Assume that $j \leq k$ in J. Define $G_{jk} = \{(g_l)_{l \in J} \in P \mid f_{jk}(g_k) = g_j\}$. Since f_{jk} is a morphism of groups, this set is a subgroup of P, and since f_{jk} is continuous, it is a closed subgroup. But $G = \bigcap_{(j,k) \in J \times J,\, j \leq k} G_{jk}$. Hence G is a closed subgroup. The remainder is straightforward.

(ii) If all G_j are compact, then P is compact by Tychonoff's Theorem, and thus G as a closed subgroup of P is compact, too. □

Definitions 4.3. If $\mathcal{P} = \{f_{jk}: G_k \to G_j \mid (j,k) \in J \times J,\, j \leq k\}$ is a projective system of topological groups, then the group G of Lemma 4.2 is called its *projective limit* and is written $G = \lim \mathcal{P}$. As a rule it suffices to remind oneself of the entire projective system by recording the family of groups G_j involved in it; therefore the notation $G = \lim_{j \in J} G_j$ is also customary. The morphisms $f_j: G \to G_j$ are called *limit maps* and the morphisms $f_{jk}: G_k \to G_j$ are called *bonding maps.* □

Example 4.4. Assume that we have a sequence $\varphi_n: G_{n+1} \to G_n$, $n \in \mathbb{N}$ of morphisms of compact groups:

$$G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 \xleftarrow{\varphi_3} G_4 \xleftarrow{\varphi_4} \cdots$$

Then we obtain a projective system of compact groups by defining, for natural numbers $j \leq k$, the morphisms

$$f_{jk} = \varphi_j \circ \varphi_{j+1} \circ \cdots \circ \varphi_{k-1}: G_k \to G_j.$$

Then $G = \lim_{n \in \mathbb{N}} G_n$ is simply given by $\{(g_n)_{n \in \mathbb{N}} \mid (\forall n \in \mathbb{N})\, \varphi_n(g_{n+1}) = g_n\}$.

(i) Choose a natural number p and set $G_n = \mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. Define $\varphi_n: \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ by $\varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}$:

$$\mathbb{Z}(p) \xleftarrow{\varphi_1} \mathbb{Z}(p^2) \xleftarrow{\varphi_2} \mathbb{Z}(p^3) \xleftarrow{\varphi_3} \mathbb{Z}(p^4) \xleftarrow{\varphi_4} \cdots$$

The projective limit of this system is none other than our *group* $\mathbb{Z}_p$ *of* p*-adic integers.*

(ii) Set $G_n = \mathbb{T}$ for all $n \in \mathbb{N}$ and define $\varphi_n(g) = p{\cdot}g$ for all $n \in \mathbb{N}$ and $g \in \mathbb{T}$. (It is customary, however, to write p in place of φ_p):

$$\mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \mathbb{T} \xleftarrow{p} \cdots$$

The projective limit of this system is called the p*-adic solenoid* $\mathbb{T}_p$. □

Proposition 4.5. *Assume that* $G = \lim_{j \in J} G_j$ *for a projective system* $f_{jk}: G_k \to G_j$ *of compact groups,* $j \leq k$ *in* J*, and denote with* $f_j: G \to G_j$ *the limit maps. Then the following statements are equivalent:*

(1) *All bonding maps* f_{jk} *are surjective.*
(2) *All limit maps* f_j *are surjective.*

Proof. (1)⇒(2) Fix $i \in J$. Let $h \in G_i$; we must find an element $g = (g_j)_{j \in J} \in G$ with $g_i = f_i(g) = h$. For all $k \in J$ with $i \leq k$ we define $C_k \subseteq \prod_{j \in J} G_j$ by

$$\{(x_j)_{j \in J} \mid (\forall j \leq k)\, x_j = f_{jk}(x_k) \text{ and } x_i = h\}.$$

Since f_{ik} is surjective, $C_k \neq \emptyset$. If $i \leq k \leq k'$ then we claim $C_{k'} \subseteq C_k$. Indeed $(x_j)_{j \in J} \in C_{k'}$ implies $f_{jk}(x_k) = f_{jk}f_{kk'}(x_{k'}) = f_{jk'}(x_{k'}) = x_j$ and $x_i = h$. Thus $(x_j)_{j \in J} \in C_k$ and the claim is established. Now $\{C_k \mid k \in J, i \leq k\}$ is a filter basis of compact sets in $\prod_{j \in J} G_j$ and thus has nonempty intersection. Assume that $g = (g_m)_{m \in J}$ is in this intersection. Then, firstly, $g_i = h$. Secondly, let $j \leq k$. Since J is directed, there is a k' with $i, k \leq k'$. Then $(g_m)_{m \in J} \in C_{k'}$. Hence $g_j = f_{jk'}(g_{k'}) = f_{jk}f_{kk'}(g_{k'}) = f_{jk}(g_k)$ by the definition of $C_{k'}$. Hence $g \in \lim_{j \in J} G_j$. Thus g is one of the elements we looked for.

