Solutions to Midterm II Review Problems

1) *Two teams, A and B, play a series of up to 5 games that ends when either team wins 3 games. If team A has probability 0.4 of winning each game, what is the probability that it will win the series?*

The series will either last 3, 4, or 5 games, ending when either team wins 3 games. To compute the probability that team A wins, we can consider this as the union of three events:

$$P(A \text{ wins}) = P(A \text{ wins } \cap \text{ 3 games}) + P(A \text{ wins } \cap \text{ 4 games}) + P(A \text{ wins } \cap \text{ 5 games})$$

In the first case, $P(A \text{ wins } \cap \text{ 3 games}) = (0.4)^3 = 0.064.$

In the second case, A must win two of the first 3 games and win the fourth game:

$$\begin{aligned} P(A \text{ wins } \cap \text{ 4 games}) &= P(A \text{ wins 2 of first 3 games, wins 4th game}) \\ &= P(A \text{ wins 2 of first 3 games}) P(A \text{ wins 4th game}) \\ &= \binom{3}{2}(0.4)^2(0.6) * 0.4 = 0.1152. \end{aligned}$$

Similarly, if the series goes 5 games team A must win two of the first four and then win the 5^{th} game:

$$\begin{aligned} P(A \text{ wins } \cap \text{ 5 games}) &= P(A \text{ wins 2 of first 4 games, wins 5th game}) \\ &= P(A \text{ wins 2 of first 4 games}) P(A \text{ wins 5th game}) \\ &= \binom{4}{2}(0.4)^2(0.6)^2 * 0.4 = 0.1382. \end{aligned}$$

Summing these values together gives the final answer of 0.3174.

2) *A four-year-old girl is getting dressed for school one morning without supervision. She is choosing from her wardrobe of 3 pink shirts, 2 purple shirts, 2 blue shirts, 1 pair of blue pants, 2 pairs of purple pants, and 3 pairs of pink pants. If she chooses her shirt and pants at random, what is the probability that she'll pick a pink shirt and a blue pair of pants?*

She has 7 shirts, of which 3 are pink. The probability that she will choose a pink shirt is therefore given by the following hypergeometric calculation:

P(pink shirt)= $\dfrac{\binom{3}{1}\binom{4}{0}}{\binom{7}{1}} = \dfrac{3}{7}$. She also has 6 pairs of pants, only one of which is blue, so

the probability that she chooses her blue pants at random is given by $\dfrac{\binom{1}{1}\binom{5}{0}}{\binom{6}{1}} = \dfrac{1}{6}$. Since

her choice of shirt and pants are independent, P(pink shirt, blue pants)=(3/7)*(1/6) = 1/14.

3) *A scientist is running an experiment and is interested in the probability that a certain chemical reaction will occur within 1 minute of the start of the experiment. Let X be the length of time until the reaction occurs. What is P(X≤1) if X is uniformly distributed on the interval (0,3)?*

If X is uniform on (0,3), then f(x) = 1/3 for values of x between 0 and 3.
Then P(*X*≤1) = 1/3*1 = 1/3.

4) *Suppose that in a city the number of suicides can be approximated by a Poisson process with λ = 0.33 per month. What is the probability of two suicides in a month?*

The number of suicides *X* is Poisson with mean λ = 0.33 per month. Then

$$P(X=2)=\frac{e^{-0.33}\left(0.33\right)^2}{2!}=0.039.$$

5) *Let X be a discrete random variable that takes on values 0, 1, and 2 with respective probabilities* $1/2$, $3/8$, *and* $1/8$. *Find E(X) and Var(X).*

$$E(X)=\sum xp(x)=\left(\frac{1}{2}\right)(0)+\left(\frac{3}{8}\right)(1)+\left(\frac{1}{8}\right)(2)=\frac{5}{8}.$$

$$Var(X)=E(X^2)-\left[E(X)\right]^2=\sum x^2p(x)-\left(\frac{5}{8}\right)^2=\left(\frac{1}{2}\right)(0)+\left(\frac{3}{8}\right)(1)+\left(\frac{1}{8}\right)(4)-\frac{25}{64}=\frac{7}{8}-\frac{25}{64}=\frac{41}{64}.$$

