On Polyhedral Products and Spaces of Commuting Elements in Lie Groups

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Biographical Sketch

Mentor Stafa was born in Durrës, Albania, in 1987. He enrolled in the Department of Mathematics at the Middle East Technical University in Ankara, Turkey, in 2005 and graduated with high honors, earning a Bachelor of Science in Mathematics in 2008. That same year he began his doctoral studies in pure mathematics at the University of Rochester. He received a Master of Arts in Mathematics from the University of Rochester in 2010. He pursued his research in algebraic topology under the supervision of Professor Frederick R. Cohen.

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Abstract

This thesis consists of two parts. The first part concentrates on polyhedral products. Certain homotopy theoretic properties of polyhedral products, such as the fundamental group, are investigated, and the results are used to compute certain monodromy representations. Partial topological characterizations of transitively commutative groups are also obtained using polyhedral products. The second part concentrates on the spaces of commuting $n-$tuples in compact and connected Lie groups. A new space is introduced, called $X(2,G)$. The homology of the space $X(2,G)$ is computed with integer coefficients with the order of the Weyl group inverted, and connections with classical representation theory are explored.

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1 Introduction

1.1 Structure

The mathematics developed in this dissertation is separated into two main parts. The first part addresses various problems related to polyhedral products of topological spaces. The polyhedral product of topological spaces is a combinatorial way of constructing a topological space out of a given collection of topological spaces. The process of construction or *gluing* is described by a simplicial complex, see [Bahri *et al.*, 2010; Buchstaber and Panov, 2000]. In this dissertation, basic homotopy theoretic problems are addressed, together with applications in representation theory.

The second part of the dissertation concerns the spaces of commuting n–tuples in a compact Lie group G. A new space is constructed to study the invariants of these spaces, such as homology and cohomology.

These two subjects, polyhedral products and commuting elements, are introduced in the next 2 sections.

1.2 Polyhedral products

The topological spaces called *polyhedral products* or *generalized moment–angle complexes*, have been studied since the 60's, if not earlier. Not surprisingly, at the time, there was no general name associated to these spaces. They merely occurred naturally in different problems that were being studied at the time. One of the early examples appears in a paper by Gerald J. Porter [Porter, 1966]. In his paper G. J. Porter studied properties of the spaces

$$T_i(X_1, ..., X_n)$$

which are the subspaces of the product $X_1 \times \cdots \times X_n$, that consist of elements with at least i coordinates being the basepoint. In the language of polyhedral products (see Chapter 2), these spaces are exactly the spaces

$$T_i(X_1, ..., X_n) = Z_K(X_i),$$

where K is the $(n-i-1)$–skeleton of the $(n-1)$–simplex $\Delta[n-1]$. G. J. Porter finds a homotopy equivalent space for $\Omega T_{n-i}(\Sigma X_1, ..., \Sigma X_n)$, which in turn classifies certain $i-$ary homotopy operations on n variables.

An important example of polyhedral products appears in seminal work of Mike Davis and Tadeusz Januszkiewicz [Davis and Januszkiewicz, 1991]. In one of the constructions, they introduce a space BP^n and they prove that the cohomology ring of BP^n is the Stanley-Reisner face ring of a simplicial complex K, denoted $\mathbb{Z}[K]$, and the space BP^n is now called the Davis-Januszkiewicz space, denoted $\mathcal{DJ}(K)$. It also follows from their work that any toric manifold M^{2n} can be realized as the quotient of an $(m+n)$–dimensional moment–angle complex by the action of a real $(m-n)$–torus T^{m-n}.

It was proved later by Victor Buchstaber and Taras Panov [Buchstaber and Panov, 2000] that there is a space $Z_K = Z_K(D^2, S^1)$, called *the moment–angle complex*, such that

$$\mathcal{DJ}(K) = E(S^1)^n \times_{(S^1)^n} Z_K(D^2, S^1) \cong Z_K(\mathbb{C}P^\infty, *)$$

which fits into a fibration

$$Z_K \longrightarrow \mathcal{DJ}(K) \longrightarrow (\mathbb{C}P^\infty)^n.$$

This also gives a cellular model for the ring $\mathbb{Z}[K]$. It is to be emphasized that the spaces $\mathcal{DJ}(K)$ and Z_K are key objects in the study of *toric topology*, an emerging field of algebraic topology, brought to prominence by work of V. Buchstaber and T. Panov [Buchstaber and Panov, 2000].

Work of T. Bahri, M. Bandersky, F. Cohen and S. Gitler [Bahri *et al.*, 2010] made it possible that polyhedral products be studied systematically. The importance of polyhedral products was emphasized strongly when many classical theorems followed from studying these spaces. A stable decomposition of these spaces was given in [Bahri *et al.*, 2010], that is, there is a homotopy equivalence

$$\Sigma Z_K(\underline{X}, \underline{A}) \simeq \Sigma \bigvee_{I \subseteq [n]} \widehat{Z}_{K_I}(\underline{X}, \underline{A}),$$

where I runs over all subsequences of $[n] = \{1, ..., n\}$, $K_I = K \cap I$ and $(\underline{X}, \underline{A}, \underline{*_i})$ is a sequence of CW triples. This decomposition implies the homological decomposition in Goresky-MacPherson [Goresky and MacPherson, 1988], Hochster [Hochster, 1977], Baskakov [Baskakov, 2002], Panov [Panov, 2008], and Buchstaber-Panov [Buchstaber and Panov, 2000] and also implies certain homotopy theoretic results of Porter [Porter, 1966] and Ganea [Ganea, 1965].

The study of the homotopy theory of polyhedral products has important implications also within ring theory, and homological algebra, see for instance [Grbić *et al.*, 2012].

Another example is the connection with the complement of a complex *hyperplane subspace arrangement* $\mathcal{U}(K)$

$$\mathcal{U}(K) = \mathbb{C}^n \setminus \bigcup_{I \notin K} \{z \in \mathbb{C}^n | z_i = 0 \text{ for } i \in I\}.$$

This space turns out to be the moment–angle complex $Z_K(\mathbb{C}, \mathbb{C}^*) \simeq Z_K$, where Z_K is a T^n–equivariant retraction of $\mathcal{U}(K) = Z_K(\mathbb{C}, \mathbb{C}^*)$, see [Buchstaber and Panov, 2000].

It is now evident that polyhedral products play an important role in a number of fields in mathematics. They give a combinatorial way to deal with different problems, topological, algebraic or geometric. In this thesis, one central problem concerns the computation of the monodromy representation of the fibration

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \longrightarrow \prod_{i=1}^{m} BG_i,$$

where G_i are finite groups. A partial result leads to a full understanding of the case of two cyclic groups G_1 and G_2. To compute the monodromy action for three or more groups, one runs into problems very quickly. There is no natural choice of basis for the fundamental group of the fibre $Z_K(EG_i, G_i)$, in cases where the fundamental group is free.

1.3 Spaces of commuting n–tuples in a Lie group

Let G be a Lie group. For a positive integer n, let $\mathrm{Hom}(\mathbb{Z}^n, G)$ denote the set of group homomorphisms from a free abelian group of rank n to G. Every element $f \in \mathrm{Hom}(\mathbb{Z}^n, G)$ is determined by the image of $1 \in \mathbb{Z}$, for each copy of $\mathbb{Z}$. That is, $f(1, ..., 1) = (g_1, ..., g_n)$ determines f. Moreover, since $\mathbb{Z}^n$ is commutative, the image of f should commute as well, hence, $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be seen as the set of pairwise commuting $n-$tuples in G. $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be naturally topologized with the subspace topology of G^n, making it a topological space. This space was first studied by E. Witten where he considers this space for $n = 3$, see [Witten, 1982; Witten, 1998].

More generally, the free abelian group $\mathbb{Z}^n$ can be replaced by a finitely generated discrete group on n generators, π. Then similarly, if π has n generators, the space $\mathrm{Hom}(\pi, G)$ is a subspace of G^n. A classical result of W. M. Goldman [Goldman, 1988] shows that these spaces are real algebraic varieties.

Now let K be a closed subgroup of G that lies in the center and let $H = G/K$. Then one also considers n–tuples $(h_1, ..., h_n) \in H^n$ which do not necessarily commute. But they lift to n–tuples $(\widetilde{h}_1, ..., \widetilde{h}_n)$ such that $[\widetilde{h}_i, \widetilde{h}_j] \in K$. The set of all such n–tuples in H forms a space called the space of K-*almost commuting elements* in G, denoted by $B_n(G, K)$. The work of E. Witten was followed by work of A. Borel, R. S. Friedman and J. W. Morgan [Borel *et al.*, 2002], where they study the spaces of almost commuting pairs and triples in G.

The space of commuting $n-$tuples was first studied in this generality by Adem and Cohen in [Adem and Cohen, 2007], where they also give a stable decomposition of the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ for all n and any closed subgroups $G \subseteq GL(n, \mathbb{C})$.

Theorem 1.1 (Adem & Cohen)**.** *If G is a closed subgroup $G \subseteq GL_n(\mathbb{C})$, then there are homotopy equivalences*

$$\Sigma(\mathrm{Hom}(\mathbb{Z}^n, G)) \simeq \bigvee_{1 \leq k \leq n} \Sigma(\bigvee^{\binom{n}{k}} \mathrm{Hom}(\mathbb{Z}^k, G)/S_k(G)).$$

They also work out explicitly the cohomology of $\mathrm{Hom}(\mathbb{Z}^3, G)$ with coefficients in $\mathbb{Z}$ for $G = SU(2)$. Following their paper, many other papers by various mathematicians appeared on the same subject. The case of $SU(2)$ has also been studied by Baird, Jeffrey and Selick in [T. Baird, 2010], where they work out the cohomology of $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ for all n. Cohomology computations have been carried out by Baird in [Baird, 2007] and the fundamental group of $\mathrm{Hom}(\mathbb{Z}^n, G)$ has been computed by Torres Giese and Sjerve in [Torres Giese and Sjerve, 2008] for $G = SU(2), U(2), SO(3)$, and by Gómez, Pettet and Souto in [Gómez *et al.*, 2012] for any compact and connected Lie group. A corollary in [Gómez *et al.*, 2012] shows the following isomorphism

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_1; F) \cong H^*(G/T \times T^n; F)^W,$$

where F is a field with characteristic relatively prime to the order of the Weyl group W.