(2)⇒(1) Let $j \leq k$. Then $f_j = f_{jk}f_k$. Thus the surjectivity of f_j implies that of f_{jk}. □

Definition 4.6. A projective system of topological groups in which all bonding maps and all limit maps are surjective is called a *strict projective system* and its limit is called a *strict projective limit.* □

Proposition 4.7. (i) *Let $G = \lim_{j\in J} G_j$ be a projective limit of compact groups. Let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then*
(a) *$\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$.*
(b) *$\mathcal{N}$ is a filter basis of compact normal subgroups converging to $\mathbf{1}$. (That is, given a neighborhood U of $\mathbf{1}$, there is an $N \in \mathcal{N}$ such that $N \subseteq U$.)*

(ii) *Conversely, assume that G is a compact group with a filter basis $\mathcal{N}$ of compact normal subgroups with $\bigcap \mathcal{N} = \{\mathbf{1}\}$. For $M \subseteq N$ in $\mathcal{N}$ let $f_{NM}: G/M \to G/N$ denote the natural morphism given by $f_{NM}(gM) = gN$. Then the f_{NM} constitute a strict projective system whose limit is isomorphic to G under the map $g \mapsto (gN)_{N\in\mathcal{N}}: G \to \lim_{N\in\mathcal{N}} G/N$. With this isomorphism, the limit maps are equivalent to the quotient maps $G \to G/N$.*

Proof . (i)(a) Let $V \in \mathcal{U}$. Then by the definition of the projective limit there is an identity neighborhood of $\prod_{j\in J} G_j$ of the form $W = \prod_{j\in J} W_j$ with $W_j \in \mathcal{U}_j$ for which there is a finite subset F of J such that $j \in J \setminus F$ implies $W_j = G_j$ such that $W \cap \lim_{j\in J} G_j \subseteq V$. Since J is directed, there is an upper bound $k \in J$ of F. There is a $U \in \mathcal{U}_k$ such that $f_{jk}(U) \subseteq W_j$ for all $j \in J$. Then $f_k^{-1}(U) \subseteq W \cap \lim_{j\in J} G_j \subseteq V$.

(i)(b) Evidently, each $\ker f_j$ is a compact normal subgroup. Since $i, j \le k$ implies $\ker f_k \subseteq \ker f_i \cap \ker f_j$ and J is directed, $\mathcal{N}$ is a filter basis. For each $j \in J$ we have $\ker f_j = f_j^{-1}(1) \subseteq f_j^{-1}(U)$ for any $U \in \mathcal{U}_j$. Since $f_j^{-1}(U)$ is a basic neighborhood of the identity by (a), we are done.

(ii) It is readily verified that the family of all morphisms $f_{NM}: G/M \to G/N$ for $M \subseteq N$ in $\mathcal{N}$ constitutes a strict projective system of compact groups. An element $(g_N N)_{N\in\mathcal{N}} \in \prod_{N\in\mathcal{N}} G/N$ with $g_N \in G$ is in its limit L if and only if for each pair $M \supseteq N$ in $\mathcal{N}$ we have $f_{MN}(g_N N) = g_M M$, that is, $g_M^{-1} g_N \in M$. Thus for each $g \in G$ certainly $(gN)_{N\in\mathcal{N}} \in L$. The kernel of the morphism $\varphi = (g \mapsto (gN)_{N\in\mathcal{N}}): G \to L$ is $\bigcap \mathcal{N} = \{\mathbf{1}\}$. Hence φ is injective. Assume $\gamma = (g_N N)_{N\in\mathcal{N}} \in L$. Then $\{g_N N \mid N \in \mathcal{N}\}$ is a filter basis of compact sets in G, for if $M \supseteq N$ then $g_M^{-1} g_N \in M$, and thus $g_N \in g_M M \cap g_N N$. Hence its intersection contains an element g and then $g \in g_N N$ is equivalent to $gN = g_N N$. Thus $\varphi(g) = \gamma$. We have shown that φ is also surjective and thus is an isomorphism of compact groups as a bijective continuous map between compact Hausdorff spces is a homeomorphism. If $q_N: G \to G/N$ is the quotient map, and if $f_N: L \to G/N$ is the limit map defined by $f_N\big((g_N N)_{N\in\mathcal{N}}\big) = g_N N$, then clearly $q_N = f_N \circ \varphi$. The proof of the proposition is now complete. □