6) *A landscaper plants bushes in poor soil in a corner of a client's yard. She estimates that the probability that any bush planted in this area will survive is about 0.4. She decides to plant six bushes. What is the probability that at least two of the six planted bushes will live?*

The number of bushes that survive X is a binomial random variable with *n*=6 and *p*=0.4, so we just calculate $P(X\geq 2)=1-P(X\leq 1)=1-0.233=0.767$.

7) *The random variable X is normally distributed with mean 10 and variance 100.*

   a) *What is P(X > 10)?*
   b) *What is P(-5 ≤ X ≤ 5)?*
   c) *What is P(X > 20 or X < 0)?*

X has mean 10, so clearly P(*X*>10) = 0.5.

$$P(-5\leq X\leq 5)=P\left(\frac{-5-10}{10}\leq Z\leq\frac{5-10}{10}\right)=P\left(-1.5\leq Z\leq -0.5\right)=0.2417.$$

$$P(X>20\text{ or }X<0)=P\left(Z>\frac{20-10}{10}\right)+P\left(Z<\frac{0-10}{10}\right)=P\left(Z>1\right)+P(Z<-1)=2P(Z<-1)=0.3173.$$

8) *Let X equal the number out of n=48 mature aster seeds that will germinate when p=0.75 is the probability that a particular seed germinates. Using the continuity correction, determine* $P(35 \le X \le 40)$, *approximately.*

X is a binomial random variable, so E(*X*)=μ= np=36 and
$\sigma = \sqrt{np(1-p)} = \sqrt{48*.75*.25} = 3$. Then
$P(35 \le X \le 40) = P(34.5 \le X \le 40.5) \approx P(34.5 \le X' \le 40.5)$, where X' is a normally distributed random variable with mean 36 and standard deviation 3. Finishing the calculation gives the answer $P(34.5 \le X \le 40.5) \approx 0.625$.

9) *In roulette, the probability of winning with a bet on red is p = 18/38. Let Y equal the number of winning bets out of 1000 independent bets that are placed. Find* $P(Y > 500)$ *approximately, using the continuity correction.*

As in the previous problem, *Y* is a binomial random variable with
mean $\mu = np = 1000 * 18/38 = 473.68$ and standard deviation
$\sigma = \sqrt{np(1-p)} = \sqrt{1000 * 18/38 * 20/38} = 15.79$. Using the continuity correction,
$P(Y > 500) = P(Y > 500.5) \approx P(Y' > 500.5)$ where Y' is a normally distributed random variable with mean 473.68 and standard deviation 15.79, so we find
$P(Y' > 500.5) = P\left(Z > \frac{500.5 - 473.68}{15.79}\right) = P(Z > 1.70) = 0.0446.$

10) *55% of the registered voters in a given town favor their incumbent mayor in her reelection campaign, but many of them aren't likely to vote. If only 400 voters go to the polls on election day, approximate the probability that the challenger will win the election. Use the continuity correction.*

The probability that the challenger wins the election is equal to P(*X*<200), where *X* is the number of votes the incumbent receives. *X* is a binomial random variable with *n*=400 and *p*=0.55, so $\mu = np = 400 * 0.55 = 220$ and $\sigma = \sqrt{400 * 0.55 * 0.45} = 9.95$. Using the continuity correction, $P(X < 200) = P(X \le 199.5) \approx P(X' \le 199.5)$ where X' is a normally distributed random variable with mean 220 and standard deviation 9.95. It follows that $P(X' \le 199.5) = P\left(Z \le \frac{199.5 - 220}{9.95}\right) = P(Z \le -2.06) = 0.0197.$