In this thesis a new space, called $X(2, G)$, is introduced. This space assembles all the spaces of commuting elements in G, into a single space. One of the main results is the computation of the homology groups of the space $X(2, G)$ with coefficients in the ring of integers with the inverse of $|W|$ adjoined. Other results also include stable decompositions of the spaces $X(2, G)$ and $G \times_{NT} J(G)$, where $J(G)$ is the James reduced product on G.

A computation of the homology of $X(2, G)$ reduces to understanding tensor products of representations of the Weyl group W, and these assemble to give all of the homology of $\mathrm{Hom}(\mathbb{Z}^n, G)$ at once by a description in terms of partitions. If ungraded homology is considered throughout, then the homology of $X(2, G)$ is given explicitly and is well–understood.

1.4 Background

The two topics that have been developed in this thesis seem unrelated at first. However, as shown by the application in Section 2.9 and Section 3.3, the study of polyhedral products and spaces of commuting elements was motivated by studying the sets of homomorphisms $\mathrm{Hom}(F_n/\Gamma^k(F_n), G)$, where F_n is a free group on n letters, $\Gamma^k(F_n)$ is the k–th stage in the descending central series of F_n (see Section 2.7.3) and G is a topological group.

If G is a finite discrete group, the spaces $\mathrm{Hom}(F_n/\Gamma^k(F_n), G)$ are used to construct simplicial spaces, resulting in a filtration of the classifying space BG, given by

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(, \infty, G) = BG,$$

where the space $B(q, G)$ for $q \geq 2$, is the geometric realization of the semi–simplicial complex formed by the simplicial spaces $\mathrm{Hom}(F_n/\Gamma^q(F_n), G)$ for varying n, see [Adem *et al.*, 2011]. In Section 2.9, we mention that for certain classes of finite discrete groups G, the space $B(2, G)$ is a polyhedral product. This fact together with the other properties of polyhedral products are used to study the structure of the space $B(2, G)$.

If G is a classical Lie group, then the sets of homomorphisms form topological spaces $\mathrm{Hom}(F_n/\Gamma^k(F_n), G)$. In particular, the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ are the spaces of pairwise commuting n-tuples in G, which are studied in Chapter 3.

2 Polyhedral Products

2.1 Introduction

In this chapter the term *moment–angle complex* will mean the polyhedral product for the pair (D^2, S^1) and *polyhedral product* will be used for other pairs. This construction will be defined in the next section, and it gives a way to obtain new topological spaces from a finite set of pointed CW-pairs. The gluing process of the pairs is described by a simplicial set K.

The first main result of this chapter gives a necessary and sufficient condition for when the polyhedral products $Z_K(BG_i, *_i)$ are of the homotopy type of Eilenberg–Mac Lane spaces.

Theorem 2.1. *Let G_i be non–trivial groups with $2 \leq |G_i| \leq \aleph_0$ and endowed with the discrete topology, for all i. Let K be a simplicial complex on a finite set of vertices. Then $Z_K(BG_i, *_i)$ is an Eilenberg–Mac Lane space if and only if K is a flag complex.*

The fundamental group of the polyhedral products $Z_K(X, *)$ is also investigated for a 1–connected space X. Let CY denote the cone on the topological space Y.

Theorem 2.2. *Let X be a 1–connected CW–complex. Then the polyhedral product $Z_K(C\Omega X, \Omega X)$ is 1–connected.*

It turns out that if the homotopy types of the pairs (X, A) and (Y, A) are known in relation to each other then the following lemma holds

Lemma 2.3. *If $A \subset X \subset Y$ with A discrete, X and Y path–connected, such that the induced map $\pi_i(X) \xrightarrow{i_*} \pi_i(Y)$ is an isomorphism for $i = 0, 1$, then*

$$\pi_1(Z_K(X, A)) \cong \pi_1(Z_K(Y, A)).$$

The content of these theorems is essential to applications, especially the case when the pairs (X, A) are of the form (EG_i, G_i) and when K is the 0–simplicial complex.

The organization of this chapter is as follows. First, introductory material is given, such as definitions and examples of polyhedral products. Then in Section 2.3 following definitions, homotopy theoretic properties of these spaces are investigated for a finite sequence of classifying spaces of topological groups. The invariants studied are the fundamental group, and sometimes the higher homotopy groups. The main result of the section is Theorem 2.23 which is mentioned above. The discussion of the same invariants is continued in Section 2.4. The results from Section 2.3 are used in Section 2.5 to compute the monodromy representation corresponding to a certain fibration, which is given in Section 2.3. The importance of this monodromy representation is that it gives some information about the outer automorphism group $\mathrm{Out}(F_n)$ for a free group F_n and for certain values of n. Other representations obtained from this monodromy representation are discussed in Section 2.8.

In Section 2.9, *transitively commutative groups* are studied in relation to polyhedral products. The main goal, yet to be achieved, is a topological characterization of these groups via extension properties involving polyhedral products. Transitively commutative groups were studied intensively before the proof of the well-known Feit–Thompson theorem, which states that every finite group of odd order is solvable, see [Feit and Thompson, 1963].

2.2 Definitions and examples

Polyhedral products can be regarded as functors from abstract simplicial complexes with values in the category of topological spaces. These spaces were given the name "polyhedral products" or "polyhedral product functors", names which were suggested by W. Browder. Alternatively, for fixed K, they can be seen as functors from the category of topological spaces to the category of topological spaces. Here the discussion is confined to pointed CW–pairs (X, A).

Let $[n]$ denote the set of integers 1 through n, $\{1, 2, ..., n\}$, and let $2^{[n]}$ denote the power set of $[n]$.

Definition 2.4. An *abstract simplicial complex* K on n vertices is a subset of the power set $2^{[n]}$ such that, if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.

Hence, any element $\sigma \in K$, called a *simplex*, is given by a sequence of integers $\sigma = \{i_1, i_2, ..., i_q\}$ where $1 \leq i_1 < i_2 < \cdots < i_q \leq n$. In particular, the empty subset of $[n]$ is an element of K. The *geometric realization* $|K|$ of K is a simplicial complex inside $\Delta[n-1]$.

Now let $(\underline{X}, \underline{A})$ denote the sequence of triples of $CW-$complexes $\{X_i, A_i, x_i\}_{i=1}^n$, where x_i are the basepoints of X_i. Define a functor D from an abstract simplicial complex K to the category of pointed CW–complexes, $D : K \longrightarrow CW_*$, as follows: For any $\sigma \in K$ let

$$D(\sigma) = \prod_{i=1}^{n} Y_i = Y_1 \times \cdots \times Y_n \quad \text{where} \quad Y_i = \begin{cases} A_i & : i \notin \sigma, \\ X_i & : i \in \sigma. \end{cases}$$

The polyhedral product or generalized moment-angle complex, denoted by $Z_K(\underline{X}, \underline{A})$, is defined as follows:

Definition 2.5. The *polyhedral product* or *generalized moment-angle complex* $Z_K(\underline{X}, \underline{A})$ is the space

$$Z_K(\underline{X}, \underline{A}) = \operatorname*{colim}_{\sigma \in K} D(\sigma) = \bigcup_{\sigma \in K} D(\sigma).$$

That means the polyhedral product is the colimit of the diagram of spaces $D(\sigma)$. Note that if K is an abstract simplicial complex on n vertices, then $Z_K(\underline{X}, \underline{A}) \subseteq X_1 \times \cdots \times X_n$. Different notations are used in the literature to denote the polyhedral products, such as $Z_K(X_i, A_i)$, $Z(K; (\underline{X}, \underline{A}))$ and $(\underline{X}, \underline{A})^K$. Whenever A_i is the basepoint x_i, the notation will be slightly simplified and instead of writing $Z_K(\underline{X}, \underline{x})$ we will write $Z_K(X_i)$.

In our discussion, we will drop the expression "generalized moment–angle complex" and refer to these spaces as polyhedral products.

Example 2.6. Some examples of polyhedral products are the following

1. Let $K = \{\{1\}, ..., \{n\}\}$ and $X_i = X, A_i = x_i$. Then $Z_K(\underline{X}, \underline{A}) = X \vee ... \vee X$, the n–fold wedge sum of the space X.
2. Let $K = 2^{[n]}$, then $Z_K(\underline{X}, \underline{A}) = X_1 \times \cdots \times X_n$.
3. Let $K = \{\{1\}, \{2\}\}$ and $(\underline{X}, \underline{A}) = (D^n, S^{n-1})$. Then

$$Z_K(\underline{X}, \underline{A}) = D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial D^{2n} = S^{2n-1}.$$

Definition 2.7. Given a simplicial graph Γ with vertex set S and a family of groups $\{G_s\}_{s \in S}$, their *graph product*

$$\prod_{\Gamma} G_s$$

is the quotient of the free product of the G_s by the relations that elements of G_s and G_t commute whenever $\{s, t\}$ is an edge of Γ.

Definition 2.8. $|K|$ is a *flag complex* if any finite set of vertices, which are pairwise connected by edges, spans a simplex in $|K|$.

There are many non–trivial examples that arise from polyhedral products. A few examples are mentioned below.

Example 2.9. As mentioned in the introduction, the Davis-Januszkiewicz space $\mathcal{DJ}(K)$ can be realized as the space $ET^m \times_{T^m} Z_K(D^2, S^1)$, which is homotopy equivalent to $Z_K(\mathbb{C}P^\infty, *)$, see [Buchstaber and Panov, 2000].

Example 2.10. The fundamental group of $Z_K(S^1,*)$ is the right-angled Artin group (RAAG) given by the graph product $\prod_{SK_1}\mathbb{Z}$, where SK_1 is the 1-skeleton of K. If K is a flag complex, then $Z_K(S^1,*)$ is the classifying space of the right-angled Artin group, see [Davis and Okun, 2012].

Example 2.11. If K is the boundary complex of a simplicial polytope, then $Z_K([0,1],1)$ is the dual polytope of K, see [Buchstaber and Panov, 2000].