The significance of the preceding proposition is that we can think of a strict projective limit G as a compact group which is approximated by factor groups

G/N modulo smaller and smaller normal subgroups N. This is not a bad image. The group G is decomposed into cosets gN whose size can be made as small as we wish using the normal subgroups in the filter basis $\mathcal{N}$.

Chapter 5
Finishing Duality

Let A be an arbitrary abelian group. Let $\mathcal{F}$ denote the family of all finitely generated subgroups. This family is directed, for if $F, E \in \mathcal{F}$ then $F + E \in \mathcal{F}$. Also, $A = \bigcup_{F\in\mathcal{F}} F$. If $E, F \in \mathcal{F}$ and $E \subseteq F$ then the inclusion $E \to F$ induces a morphism $f_{EF}: \widehat{F} \to \widehat{E}$ via $f_{EF}(\chi) = \chi|E$ for $\chi: F \to \mathbb{T}$. The family $\{f_{EF}: \widehat{F} \to \widehat{E} \mid E, F \in \mathcal{F}, E \subseteq F\}$ is a projective system of compact abelian groups. By the divisibility of $\mathbb{T}$, each character on $E \subseteq F$ extends to one on F and so this system is strict. The inclusion $F \to A$ induces a morphism $f_F: \widehat{A} \to \widehat{F}$ by $f_F(\chi) = \chi|F$ for each character $\chi: A \to \mathbb{T}$.

Proposition 5.1. *The map* $\chi \mapsto (\chi|F)_{F\in\mathcal{F}}: \widehat{A} \to \lim_{F\in\mathcal{F}} \widehat{F}$ *is an isomorphism of compact abelian groups.*

Proof . Define $\varphi: \mathrm{Hom}(A, \mathbb{T}) \to \lim_{F\in\mathcal{F}} \mathrm{Hom}(F, \mathbb{T})$ by $\varphi(\chi) = (\chi|F)_{F\in\mathcal{F}}$. This definition yields a morphism of compact groups. A character χ of A is in its kernel if and only if $\chi|F = 0$ for all $F \in \mathcal{F}$. But since $A = \bigcup_{F\in\mathcal{F}} F$ this is the case if and only if $\chi = 0$. Thus φ is injective. Now let $\gamma = (\chi_F)_{F\in\mathcal{F}} \in \lim_{F\in\mathcal{F}} \widehat{F}$. By the definition of the bonding maps, this means that for every pair of finitely generated subgroups $E \subseteq F$ in A we have $\chi_F|E = \chi_E$. Now we can unambiguously define a function $\chi: A \to \mathbb{T}$ as follows. We pick for each $a \in A$ an $F \in \mathcal{F}$ with $a \in F$ (for instance, $F = \mathbb{Z}{\cdot}a$). By the preceding, the element $\chi_F(a)$ in $\mathbb{T}$ does not depend on the choice of F. Hence we define a function $\chi: A \to \mathbb{T}$ by $\chi(a) = \chi_F(a)$. If $a, b \in A$, take $F = \mathbb{Z}{\cdot}a+\mathbb{Z}{\cdot}b$ and observe $\chi(a+b) = \chi_F(a+b) = \chi_F(a)+\chi_F(b) = \chi(a)+\chi(b)$. Thus $\chi \in \mathrm{Hom}(A, \mathbb{T})$ and $\chi|F = \chi_F$. Hence $\varphi(\chi) = \gamma$. Thus φ is bijective and hence an isomorphism of compact groups, since a bijective continuous function between Hausdorff spaces with compact domain is a homeomorphism. ☐

In short: *The character group $\widehat{A}$ of any abelian group A is the strict projective limit of the character groups $\widehat{F}$ of its finitely generated subgroups F.* We know that $\widehat{F}$ is a direct product of a finite group and a finite-dimensional torus group (see Proposition 3.7). In particular, every character group of an abelian group is approximated by compact abelian groups on manifolds.

Assume that $G = \lim_{j\in J} G_j$ is a strict projective limit of compact abelian groups with limit maps $f_j: G \to G_j$. Every character $\chi: G_j \to \mathbb{T}$ gives a character $\chi \circ f_j: G \to \mathbb{T}$ of G. Since f_j is surjective, $\chi \mapsto \chi \circ f_j: \widehat{G_j} \to \widehat{G}$ is injective. Under this map, we identify $\widehat{G_j}$ with a subgroup of $\widehat{G}$.

Proposition 5.2. *If G is a strict projective limit* $\lim_{j\in J} G_j$ *then* $\widehat{G} = \bigcup_{j\in J} \widehat{G_j}$.

Proof . With our identification of $\widehat{G}_j$ as a subgroup of $\widehat{G}$, the right side is contained in the left one. Now assume that $\chi\colon G \to \mathbb{T}$ is a character of G. If we denote with V the image of $]-\frac{1}{3}, \frac{1}{3}[$ in $\mathbb{T}$, then $\{\mathbf{0}\}$ is the only subgroup of $\mathbb{T}$ which is contained in V. Now $U = \chi^{-1}(V)$ is an open neighborhood of 0 in G. Hence by Proposition 1.31(i) there is a $j \in J$ such that $\ker f_j \subseteq U$. Hence $\chi(\ker f_j)$ is a subgroup of $\mathbb{T}$ contained in V and therefore is $\{\mathbf{0}\}$. Thus $\ker f_j \subseteq \ker \chi$ and there is a unique morphism $\chi_j\colon G_j \to \mathbb{T}$ such that $\chi = \chi_j \circ f_j$. With our convention, this means exactly $\chi \in \widehat{G_j}$. Thus $\widehat{G} \subseteq \bigcup_{j\in J} \widehat{G_j}$. □

The next theorem is one half of the famous Pontryagin Duality Theorem for compact abelian groups.

Theorem 5.3. *For any abelian group A the morphism $\eta_A\colon A \to \widehat{\widehat{A}}$ is an isomorphism.*

Proof . The injectivity of the morphism η_A was established in Lemma 3.12(i) and we have to prove its surjectivity. We know that $\widehat{A}$ is the strict projective limit $\lim_{F\in\mathcal{F}} \widehat{F}$ with the directed family $\mathcal{F}$ of finitely generated subgroups of A. (See Proposition 5.1.) The limit maps $f_F\colon \widehat{A} \to \widehat{F}$ are given by $f_F(\chi) = \chi|F$, and these surjective maps induce injective morphisms $\mathrm{Hom}(f_F, \mathbb{T})\colon \mathrm{Hom}(\widehat{F}, \mathbb{T}) \to \mathrm{Hom}(\widehat{A}, \mathbb{T})$ with $\mathrm{Hom}(f_F, \mathbb{T})(\Sigma) = \Sigma \circ f_F$. By Proposition 5.2, $\mathrm{Hom}(\widehat{A}, \mathbb{T})$ is the union of the images of the injective morphisms $\mathrm{Hom}(f_F, \mathbb{T})$. Thus for any $\Omega \in \mathrm{Hom}(\widehat{A}, \mathbb{T})$ there is an $F \in \mathcal{F}$ such that Ω is in the image of $\mathrm{Hom}(f_F, \mathbb{T})$. Hence there is a $\Sigma \in \mathrm{Hom}(\widehat{F}, \mathbb{T})$ such that $\Omega = \mathrm{Hom}(f_F, \mathbb{T})(\Sigma) = \Sigma \circ f_F$. But $\eta_F\colon F \to \mathrm{Hom}(\widehat{F}, \mathbb{T})$ is an isomorphism by Remark 3.13(iii). Hence there is an $a \in F$ such that $\Sigma = \eta_F(a)$. Thus $\Omega = \eta_F(a) \circ f_F$. Therefore, for any character $\chi\colon A \to \mathbb{T}$ of A we have $\Omega(\chi) = \eta_F(a)\big(f_F(\chi)\big) = \eta_F(a)(\chi|F) = (\chi|F)(a) = \chi(a) = \eta_A(a)(\chi)$. Thus η_A is surjective and the theorem is proved. □

It is helpful to visualize our argument by diagram chasing:

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & \mathrm{Hom}(\widehat{F}, \mathbb{T}) \\ {\scriptstyle \mathrm{inc}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\widehat{\mathrm{inc}}, \mathbb{T})} \\ A & \xrightarrow[\eta_A]{} & \mathrm{Hom}(\widehat{A}, \mathbb{T}). \end{array}$$

The other half of the Pontryagin Duality Theorem claims that $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism for any compact abelian group G, too. We shall prove this with the aid of information provided in Theorem 3.11.