Before proceeding to the next sections let us establish some notation. From now on assume that the homotopy category of pointed CW–complexes is the underlying category, unless otherwise stated. That means that all topological spaces considered are CW–complexes with non–degenerate basepoints and maps between them are homotopy classes of basepoint preserving continuous maps. For any topological group G, BG and EG will stand for its classifying space and the universal cover of the classifying space, respectively. In the case of a discrete group G, BG and EG can be thought of as the Eilenberg–Mac Lane space $K(G,1)$ and its universal cover, respectively. Hence, EG is a contractible space. In the following sections, K will frequently stand for the geometric realization of K, but it will be clarified wherever ambiguity might occur, since formally K is the abstract simplicial complex. It is easier to think of K geometrically, instead of thinking of a collection of subsets of $[n]$ as defined above.

The exposition in this chapter is not self–contained and some acquaintance with notions of algebra, topology and algebraic topology are required. Specific references will be given where necessary. The necessary preliminary background in algebra can be found in [Lang, 2002], point–set topology can be found in [Munkres, 2000] and algebraic topology in [Hatcher, 2002].

2.3 Homotopy groups of $Z_K(BG_i)$

A natural problem is the study of homotopy groups of the polyhedral products $Z_K(\underline{X}, \underline{A})$. In this section the problem is restricted to the study of the homotopy groups of $Z_K(BG_i)$, where G_i are finite discrete groups.

In general, for G a topological group and K a simplicial complex on n vertices, there is a fibration due to G. Denham and A. Suciu [Denham and Suciu, 2007]. Their theorem will be referred to extensively in this chapter, especially in applications, so it is stated next.

Theorem 2.12 (Denham & Suciu)**.** *Let G be a topological group and K a simplicial complex on n vertices. Then the following hold:*

1. $EG^n \times_{G^n} Z_K(EG, G) \simeq Z_K(BG)$.
2. *The homotopy fiber of the inclusion $Z_K(BG) \hookrightarrow BG^n$ is $Z_K(EG, G)$.*

Their theorem is a result of studying the bundle

$$Z_K(EG, G) \longrightarrow EG^n \times_{G^n} Z_K(EG, G) \longrightarrow BG^n,$$

where $EG^n \times_{G^n} Z_K(EG, G)$ is the quotient of $EG^n \times Z_K(EG, G)$ by the diagonal action of G^n, and proving that the total space $EG^n \times_{G^n} Z_K(EG, G)$ is homotopy equivalent to $Z_K(BG)$. This theorem can be extended directly to the sequence of CW–pairs $\{(BG_i, *_i)\}_{i=1}^n$, where G_i are topological groups for all i. Hence, there is a fibration

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \longrightarrow BG_1 \times \cdots \times BG_n,$$

where similarly, the total space in this fibration is homotopy equivalent to the twisted product of spaces

$$(EG_1 \times \cdots \times EG_n) \times_{G_1 \times \cdots \times G_n} Z_K(EG_i, G_i).$$

Now consider the sequence of CW–pairs $\{(BG_i, *_i)\}_{i=1}^n$, where G_i are finite discrete groups for all i. Let K be a simplicial complex on n vertices and let SK_q denote the q-skeleton of K. Consider the Denham and Suciu fibration

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \longrightarrow BG_i \times \cdots \times BG_n. \tag{2.1}$$

Lemma 2.13. *Assume that G_i are finite discrete groups for all i and K is a simplicial complex on n vertices. The following hold:*

1. *$Z_K(BG_i)$ is path connected for any K.*

2. *$\pi_q(Z_K(EG_i, G_i)) \cong \pi_q(Z_K(BG_i))$ for $q \geq 2$.*

3. *There is a short exact sequence of groups*

$$1 \to \pi_1(Z_K(EG_i, G_i)) \to \pi_1(Z_K(BG_i)) \to \pi_1(BG_1 \times \cdots \times BG_n) \to 1.$$

Proof. Run the long exact sequence in homotopy for the Denham and Suciu fibration (2.1) to get

$$\cdots \to \pi_q(Z_K(EG_i, G_i)) \to \pi_q(Z_K(BG_i)) \to \pi_q(BG_1 \times \cdots \times BG_n) \to$$
$$\to \pi_{q-1}(Z_K(EG_i, G_i)) \to \cdots \to \pi_2(BG_1 \times \cdots \times BG_n) \to$$
$$\to \pi_1(Z_K(EG_i, G_i)) \to \pi_1(Z_K(BG_i)) \to \pi_1(BG_1 \times \cdots \times BG_n) \to \pi_0(Z_K(EG_i, G_i))$$

The space $BG_1 \times \cdots \times BG_n$ is an Eilenberg–Mac Lane space with

$$\pi_1(BG_1 \times \cdots \times BG_n) = G_1 \times \cdots \times G_n$$

and

$$\pi_q(BG_1 \times \cdots \times BG_n) = 0$$

for $q \geq 2$ and $q = 0$. Hence, $Z_K(EG_i, G_i)$ is path connected for any K and part 1 and 2 follow. Finally, there is a short exact sequence of groups

$$1 \to \pi_1(Z_K(EG_i, G_i)) \to \pi_1(Z_K(BG_i)) \to \pi_1(BG_1 \times \cdots \times BG_n) \to 1.$$

□

The fundamental group of the spaces $Z_K(BG_i)$ is calculated next. This calculation will precede the fact that the polyhedral product $Z_K(BG_i)$ is an Eilenberg–Mac Lane space if and only if K is a flag complex. An important feature that distinguishes flag complexes from other simplicial complexes are the objects defined next.

Definition 2.14. Let K be a simplicial complex on n vertices. A *minimal non–face* in K is the boundary of a full simplex on 3 or more vertices not in K. That is, if $\sigma = \partial\tau$ and $\tau = \Delta[q]$ for $2 \leq q \leq n-1$, and σ is a subcomplex of K but $\tau \notin K$, then σ is a *minimal non-face* in K.

Clearly if K has a minimal non–face, then it cannot be a flag complex. Therefore, a flag complex can be redefined to be a simplicial complex with no minimal non-faces.

Proposition 2.15. *Let K be a simplicial complex on n vertices and let SK_1 be the 1–skeleton of K. Then*

$$\pi_1(Z_K(BG_i)) \cong \pi_1(Z_{SK_1}(BG_i)) \cong \prod_{SK_1} G_i.$$

Proof. If K is a flag complex, then this is true since $Z_K(BG_i)$ is a $K(\pi, 1)$ with $\pi = \prod_{SK_1} G_i$, see [Davis and Okun, 2012].

Now assume that K is not a flag complex. Then K has a minimal non-face σ on k vertices. Hence, σ contains the complete graph Γ on its vertices such that $\Gamma \subseteq \sigma = \partial\Delta[k-1] \subset \Delta[k-1]$ on these k vertices, for $3 \leq k \leq n$ (trivial for $k = 1, 2$). First we claim that

$$\pi_1(Z_\sigma(BG_i)) \cong \pi_1(Z_{\Delta[k-1]}(BG_i)) \cong \prod_{\Gamma} G_i = \prod_{i=1}^{k} G_i.$$

Note that if $k = 3$ and $G_{j_1}, G_{j_2}, G_{j_3}$ are subgroups of a finite group G such that they pairwise commute in G, then any product of elements commutes in G. Hence, if σ is the boundary of the 2–simplex, then

$$Z_\sigma(BG_{j_i}) = (BG_{j_1} \times BG_{j_2} \times 1) \cup (BG_{j_1} \times 1 \times BG_{j_3}) \cup (1 \times G_{j_2} \times BG_{j_3})$$

and using the Seifert–van Kampen theorem twice it follows that

$$\pi_1(Z_\sigma(BG_{j_i})) \cong \pi_1(Z_{\Delta[2]}(BG_{j_i})) \cong \prod_\Gamma G_{j_i} \cong G_{j_1} \times G_{j_2} \times G_{j_3}.$$

Note that for $k \geq 3$, the polyhedral product can be written as follows

$$Z_\sigma(BG_i) = \bigcup_{1\leq j\leq k} (BG_1 \times \cdots \times BG_{j-1} \times 1 \times BG_{j+1} \times \cdots \times BG_n).$$

Using the Seifert–van Kampen theorem finitely many times, more exactly $k-1$ times, it follows that

$$\pi_1(Z_\sigma(BG_i)) \cong \pi_1(Z_{\Delta[k-1]}(BG_i)) \cong \prod_\Gamma G_i \cong G_1 \times \cdots \times G_k.$$

Now Γ is the complete graph on the vertices of σ. The graph product $\prod_\Gamma G_i$ of the groups $G_1, \ldots, G_k$ is actually isomorphic to the product $\prod_{i=1}^k G_i$, since all the groups pairwise commute. Also note that adding a 2–dimensional face to Γ does not introduce a new generator in the fundamental group of $Z_\Gamma(BG_i)$. Hence, if the groups G_i are subgroups of G and Γ is the complete graph on its vertices, the computation above is equivalent to saying that the map

$$Z_\Gamma(BG_i) \longrightarrow BG$$

factors through $Z_{\Delta[k-1]}(BG_i) = \prod_{i=1}^k G_i$, so that we have the following commutative diagram

$$\begin{tikzcd}
Z_\Gamma(BG_i) \arrow[r] \arrow[d] & BG \\
Z_{\Delta[k-1]}(BG_i) \arrow[ur] &
\end{tikzcd}$$

Hence, on the level of fundamental groups the following diagram commutes

$$\begin{tikzcd}
\pi_1(Z_\Gamma(BG_i)) \arrow[r] \arrow[d] & \pi_1(BG) \\
\pi_1(Z_\sigma(BG_i)) \arrow[ur] \arrow[d] & \\
\pi_1(Z_{\Delta[k-1]}(BG_i)) \arrow[uur] &
\end{tikzcd}$$

Therefore,

$$\pi_1(Z_\sigma(BG_i)) \cong \pi_1(Z_{\Delta[k-1]}(BG_i)) \cong \prod_{\Gamma} G_i \cong \prod_{i=1}^{k} G_i.$$