Theorem 5.4. *For any compact abelian group G the morphism $\eta_G\colon G \to \widehat{\widehat{G}}$ is an isomorphism.*

Proof . In Lemma 3.12(ii) we already saw that $\eta_G\colon G \to \widehat{\widehat{G}}$ is injective. Hence the corestriction $g \mapsto \eta_G(g) : G \to \Gamma \stackrel{\text{def}}{=} \eta_G(G)$ is an isomorphism onto the subgroup $\Gamma \subseteq \widehat{\widehat{G}}$. We claim that $\Gamma = \widehat{\widehat{G}}$; a proof of this claim will finish the proof. By Lemma 3.12(ii) once again, the claim is proved if every character of $\widehat{\widehat{G}}/\Gamma$ is zero, that is, if every character of $\widehat{\widehat{G}}$ which vanishes on Γ is zero. By Theorem 5.3, we may identify $\widehat{G}$ with the character group of $\widehat{\widehat{G}}$ under the evaluation isomorphism. Thus a character f of $\widehat{\widehat{G}}$ vanishing on Γ is given by an element $\chi \in \widehat{G}$ such that $f(\Omega) = \Omega(\chi)$. But we have $0 = f\bigl(\eta_G(g)\bigr) = \eta_G(g)(\chi)$ for all $g \in G$ since f annihilates Γ. By the definition of η_G we then note $\chi(g) = \eta_G(g)(\chi) = 0$ for all $g \in G$, that is, $\chi = 0$ and thus $f = 0$. □

Theorems 5.3 and 5.4 constitute the object portion of the *Pontryagin Duality Theorem for discrete and compact abelian groups.* Up to natural isomorphism it sets up a bijection between the class of discrete and that of compact abelian groups. It reveals its true power when it is complemented by the morphism part which sets up a similar bijection between morphisms.

Proposition 5.5. (i) *Let $f\colon A \to B$ be a morphism of abelian groups. Then the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\ \eta_A\ } & \widehat{\widehat{A}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f}}} \\ B & \xrightarrow[\ \eta_B\ ]{} & \widehat{\widehat{B}}. \end{array}$$

(ii) *Let $f\colon G \to H$ be a morphism of compact abelian groups. Then the following diagram is commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\ \eta_G\ } & \widehat{\widehat{G}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\widehat{f}}} \\ H & \xrightarrow[\ \eta_H\ ]{} & \widehat{\widehat{H}}. \end{array}$$

Proof . Exercise E5.1 □

Exercise E5.1. Prove Proposition 5.5.

[Solution. [Gail Blaustein] Let A and B be abelian groups and $f\colon A \to B$ a morphism. Then $\widehat{f}\colon \widehat{B} \to \widehat{A}$ is defined by $\widehat{f}(\chi) = \chi \circ f$ for $\chi \in \widehat{B}$; accordingly, $\widehat{\widehat{f}}\colon \widehat{\widehat{A}} \to \widehat{\widehat{B}}$ is defined by $\widehat{\widehat{f}}(\Omega) = \Omega \circ \widehat{f}$ for $\Omega \in \widehat{\widehat{A}}$. Also, the evaluation morphisms $\eta_A\colon A \to \widehat{\widehat{A}}$ is given by $\eta_A(a)(\chi) = \chi(a)$, $a \in A$, $\chi \in \widehat{A}$, respectively, $\eta_B\colon B \to \widehat{\widehat{B}}$ by $\eta_B(b)(\varphi) = \varphi(b)$, $b \in B$, $\varphi \in \widehat{B}$.

Now we prove the claim. For this purpose we take an $a \in A$. Then $(\widehat{\widehat{f}} \circ \eta_A)(a)$ and $(\eta_B \circ f)(a)$ are elements of $\widehat{\widehat{B}} = \mathrm{Hom}(\widehat{B}, \mathbb{T})$; so we evaluate these functions at an element $\chi \in \widehat{B}$. Then $(\widehat{\widehat{f}} \circ \eta_A)(a)(\chi) = (\eta_A(a) \circ \widehat{f})(\chi) = \eta_A(a)(\chi \circ f) = (\chi \circ f)(a) = \chi\big(f(a)\big) = \eta_B\big(f(a)\big)(\chi) = (\eta_B \circ f)(a)(\chi)$. So $(\eta_B \circ f)(a) = (\widehat{\widehat{f}} \circ \eta_A)(a)$ for all $a \in A$. Thus $\widehat{\widehat{f}} \circ \eta_A = \eta_B \circ f$ as asserted.