If α is a simplex in K, then α can intersect σ at most at a single top dimensional face. Then,

$$Z_\sigma(BG_i) \cap Z_\alpha(BG_i) = \prod_{i_j \in \sigma\cap\alpha} BG_{i_j}$$

where $|\sigma \cap \alpha| \leq k-1$. Now let $\sigma_1, ..., \sigma_p$ be all the minimal non–faces of K. For σ_1 there is a simplicial subcomplex $K_1 \subseteq K$ not containing σ_1 such that $\sigma_1 \cup K_1 = K$ and $\sigma_1 \cap K_1$ is a flag complex. Since the CW–pairs are $(BG_i, *)$, it follows that

$$Z_{\sigma_1}(BG_i) \cup Z_{K_1}(BG_i) = Z_K(BG_i)$$

and

$$Z_{\sigma_1}(BG_i) \cap Z_{K_1}(BG_i) = Z_{\sigma_1\cap K_1}(BG_i),$$

where $\sigma_1 \cap K_1$ is a flag complex. Thus,

$$\pi_1(Z_{\sigma_1\cap K_1}(BG_i)) = \prod_{S[\sigma_1\cap K_1]_1} G_i.$$

Hence, using Seifert–van Kampen theorem

$$\pi_1(Z_{\sigma_1}(BG_i)) *_{(\prod_{S[\sigma_1\cap K_1]_1} G_i)} \pi_1(Z_{K_1}(BG_i)) = \pi_1(Z_K(BG_i)).$$

Now, to find $\pi_1(Z_{K_1}(BG_i))$ we repeat the same process. Clearly, $\sigma_2 \subseteq K_1$. Then, there is $K_2 \subseteq K_1$ not containing σ_2 such that $\sigma_2 \cup K_2 = K_1$ and $\sigma_2 \cap K_2$ is a flag complex. Similarly,

$$Z_{\sigma_2}(BG_i) \cup Z_{K_2}(BG_i) = Z_{K_1}(BG_i)$$

and

$$Z_{\sigma_2}(BG_i) \cap Z_{K_2}(BG_i) = Z_{\sigma_2 \cap K_2}(BG_i),$$

where $\sigma_2 \cap K_2$ is a flag complex. Thus,

$$\pi_1(Z_{\sigma_2 \cap K_2}(BG_i)) = \prod_{S[\sigma_2 \cap K_2]_1} G_i.$$

Hence, using Seifert–van Kampen theorem

$$\pi_1(Z_{\sigma_2}(BG_i)) *_{(\prod_{S[\sigma_2 \cap K_2]_1} G_i)} \pi_1(Z_{K_2}(BG_i)) = \pi_1(Z_{K_1}(BG_i)).$$

Hence, for σ_q there is

$$K_q \subseteq K_{q-1} \subseteq \cdots \subseteq K_1 \subseteq K$$

not containing $\sigma_q, \sigma_{q-1}, ..., \sigma_1$, such that $\sigma_q \cup K_q = K_{q-1}$ and $\sigma_q \cap K_q$ is a flag complex, for all $1 \leq q \leq p$. Similarly,

$$Z_{\sigma_q}(BG_i) \cup Z_{K_q}(BG_i) = Z_{K_{q-1}}(BG_i)$$

and

$$Z_{\sigma_q}(BG_i) \cap Z_{K_q}(BG_i) = Z_{\sigma_q \cap K_q}(BG_i),$$

where $\sigma_q \cap K_2$ is a flag complex. Thus,

$$\pi_1(Z_{\sigma_q \cap K_q}(BG_i)) = \prod_{S[\sigma_q \cap K_q]_1} G_i.$$

Hence, using Seifert–van Kampen theorem

$$\pi_1(Z_{\sigma_q}(BG_i)) *_{(\prod_{S[\sigma_q \cap K_q]_1} G_i)} \pi_1(Z_{K_q}(BG_i)) = \pi_1(Z_{K_{q-1}}(BG_i)).$$

Let us denote $\prod_{S[\sigma_q \cap K_q]_1} G_i$ by N_q. Therefore, combining all the steps we get

$$\pi_1(Z_K(BG_i)) = \pi_1(Z_{\sigma_1}(BG_i)) *_{N_1} (\pi_1(Z_{\sigma_2}(BG_i) *_{N_2} (\cdots *_{N_p} \pi_1(Z_{K_p}(BG_i)) \cdots)).$$

Note that K_p is a flag complex since it does not contain any of the minimal non–faces $\sigma_1, ..., \sigma_p$, so its fundamental group is $\prod_{i\in S(K_p)_1} G_i$. Let us denote

$$M_q = \pi_1(Z_{\sigma_q}(BG_i)) = \prod_{i\in S(\sigma_q)_1} G_i.$$

Hence,

$$\pi_1(Z_K(BG_i)) = M_1 *_{N_1} (M_2 *_{N_2} (\cdots (M_p *_{N_p} (\prod_{i\in S(K_p)_1} G_i)) \cdots)).$$

Therefore,

$$\pi_1(Z_K(BG_i)) \cong \prod_{SK_1} G_i.$$

□

Another way to prove Proposition 2.15 is by comparing fibrations.

Second proof of Proposition 2.15 . : Consider the following commutative diagram of spaces

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & Z_{SK_q}(EG_i, G_i) \arrow[r] \arrow[d] & Z_K(EG_i, G_i) \arrow[d] \\
F \arrow[r] \arrow[d] & Z_{SK_q}(BG_i) \arrow[r] \arrow[d] & Z_K(BG_i) \arrow[d] \\
* \arrow[r] & \prod_{i=1}^n BG_i \arrow[r] & \prod_{i=1}^n BG_i
\end{tikzcd}$$

where F is the homotopy fibre and SK_q is the q–skeleton of K and $1 \leq q \leq n$. Hence, the homotopy fibre F of the inclusion

$$Z_{SK_q}(EG_i, G_i) \hookrightarrow Z_K(EG_i, G_i)$$

is the same as the homotopy fibre F of the inclusion

$$Z_{SK_q}(BG_i) \hookrightarrow Z_K(BG_i).$$

It suffices to show that on the level of fundamental groups, the induced map

$$\pi_1(Z_{SK_1}(EG_i, G_i)) \hookrightarrow \pi_1(Z_K(EG_i, G_i))$$

is an injection. This is true since adding 2–dimensional or higher dimensional faces to the q–skeleton, for $q \geq 1$, adds 3–dimensional or higher dimensional cells to the space $Z_{SK_q}(EG_i, G_i)$, and adding these cells to the space does not introduce new elements in the fundamental group, see [Hatcher, 2002]. This clearly holds for the case when K is a flag complex, since in that case K has no minimal non–faces. □

A special case is the 0–simplex on n vertices, that is, the simplicial complex consisting only of vertices 1 through n. This case will be treated next in detail.

Before stating the theorem, it is important to note that the functor Z_K is a homotopy functor, which means that for fixed K the homotopy type of $Z_K(\underline{X}, \underline{A})$ depends only on the relative homotopy type of the pairs $(\underline{X}, \underline{A})$. This fact was also observed in [Denham and Suciu, 2007].

Let K be the 0–skeleton of $\Delta[n-1]$, that is, $K = \{\{1\}, ..., \{n\}\}$. Let $X_i = EG_i$ and $A_i = G_i$, where G_i are finite discrete groups for all i. Note that the CW-pairs (EG_i, G_i) have the relative homotopy type (not G_i-equivariant) of the pairs $([0,1], F_i)$, for all i, where $|F_i| = |G_i|$ and F_i is a finite subset of the unit interval [0,1], for all i. The following result will be used when talking about monodromy.

Proposition 2.16. *Let K be the zero skeleton of $\Delta[r-1]$ and F_i be finite subsets of $[0,1]$ with $|F_i| = m_i$, for $1 \leq i \leq r$. Then $Z_K([0,1], F_i) \simeq \vee_{N_r} S^1$, where N_r is defined inductively as follows:*

$$N_2 = (m_1 - 1)(m_2 - 1)$$

$$N_r = m_r N_{r-1} + (m_r - 1)\left(\prod_{i=1}^{r-1} m_i - 1\right) \text{ for } r \geq 3.$$

Proof. Recall that if T is a spanning tree of a connected graph Γ on finitely many vertices, then collapsing T to a point does not change the homotopy type of Γ. Γ/T has only one vertex and has the homotopy type of a finite wedge of circles, and so does Γ; see [Hatcher, 2002]. It suffices to find the number of circles N_r and the proof is given by induction on r.

$r = 2$: $Z_K(([0,1], F_1), ([0,1], F_2))$ contains a maximal tree which we call T_2 defined in the following way: It starts at the point $(0,0) \in [0,1] \times [0,1]$ and runs parallel to the first coordinate and goes to the next level by using one of the extreme vertical edges. T_2 contains all the vertices and has no loops, hence it is a spanning tree. There are $N_2 = (m_1 - 1)(m_2 - 1)$ edges not in T_2. Figure 2.1 shows a version of T_2 for given m_1 and m_2.

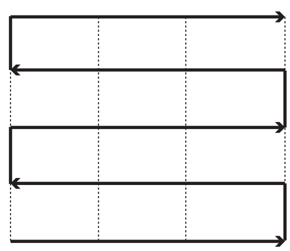

Figure 2.1: $T_2, r = 2$

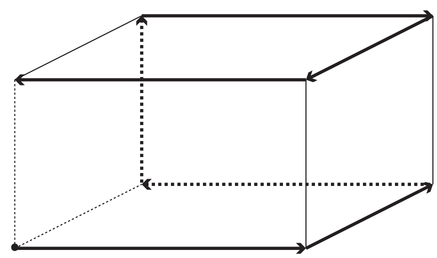

Figure 2.2: $T_3, r = 3$

$r = 3$: $Z_K(([0,1], F_1), ([0,1], F_2), ([0,1], F_3))$ contains a maximal tree called T_3 (see figure 2.2) defined in the similar way as above: on each level parallel to the xy-plane it is the same as T_2 and it needs a vertical edge to jump to the next dimension each time. There are m_3 levels, each having N_2 edges not in T_3, and there are $m_3 - 1$ spaces between levels, each having $\prod_{i=1}^{2}(m_i) - 1$ edges not in T_3. Therefore there are $N_3 = m_3 N_2 + (m_3 - 1)(\prod_{i=1}^{2}(m_i) - 1)$.

Assume true for $r = n$, that is, $N_n = m_n N_{n-1} + (m_n - 1)(\prod_{i=1}^{n-1}(m_i) - 1)$. For $r = n+1$ there is an inclusion $Z_K(([0,1], F_1), ([0,1], F_2), ..., ([0,1], F_{n+1})) \subseteq \mathbb{R}^{n+1}$. Set

$$A_n = Z_K(([0,1], F_1), ([0,1], F_2), ..., ([0,1], F_n)) \subseteq \mathbb{R}^n,$$

then $A_n \simeq \bigvee_{N_n} S^1$. The following diagram describes A_{n+1}, where between any

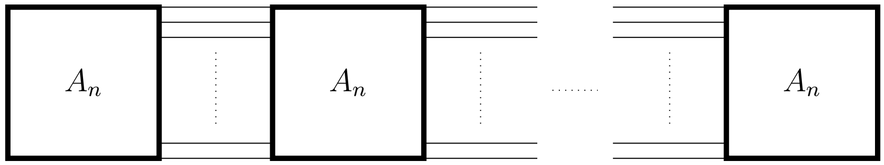

Figure 2.3: A_{n+1}

two consecutive A_n's there are $\prod_{i=1}^{n} m_i$ edges. Each A_n has the maximal tree T_n and the number of edges in A_n not in T_n is N_n. One edge is used between two consecutive A_n's to complete the graph T_{n+1}, so there are $(\prod_{i=1}^{n} m_i) - 1$ edges not in T_{n+1}. Hence the total number of edges not in T_{n+1} is $N_{n+1} = m_{n+1}N_n + (m_{n+1} - 1)(\prod_{i=1}^{n} m_i - 1)$. $\square$

Corollary 2.17. *The value of N_r in Proposition 2.16 is*

$$N_r = (r-1)\prod_{i=1}^{r} m_i - \sum_{i=1}^{r}(\prod_{j\neq i} m_j) + 1.$$

Proof. The proof follows by induction on r. For $r = 2$ then $(m_1 - 1)(m_2 - 1) = m_1m_2 - (m_1 + m_2) + 1$. Now assume this is true for $r = n$, then for $r = n+1$ it follows from Lemma 2.13 that

$$N_{n+1} = m_{n+1}N_n + (m_{n+1} - 1)(\prod_{i=1}^{n} m_i - 1),$$

where by assumption N_n equals

$$N_n = (n-1)\prod_{i=1}^{n} m_i - \sum_{i=1}^{n}(\prod_{j\neq i} m_j) + 1.$$

Substituting this value for N_n and rearranging the terms shows that

$$N_{n+1} = (n)\prod_{i=1}^{n+1} m_i - \sum_{i=1}^{n+1}(\prod_{j\neq i} m_j) + 1.$$

$\square$

Consider the Denham and Suciu (2.1) for K the 0–simplex, to get the following fibration

$$Z_K(EG_i, G_i) \simeq \bigvee_{N_n} S^1 \longrightarrow \bigvee_{1\leq i\leq n} BG_i \longrightarrow BG_i \times \cdots \times BG_n.$$

Each of the spaces in the fibration is an Eilenberg–Mac Lane space, hence there is a short exact sequence of groups

$$1 \longrightarrow F[x_1, ..., x_{N_n}] \longrightarrow G_1 * \cdots * G_n \longrightarrow G_1 \times \cdots \times G_n \longrightarrow 1.$$

where the rank of the free group in the kernel is N_n. This gives a topological proof of an early result of J. Nielsen [Nielsen, 1948], that computes the rank of free group in the kernel of the short exact sequence above.

Now we turn our focus to the case when K is a flag complex. As stated in the proof of Proposition 2.15, if K is a flag complex and G_i are finite discrete groups for $1 \leq i \leq n$, then $Z_K(BG_i)$ is an Eilenberg–Mac Lane space. By a simple argument, the converse of this statement is also true. The following lemma proves it.

Lemma 2.18. *If K is not a flag simplicial complex, then $Z_K(BG_i)$ is not an Eilenberg–Mac Lane space.*

Proof. Recall that from Lemma 2.13 it follows that $\pi_q(Z_K(EG_i, G_i)) \cong \pi_q(Z_K(BG_i))$ for $q \geq 2$. So it suffices to show that for K not a flag complex, there is a non–trivial higher homotopy group of $Z_K(EG_i, G_i)$. Also recall that if K is not a flag complex, then it has a minimal non–face σ, say on k vertices for $3 \leq k \leq n$. There is an inclusion

$$i : Z_\sigma(EG_i, G_i) \hookrightarrow Z_K(EG_i, G_i),$$

where for each vertex of K missing in σ, we put the basepoint of the space BG_i. There is also a surjection

$$s : Z_K(EG_i, G_i) \twoheadrightarrow Z_\sigma(EG_i, G_i)$$

obtained by projecting the coordinates corresponding to the vertices not in σ to the basepoint. The composition $s \circ i$ is the identity on $Z_\sigma(EG_i, G_i)$ and hence, induces the identity on π_q, the q–th homotopy group, for all q. It suffices now to show that $Z_\sigma(EG_i, G_i)$ has at least one non–trivial higher homotopy group. Next we show that $Z_\sigma(EG_i, G_i)$ is actually a wedge of $\prod_{i=1}^{k} m_i - 1$ copies of $(k-1)$–dimensional spheres, where $|G_i| = m_i$. This follows by induction on $k \geq 3$.

For $k = 3$, let $|G_i| = m_i$ for $i = 1, 2, 3$. Then σ is the boundary of the 2–simplex, that is , $\sigma = \{\{1,2\},\{1,3\},\{2,3\}\}$ and

$$\begin{aligned} Z_\sigma(EG_i, G_i) &= (D^1 \times D^1 \times G_3) \cup (D^1 \times G_2 \times D^1) \cup (G_1 \times D^1 \times D^1) \\ &= D^1 \times (\underbrace{(D^1 \times G_3) \cup (G_2 \times D^1)}_{\text{wedge of circles}}) \cup (G_1 \times D^1 \times D^1). \end{aligned}$$

The wedge of circles in the union above consists of $(m_2 - 1)(m_3 - 1)$ circles (see figure 2.1). The union is equivalent to a wedge of $(m_1 - 1)$ copies of the suspension of $(D^1 \times G_3) \cup (G_2 \times D^1)$. One can think of the simple case when the groups are of order 2 and in that case a single sphere is obtained. That is, in general

$$\begin{aligned} Z_\sigma(EG_i, G_i) &= \bigvee_{(m_1-1)} \Sigma((D^1 \times G_3) \cup (G_2 \times D^1)) \\ &= \bigvee_{(m_1-1)} \Sigma(\bigvee_{(m_2-1)(m_3-1)} S^1) \\ &= \bigvee_{(m_1-1)(m_2-1)(m_3-1)} S^2. \end{aligned}$$

Assume for n, then for $n+1$ σ has $n+1$ vertices and the polyhedral product becomes

$$\begin{aligned} Z_\sigma(EG_i,G_i) &= (D^1\times\cdots\times D^1\times G_{n+1})\cup(D^1\times\cdots\times D^1\times G_n\times D^1)\cup\cdots\cup\\ &\cup(G_1\times D^1\times\cdots\times D^1)\\ &= D^1\times(\underbrace{(D^1\times\cdots\times D^1\times G_n)\cup\cdots\cup(G_1\times D^1\times\cdots\times D^1)}_{Z_{\sigma'}(EG_i,G_i)})\cup\\ &\cup(G_1\times D^1\times\cdots\times D^1)\\ &= (D^1\times Z_{\sigma'}(EG_i,G_i))\cup(G_1\times D^1\times\cdots\times D^1), \end{aligned}$$

where σ' is the boundary of the simplex $\Delta[n-1]$ on vertices $\{2,...,n+1\}$. By assumption $Z_{\sigma'}(EG_i,G_i)$ is homotopy equivalent to a wedge of $\prod_{i=2}^{n+1} m_i - 1$ copies of $(n-1)$–dimensional spheres. Therefore, $Z_\sigma(EG_i,G_i)$ is homotopy equivalent to the wedge sum of (m_1-1) copies of the suspension of $\prod_{i=2}^{n+1} m_i - 1$ copies of $(n-1)$–dimensional spheres. That is

$$Z_\sigma(EG_i,G_i)\simeq\bigvee_{m_1-1}\Sigma(\bigvee_{\prod_{i=2}^{n+1}m_i-1}S^{n-1})\simeq\bigvee_{\prod_{i=1}^{n+1}m_i-1}S^n$$

This proves the claim.

Now, it follows that for $k\geq 3$ the space $Z_\sigma(EG_i,G_i)$ has non–trivial higher homotopy groups. Therefore, the lemma follows. $\square$

Combining this result and the result of Davis and Okun [Davis and Okun, 2012], the following is an immediate corollary

Corollary 2.19. *Let G_i be finite discrete groups and K be a simplicial complex on n vertices. Then $Z_K(BG_i)$ is an Eilenberg-Mac Lane space if and only if K is a flag complex.*

The following corollary also follows immediately from the previous corollary and Lemma 2.13.

Corollary 2.20. *Let G_i be finite discrete groups and K be a simplicial complex on n vertices. Then $Z_K(EG_i, G_i)$ is an Eilenberg-Mac Lane space if and only if K is a flag complex.*

If K is flag, then $Z_K(EG_i, G_i)$ is the classifying space of the kernel of the projection

$$\prod_{SK_1} G_i \longrightarrow\!\!\!\!\!\rightarrow \prod_{i=1}^{n} G_i.$$

Hence, Corollary 2.20 shows that the kernel of this projection is torsion free, since $Z_K(EG_i, G_i) = K(\pi_1(Z_K(EG_i, G_i)), 1)$ is of finite type.