The proof of Part (iii) of Proposition 5.5 follows in a completely analogous way.] □

The following consequence of the duality theorem turns out to be very useful.

Proposition 5.6. (i) *Let G be a compact abelian group and A a subgroup of the character group $\widehat{G}$. The following two conditions are equivalent:*
(1) *A separates the points of G.*
(2) *$A = \widehat{G}$.*

(ii) (The Extension Theorem for Characters) *If H is a closed subgroup of G, then every character of H extends to a character of G.*

Proof . (i) Proposition 3.11 says that (2) implies (1), and so we have to prove that (1) implies (2). Since the characters of the discrete group $\widehat{G}/A$ separate the points by Proposition 3.10, in order to prove (2) it suffices to show that every character of $\widehat{G}$ vanishing on A must be zero. Thus let Ω be a character of $\widehat{G}$ vanishing on A. By Theorem 5.4, there is a $g \in G$ with $\eta_G(g) = \Omega$. Thus $\chi \in A$ implies $0 = \Omega(\chi) = \eta_G(g)(\chi) = \chi(g)$. From (1) we now conclude $g = 0$. Hence $\Omega = \eta_G(g) = 0$.

(ii) The collection of all restrictions $\chi|H$ of characters of G to H separates the points of H since the characters of G separate the points of G by Theorem 3.11. Then (i) above shows that the function $\chi \mapsto \chi|H\colon \widehat{G} \to \widehat{H}$ is surjective, and this proves the assertion. □

Proposition 5.7. *For every compact abelian group G there is a filter basis $\mathcal{N}$ of compact subgroups such that G is the strict projective limit $\lim_{N\in\mathcal{N}} G/N$ of factor groups each of which is a character group of a finitely generated abelian group.*

Proof . Let $A = \widehat{G}$ denote the character group of G and $\mathcal{F}$ the family of finitely generated subgroups. If $F \in \mathcal{F}$, let $N_F = F^\perp$ denote the annihilator $\{g \in G \mid \chi(g) = 0 \text{ for all } \chi \in F\}$. Since $F \subseteq F'$ in $\mathcal{F}$ implies $N_{F'} \subseteq N_F$, the family $\mathcal{N} = \{N_F \mid F \in \mathcal{F}\}$ is a filter basis of closed subgroups. An element g is in $\bigcap \mathcal{N}$ if and only if it is in the annihilator of every finitely generated subgroup of A, hence if and only if it is annihilated by all of A, since A is the union of all of its finitely generated subgroups. Thus $g = 0$ by Theorem 3.11. By Proposition 4.7(ii), therefore, G is the strict projective limit $G = \lim_{F\in\mathcal{F}} G/N_F$.

Now we claim that the character group of G/N_F may be identified with F. This will finish the proof of the Proposition. If $q_F\colon G \to G/N_F$ denotes the quotient map, then the function $\varphi \mapsto \varphi \circ q_F\colon (G/N_F)\widehat{\ } \to \widehat{G}$ is injective as q_F is surjective. Its image is precisely the group $F^{\perp\perp}$ of all characters vanishing on N_F. Since every character $\chi \in F$ vanishes on N_F, we have $F \subseteq F^{\perp\perp}$. We shall now show equality and thereby prove the claim. But when $F^{\perp\perp}$ is identified with the Extension character group of G/N_F then the subgroup F separates the points of G/N_F since the only coset $g+N_F \in G/N_F$ annihilated by all of F is N_F by the definition of N_F. Now the Extension Theorem for Characters, Corollary 5.6(ii) shows $F = F^{\perp\perp}$. □

Proposition 5.7 yields the following remark:

Proposition 5.8. *Every compact abelian group is the strict projective limit of a projective system of groups G/N isomorphic to $\mathbb{T}^{n(N)} \times E_N$ with suitable numbers $n(N) = 0, 1, \ldots,$ and finite abelian groups E_N.* □

Thus every compact abelian groups is the strict projective limit of compact abelian groups defined on compact manifolds. Such groups are called compact abelian *Lie groups*.

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