It is possible to carry this discussion to finitely generated discrete groups G_i of infinite order. Recall that if K is a flag complex, then $Z_K(B\mathbb{Z}, *)$ is the classifying space of the right–angled Artin group $\prod_{SK_1} \mathbb{Z}$, see [Davis and Okun, 2012]. Using the Denham and Suciu fibration for $G_i = \mathbb{Z}$, the following is a fibration

$$Z_K(E\mathbb{Z}, \mathbb{Z}) \longrightarrow Z_K(B\mathbb{Z}, *) \longrightarrow \prod_{n} B\mathbb{Z},$$

where $B\mathbb{Z} = S^1$ and K is a simplicial complex on n vertices. Hence, there is a fibration

$$Z_K(E\mathbb{Z}, \mathbb{Z}) \longrightarrow Z_K(S^1, *) \longrightarrow \prod_{n} S^1.$$

The pairs $(E\mathbb{Z}, \mathbb{Z})$ have the relative homotopy type of $(\mathbb{R}, \mathbb{Z})$. Similarly, if G_i are finitely generated discrete groups of infinite order, then the pairs (EG_i, G_i) have the relative homotopy type of $(\mathbb{R}, \mathbb{Z})$. This equivalence is not $(\prod G_i)$–equivariant. Hence, $Z_K(E\mathbb{Z}, \mathbb{Z})$ is an Eilenberg–Mac Lane space if and only if $Z_K(EG_i, G_i)$ is an Eilenberg–Mac Lane space and that is true if and only if $Z_K(S^1, *)$ s an Eilenberg–Mac Lane space.

As mentioned above if K is a flag complex, then $Z_K(S^1, *)$ is an Eilenberg–Mac Lane space, thus $Z_K(EG_i, G_i)$ is such a space as well. Next by showing that if K is not a flag complex then $Z_K(E\mathbb{Z}, \mathbb{Z})$ is not an Eilenberg–Mac Lane space, it will follow that $Z_K(EG_i, G_i)$ is not an Eilenberg–Mac Lane space, either.

Proposition 2.21. *If K is not a flag complex, then $Z_K(E\mathbb{Z}, \mathbb{Z})$ is not a $K(\pi, 1)$.*

Proof. If K is a flag complex, then it has a minimal non–face σ on $k \geq 3$ vertices. Let $0 \in \mathbb{R}$ be the basepoint. If F is the finite set of integers $\{0, 1\}$, then there is an embedding

$$Z_\sigma([0,1], F) \hookrightarrow Z_\sigma(\mathbb{R}, \mathbb{Z})$$

obtained by the inclusion of pairs $([0,1], F) \hookrightarrow (\mathbb{R}, \mathbb{Z})$. There is an equivalence $Z_\sigma([0,1], F) \simeq S^{k-1}$. Therefore, $Z_K(E\mathbb{Z}, \mathbb{Z})$ has non–trivial higher homotopy groups. The rest of this proof is similar to the proof of Lemma 2.18 □

An immediate corollary of Proposition 2.21 and [Davis and Okun, 2012] is the following

Corollary 2.22. *Let G_i be finitely generated infinite discrete groups. Then, $Z_K(EG_i, G_i)$ is an Eilenberg–Mac Lane space if and only if K is a flag complex. Similarly, this is true if and only if $Z_K(BG_i, *)$ is an Eilenberg–Mac Lane.*

Note that in the proof of Proposition 2.21 it is actually not required that G_i are infinite, or that they are finitely generated. The only requirement is that G_i have at most the cardinality of $\mathbb{N}$, which is denoted by $\aleph_0$, and have the discrete topology.

Theorem 2.23. *Let G_i be groups with $2 \leq |G_i| \leq \aleph_0$ and endowed with the discrete topology, for all i. Let K be a simplicial complex on a finite set of vertices. Then $Z_K(BG_i, *)$ is an Eilenberg–Mac Lane space if and only if K is a flag complex.*

Proof. This is the same as the proof of Proposition 2.21. □

2.4 Homotopy groups of other polyhedral products

Let X be a path–connected, finite dimensional and pointed simplicial complex. Then it is a classical result that X is the classifying space of a topological group G, which can be described precisely, see for example [Milnor, 1956]. Hence, we can write $X \simeq BG$. This gives a homotopy equivalence $\Omega X \simeq G$, which implies an equivalence between the cone of the spaces $CG \simeq C(\Omega X)$. Let $*$ be the basepoint of EG. There is a commutative diagram of spaces

$$\begin{array}{ccc} EG \times C(G) & \xrightarrow{\simeq} & C(\Omega X) \\ \uparrow & & \uparrow \\ * \times G & \xrightarrow{\simeq} & \Omega X. \end{array}$$

Hence, there is a homotopy equivalence of pairs $(EG, G) \simeq (C\Omega X, \Omega X)$. The Denham and Suciu fibration (2.1) also works in this setting. Let K be a simplicial complex on n vertices. Then, there is a fibration

$$Z_K(EG, G) \longrightarrow Z_K(X, *) \longrightarrow X^n,$$

which can also be written as

$$Z_K(C\Omega X, \Omega X) \longrightarrow Z_K(X, *) \longrightarrow X^n,$$

since there is a homotopy equivalence of pairs $(EG, G) \simeq (C\Omega X, \Omega X)$. This is an instance of the more general case of a sequence of CW–pairs $(\underline{X}, \underline{*})$. Hence, there is a fibration

$$Z_K(\underline{C\Omega X}, \underline{\Omega X}) \longrightarrow Z_K(\underline{X}, \underline{*}) \longrightarrow \prod_{i=1}^{n} X_i.$$

Assume that G is path–connected. That means $\pi_0(G) = 1$ and $\pi_1(X) = \pi_0(\Omega X) = \pi_0(G) = 0$. That is, assume that X is 1–connected. The goal of this section is to prove that if X is 1–connected, then $Z_K(EG, G)$ is 1–connected.

Definition 2.24. A simplicial complex K on n vertices is *shifted* if there is a labelling of the vertices by 1 through n such that for any face, replacing any vertex of that face with a vertex of smaller label and not in that face results in a collection which is also a face.

Geometrically, a shifted complex is the geometric realization of K and it is homotopy equivalent to a wedge of spheres. In case K is a shifted complex, a conjecture of Bahri *etal.* [Bahri *et al.*, 2010] was proved by Grbic and Theriault [Grbić and Theriault,] that there is a homotopy equivalence

$$Z_K(\underline{CY}, \underline{Y}) \simeq \bigvee_{I \notin K} |K_I| * \widehat{Y}^I,$$

where I is a sequence of integers not in K, $|K_I|$ is the realization of $K_I = \{\sigma \cap I | \sigma \in K\}$ and $\widehat{Y}^I$ is the smash product of Y_i for $i \in I$. In this case it follows that if X is 1–connected, then $Z_K(\underline{C\Omega X}, \underline{\Omega X})$ is 1–connected.

Theorem 2.25. *Let X be a 1–connected CW–complex. Then the polyhedral product $Z_K(C\Omega X, \Omega X)$ is 1–connected.*

Proof. As mentioned above, if X is a 1–connected simplicial complex, then $X \simeq BG$ and G is path-connected, see [Milnor, 1956]. Moreover, there is an equivalence $Z_K(C\Omega X, \Omega X) \simeq Z_K(EG, G)$, where EG and G are path–connected. Hence, $Z_K(EG, G)$ is path–connected. In this proof we will work with $Z_K(EG, G)$.

To show that $\pi_1(Z_K(EG, G)) = 0$, the definition of the polyhedral product will be used. Recall that

$$Z_K(EG, G) = \operatorname*{colim}_{\sigma \in K} D(\sigma),$$

where $D(\sigma)$ is a product of EG's and G's. Also recall that $G \times \cdots \times G = D(\emptyset) \subset D(\sigma)$, for all $\sigma \in K$. For two simplices $\tau, \sigma \in K$ consider the pushout diagram

$$\begin{tikzcd}
D(\tau) \cap D(\sigma) \arrow[rr, "i"] \arrow[d, "j"'] & & D(\tau) \arrow[d] \\
D(\sigma) \arrow[rr] & & D(\tau) \cup_{D(\tau)\cap D(\sigma)} D(\sigma),
\end{tikzcd}$$

where $D(\tau) \cup_{D(\tau)\cap D(\sigma)} D(\sigma)$ is the colimit of the two maps emanating from the intersection $D(\tau)\cap D(\sigma)$. Using Seifert–van Kampen theorem for the fundamental group, it follows that

$$\pi_1(D(\tau) \cup_{D(\tau)\cap D(\sigma)} D(\sigma)) = \pi_1(D(\tau)) *_N \pi_1(D(\sigma)),$$

where N is the subgroup generated by the images of the fundamental group of the intersection under the induced maps of i and j in π_1. Let the set $V_{\sigma,\tau} = \{v_1, ..., v_t\}$ be the maximal set of vertices in K such that $V_{\sigma,\tau} \cap \sigma = \emptyset$ and $V_{\sigma,\tau} \cap \tau = \emptyset$. Clearly, $\pi_1(D(\tau)) *_N \pi_1(D(\sigma)) = \pi_1(G^t)$, since i and j induce monomorphisms in π_1 and the images of these two maps do not hit the coordinates corresponding to the vertices in $V_{\sigma,\tau}$.

K contains only a finite number of simplices $\tau_1, ..., \tau_k$. Let $V_{i_1,\ldots,i_l}$ be the maximal set of vertices in K such that $V_{i_1,\ldots,i_l} \cap \tau_{i_j} = \emptyset$ for $1 \leq j \leq l \leq k$. As explained above

$$\pi_1(D(\tau_1) \cup_{D(\tau_1)\cap D(\tau_2)} D(\tau_2)) = \pi_1(D(\tau_1)) *_{N_1} \pi_1(D(\tau_2)) = \pi_1(G^{|V_{1,2}|}).$$

To complete the proof, first perform the computation by taking the colimit with more simplices until all simplices $\tau_1, .., \tau_{k-1}$ are used. It follows that

$$\pi_1(\operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i)) = \pi_1(G^{|V_{1,\ldots,k-1}|}).$$

In the last step, the polyhedral product equals

$$Z_K(EG, G) = \operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i) \cup_{\operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i)\cap D(\tau_k)} D(\tau_k).$$

Thus, the fundamental group equals

$$\pi_1(\operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i) \cup_{\operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i)\cap D(\tau_k)} D(\tau_k)) = \pi_1(\operatorname*{colim}_{1\leq i\leq k-1} D(\tau_i)) *_{N_{k-1}} \pi_1(D(\tau_k)).$$

Since $V_{1,\ldots,k} = \emptyset$, it follows that $\pi_1(Z_K(EG, G)) = 0$.

$\square$

A more delicate question is the case when a topological group G acts freely and properly discontinuously on a CW–complex Y and $p : Y \to X = Y/G$ is a bundle projection. There is a lemma due to Denham and Suciu [Denham and Suciu, 2007] which describes the fibre when comparing certain fibrations involving polyhedral products.

Lemma 2.26. *Let $p : (E, E') \to (B, B')$ be a map of pairs, such that both $p : E \to B$ and $p' = p|_{E'} : E' \to B'$ are fibrations with fibres F and F' respectively. Suppose that either $F = F'$ or $B = B'$. Then the product fibration $p^{\times n} : E^n \to B^n$ restricts to a fibration*

$$Z_K(F, F') \longrightarrow Z_K(E, E') \xrightarrow{Z_K(p)} Z_K(B, B'). \tag{2.2}$$

Moreover, if $(F, F') \to (E, E') \to (B, B')$ is a relative bundle (with structure group G), and either $F = F'$ or $B = B'$, then (2.2) is also a bundle (with structure group G^n).

Now, consider the relative map $p : (Y, G) \to (Y/G, *)$ with fibres $F = F' = G$. Thus, there is a fibration

$$G^n \longrightarrow Z_K(Y, G) \longrightarrow Z_K(Y/G, *).$$

Pushing the fibre to the right by taking its classifying space, one gets a fibration

$$Z_K(Y, G) \longrightarrow Z_K(Y/G, *) \longrightarrow BG^m.$$

In this case it is not true in general that $Z_K(Y, G)$ is 1–connected, as shown in the next example.

Example 2.27. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on the 2–sphere S^2 by the antipodal map. Then the orbit space is $S^2/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}P^2$. Hence there is a fibration

$$Z_K(S^2, \mathbb{Z}/2\mathbb{Z}) \longrightarrow Z_K(\mathbb{R}P^2, *) \longrightarrow B(\mathbb{Z}/2\mathbb{Z})^m.$$

If K is the 0–skeleton of the 1–simplex, $\{\{1\}, \{2\}\}$, then $Z_K(S^2, \mathbb{Z}/2\mathbb{Z}) = (S^2 \times \mathbb{Z}/2\mathbb{Z}) \cup (\mathbb{Z}/2\mathbb{Z} \times S^2)$ is equivalent to the wedge sum $(\bigvee_4 S^2) \vee S^1$. Hence, $Z_K(S^2, \mathbb{Z}/2\mathbb{Z})$ is not simply connected.

Nevertheless, the following theorem gives that the fundamental group of these polyhedral products can be computed with the information that was obtained from Section 2.3.

Lemma 2.28. *If $A \subset X \subset Y$ with A finite discrete, X and Y path–connected, such that the induced map $\pi_i(X) \xrightarrow{i_\#} \pi_i(Y)$ is an isomorphism for $i = 0, 1$, then*

$$\pi_1(Z_K(X, A)) \cong \pi_1(Z_K(Y, A)).$$

Proof. Let $D_X(\sigma)$ denote the functor D evaluated on the pair (X, A) and $D_Y(\sigma)$ denote the functor D evaluated on the pair (Y, A). If $\sigma \in K$ is a simplex and if $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i = 0, 1$, then the inclusion map $D_X(\sigma) \hookrightarrow D_Y(\sigma)$ induces an isomorphism on the fundamental group. By definition,

$$\pi_1(Z_K(X, A)) = \pi_1(\operatorname*{colim}_{\sigma \in K} D_X(\sigma))$$

and

$$\pi_1(Z_K(Y, A)) = \pi_1(\operatorname*{colim}_{\sigma \in K} D_Y(\sigma)).$$

Recall that if X_2 is the 2–skeleton of X, then $\pi_1(X) \cong \pi_1(X_2)$. Similarly, if Y_2 is the 2–skeleton of Y, then $\pi_1(Y) \cong \pi_1(Y_2)$. So this shows that $\pi_1(X_2) \cong \pi_1(Y_2)$. Similarly, it suffices to work with the 2–skeleton of the spaces $Z_K(X, A)$ and $Z_K(Y, A)$. Denote these two spaces by $Z_K(X, A)_2$ and $Z_K(Y, A)_2$, respectively. It is clear that there are inclusions up to homotopy $Z_K(X, A)_2 \hookrightarrow Z_{SK_1}(X_2, A)$

and $Z_K(Y,A)_2 \hookrightarrow Z_{SK_1}(Y_2,A)$ since adding a simplex σ on $k \geq 3$ vertices, does not change the 2–skeleton of the polyhedral product. Both inclusions induce isomorphisms in π_1. This suffices. $\square$

Note that Lemma 2.28 can be also proved using the fact that there is a homotopy equivalence

$$Z_K(X,A)) = \operatorname*{colim}_{\sigma \in K} D_X(\sigma) \simeq \operatorname*{hocolim}_{\sigma \in K} D_X(\sigma),$$

see [Bahri *et al.*, 2010], and the fact that

$$\pi_1(\operatorname*{hocolim}_{\sigma \in K} D_X(\sigma)) \cong \operatorname*{colim}_{\sigma \in K} \pi_1(D_X(\sigma)),$$

see [Farjoun, 2004] for details.

Next consider the pair (BG, BH), where H is a closed subgroup of G. There is an inclusion $G/H \hookrightarrow C(G/H)$ which gives an inclusion $EG \times G/H \hookrightarrow EG \times C(G/H)$. Hence, G/H can be regarded as a G–equivariant subspace of EG. There is a fibration

$$Z_K(EG, G/H) \longrightarrow (EG)^n \times_{G^n} Z_K(EG, G/H) \longrightarrow (BG)^n.$$

Since $EG \times G/H \simeq G/H$ and $EG \times_G G/H \cong EG/H \cong BH$, then the space $(EG)^n \times_{G^n} Z_K(EG, G/H)$ is homotopy equivalent to $Z_K(EG/G, EG \times_G G/H) \simeq Z_K(BG, BH)$. This proves the following proposition

Proposition 2.29. *Let H be a closed subgroup of the Lie group G. There is a fibration given by*

$$Z_K(EG, G/H) \longrightarrow Z_K(BG, BH) \longrightarrow (BG)^n.$$

Proposition 2.30. *Let H be a closed subgroup of G. Then there is a splitting*

$$\Omega(Z_K(BG, BH)) \simeq G^n \times \Omega(Z_K(EG, G/H))$$

and a short exact sequence of groups

$$1 \to \pi_1(Z_K(EG, G/H)) \to \pi_1(Z_K(BG, BH)) \to G^n \to 1.$$

Proof. There is a fibration as follows

$$Z_K(EG, G/H) \longrightarrow Z_K(BG, BH) \longrightarrow (BG)^n.$$

Taking the loops of this fibration gives a fibration

$$\Omega(Z_K(EG, G/H)) \longrightarrow \Omega(Z_K(BG, BH)) \longrightarrow \Omega((BG)^n) = G^n.$$

There is a section $G^n \xrightarrow{s} \Omega(Z_K(BG, BH))$ which implies that there is a homotopy equivalence of topological spaces

$$\Omega(Z_K(BG, BH)) \simeq G^n \times \Omega(Z_K(EG, G/H)).$$

The short exact sequence follows from this equivalence. □

2.5 Monodromy representation

Let G_i be finite discrete groups of order m_i, for $1 \leq i \leq n$. In this section we are interested in describing the monodromy representation corresponding to the fibration in (2.1)

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \longrightarrow \prod_{i=1}^{n} BG_i.$$

Recall that the homotopy type of the polyhedral product $Z_K(\underline{X}, \underline{A})$ depends only on the relative homotopy type of the pairs $(\underline{X}, \underline{A})$.

Lemma 2.31. *Let G be a finite discrete group of order m. Then there is a relative homotopy equivalence $(EG, G) \sim ([0, 1], F)$, where F is a subset of $[0, 1]$ of cardinality m.*

Proof. By definition EG is contractible and the group G can be identified with the orbit of a point $x \in EG$, since G acts freely on EG. This gives an equivalence of CW–pairs $(EG, G) \simeq ([0, 1], F) = (I, F)$, where $F = \{f_1 < ... < f_m\} \approx \{(1 = g_1) \cdot x = x, g_2 \cdot x, ..., g_m \cdot x\}$ is a finite subset of I with the same cardinality as G. One can pick a path $\gamma : [0, 1] \to EG$ such that $\gamma(f_i) = g_i \cdot x$. □

Hence, there is a homotopy equivalence $Z_K(EG_i, G_i) \simeq Z_K(I, F_i)$.

If $K = K_0$ is the zero skeleton of the $(n-1)$–simplex, then $Z_{K_0}(I, F_i)$ is a graph in the space $[0,1]^n \subset \mathbb{R}^n$ (see figure 2.2) and hence, has the homotopy type of a wedge of N_n circles. The number N_n was shown in Proposition 2.16 and Corollary 2.17 to be

$$N_n = (n-1)\prod_{i=1}^{n} - \sum_{i=1}^{n} \big(\prod_{j \neq i} m_j\big) + 1.$$

It will be clear that computing the monodromy representation in general is an involved task, and we will restrict the computations to finite cyclic groups. Using the description of $Z_K(EG_i, G_i)$ from Proposition 2.16 allows for a clear description of monodromy, where possible. The importance of this representation lies in the fact that it gives some information about elements in $\mathrm{Out}(F_n)$.

2.6 The generators of the fundamental group

Let K_0 be the 0–simplicial complex on n vertices. In this section explicit loops in $Z_{K_0}(I, F_i)$ are found, whose equivalence classes constitute a generating set for the fundamental group. These loops represent classes of loops that are elements in the kernel of the following short exact sequence of groups

$$1 \longrightarrow F_{N_n} \longrightarrow G_1 * \cdots * G_n \longrightarrow G_1 \times \cdots \times G_n \longrightarrow 1,$$

where G_i are finite discrete groups of order m_i. Using these loops we will give an explicit description of the monodromy action of $G_1 \times \cdots \times G_n$ on the fiber $Z_K(EG_i, G_i)$ for cyclic groups $G_i = \langle x_i | x_i^{m_i} = 1\rangle$.

Recall that the homotopy type of $Z_K(EG_i, G_i)$ depends only on the cardinality of G_i. Hence, when finding the loops for $Z_K(EG_i, G_i)$ for cyclic groups, the same computation holds for any collection of groups with the same order, that is the same loops will be used to describe the monodromy. However, the representation depends on the structure of the groups.

The loops will be chosen as follows: Let the point $(0, 0, ..., 0) = *$ be the basepoint of $Z_{K_0}(I, F_i)$. Starting from the basepoint $*$, each path in $Z_{K_0}(I, F_i)$ will be tracked by a word $x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_r}^{j_r}$, where $x_{i_k}^{j_k} \in G_k$, each letter showing the direction of the group it belongs to, together with the distance taken in that direction. See figure 2.4 for a picture in two dimensions. Always move in the positive direction, when regarding $Z_K(EG_i, G_i)$ as a subspace of $[0, 1]^n \subset \mathbb{R}^n$.

Lemma 2.32. *The path tracked by the word $x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_r}^{j_r}$ is closed if and only if $\sum_{i=1}^{r} j_i = 0$.*

Proof. This can be seen by arguing that, to start and end at the basepoint $*$, if $x_{i_1}^{j_1}$ is a letter of the word, then the letter $x_{i_1}^{-j_1}$ should also appear in the same word, otherwise one can never come back to $*$. Conversely if the sum $\sum_{i=1}^{r} j_i = 0$, then every move forward has been compensated by a move backward. $\square$

2.6.1 The case of two finite groups

Start by first considering cyclic groups. Let G_1 and G_2 be finite cyclic groups with order m_1 and m_2, respectively. Then, $Z_{K_0}(EG_i, G_i) \simeq Z_{K_0}(I, F_i) = I \times F_2 \cup F_1 \times I$ (the dotted grid in figure 2.4). Consider the cycles given by the words $x_1^i x_2^j x_1^{-i} x_2^{-j} = [x_1^i, x_2^j]$, where $1 \leq i \leq m_1 - 1$ and $1 \leq m_2 \leq m_2 - 1$. The following lemma tells which loops suffice.

Lemma 2.33. *The set of words $\mathcal{W} = \{[x_1^i, x_2^j] | 1 \leq i \leq m_1 - 1, 1 \leq j \leq m_2 - 1\}$ generates all the cycles in $Z_{K_0}(I, F_i)$.*

Proof. It suffices to prove that a little square in the grid can be written as a product of these generators. Let γ be the cycle in $Z_{K_0}(I, F_i)$ given by $x_1^i x_2^j [x_1, x_2] x_1^{-i} x_2^{-j}$.

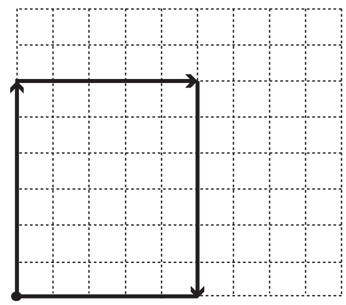

Figure 2.4: 2-dimensional case, the loop $[x_1^6, x_2^5]$

This word can also be written as

$$[x_1^i, x_2^j][x_2^j, x_1^{i+1}][x_1^{i+1}, x_2^{j+1}][x_2^{j+1}, x_1^i].$$

$\square$

Lemma 2.34. *The set of words* $\mathcal{W} = \{[x_1^i, x_2^j] | 1 \leq i \leq m_1 - 1, 1 \leq j \leq m_2 - 1\}$ *is a minimal generating set.*

Proof. First note that $|\mathcal{W}| = (m_1 - 1)(m_2 - 1) = m_1 m_2 - (m_1 + m_2) + 1$. In Proposition 2.16 and Corollary 2.17 it was shown that N_2 equals exactly this number and hence $N_2 = |\mathcal{W}|$. $\square$

Now let H_i be any finite groups with cardinality m_i, for $i = 1, 2$. That is, $H_1 = \{1, h_1, ..., h_{m_1-1}\}$ and $H_2 = \{1, g_1, ..., g_{m_2-1}\}$.

Corollary 2.35. *The set of words* $\mathcal{W}_{\mathcal{H}} = \{[h_i, g_j] | 1 \leq i \leq m_1 - 1, 1 \leq j \leq m_2 - 1\}$ *generates all the cycles in* $Z_{K_0}(EH_i, H_i) = Z_{K_0}(I, F_i)$*. Moreover, this is a minimal generating set.*

Proof. Follows from Lemma 2.33 and 2.34. $\square$

The next problem is to find the action $G_1 \times \cdots \times G_n$ on these generators. We know $G_1 \times \cdots \times G_n$ acts on the fiber by conjugation, i. e. $g \cdot \gamma = g\widetilde{\gamma}g^{-1}$, where

$\widetilde{\gamma}$ is the word corresponding to γ. Pictorially, the action shifts the loop by g, as shown in figure 2.5. Let $G_1 = C_{10}$ and $G_2 = C_9$ be the cyclic groups of order 10 and 9 respectively. The element $x_2^4 \in G_2$ acts on the word $[x_1^2, x_2^5]$ by conjugation

$$x_2^4 \cdot [x_1^2, x_2^5] = x_2^4[x_1^2, x_2^5]x_2^{-4}$$

which is the loop shifted up by $x_2^4 \in G_2$. Section 2.7 gives a thorough description

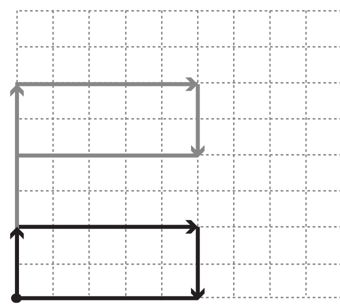

Figure 2.5: x_2^4 acting on $[x_1^2, x_2^5]$

of the action in general.

2.7 The monodromy action

Let $[\gamma] \in F_{N_n}$ be the homotopy class of a loop in $Z_{K_0}(I, F_i)$. Assume that γ corresponds to the word $\omega \in \mathcal{W}_n$. Then $G_1 \times \cdots \times G_n$ acts on the fibre by

$$g \cdot [\gamma] = [g\omega g^{-1}].$$

Call this map φ_g. The goal is to write $g\omega g^{-1}$ as a product of words in $\mathcal{W}_n$, if possible. Then any element $g \in G_1 * \cdots * G_n$ gives an automoprhism of F_{N_n}, the free group on letters the elements of $\mathcal{W}_n$

$$G_1 * \cdots * G_n \xrightarrow{\varphi} \mathrm{Aut}(F_{N_n})$$

$$g \longmapsto \varphi_g,$$

where $\mathrm{Aut}(G)$ is the group of group automorphisms of G, under composition. One example is in Section 2.7.1, where this computation is carried out explicitly.

In general, given a short exact sequence of discrete groups $1 \to A \to B \to C \to 1$, there is a map

$$B \xrightarrow{\Theta} \mathrm{Aut}(B)$$

$$\Theta(g)(h) = ghg^{-1}.$$

Also there is a map

$$A \xrightarrow{\Psi} \mathrm{Inn}(A)$$

$$\Psi(g)(h) = ghg^{-1},$$

where $\mathrm{Inn}(A)$ is the group of *inner automorphisms* of A. Since $A \hookrightarrow B$, then $\mathrm{Inn}(A) \hookrightarrow \mathrm{Aut}(B)$. Moreover, $\mathrm{Inn}(A) \trianglelefteq \mathrm{Aut}(B)$ and $\mathrm{Out}(B) := \mathrm{Aut}(B)/\mathrm{Inn}(A)$ is called the group of *outer automorphisms* of B.

One can also show that $G \cong \mathrm{Inn}(G)$ if and only if for any $g \in G$, there is $h \in G$ such that $ghg^{-1} \neq h$, that is $Z(G) = 1$. Hence, for example $F_n \cong \mathrm{Inn}(F_n)$.

For $B = F_n$ and $n \geq 2$, there is a short exact sequence of groups

$$1 \longrightarrow \mathrm{Inn}(F_n) \longrightarrow \mathrm{Aut}(F_n) \longrightarrow \mathrm{Out}(F_n) \longrightarrow 1$$

and hence, a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & F_{N_n} \arrow[r] \arrow[d] & G_1 * \cdots * G_n \arrow[r] \arrow[d] & G_1 \times \cdots \times G_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Inn}(F_{N_n}) \arrow[r] & \mathrm{Aut}(F_{N_n}) \arrow[r] & \mathrm{Out}(F_{N_n}) \arrow[r] & 1.
\end{tikzcd}$$

Hence, the map $G_1 * \cdots * G_n \to \mathrm{Aut}(F_{N_n})$ induces a map $G_1 \times \cdots \times G_n \to \mathrm{Out}(_{N_n})$. Since the kernel of the short exact sequence in the first row of the diagram above is a free group, it follows that $F_{N_n} \cong \mathrm{Inn}(F_{N_n})$.

There is also another short exact sequence

$$1 \longrightarrow \mathrm{IA}_n \longrightarrow \mathrm{Aut}(F_n) \xrightarrow{\mathrm{ab}} GL_n(\mathbb{Z}) \longrightarrow 1$$

with kernel the group IA_n, which is the subgroup of automorphisms that restrict to the identity in the abelianization of F_n, and "ab" is the map induced by the abelianization map $F_n \to F_n/[F_n, F_n] = \mathbb{Z}^n$. In the examples that will be given, none of the homomorphisms restrict to the identity in the abelianization. Thus, these elements are not elements of IA_n.

Example 2.36. Let $G_1 = \mathbb{Z}/2\mathbb{Z} := \mathbb{Z}_2 = \langle x_1 | x_1^2 = 1\rangle$ and $G_2 = \mathbb{Z}/3\mathbb{Z} := \mathbb{Z}_3 = \langle x_2 | x_2^3 = 1\rangle$. Consider the short exact sequence of groups

$$1 \longrightarrow F_2 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow 1,$$

where F_2 is the free group on two generators $\omega_1 = [x_1, x_2]$ and $\omega_2 = [x_1, x_2^2]$.

To compute the map $\Theta : \mathbb{Z}_2 * \mathbb{Z}_3 \to \mathrm{Aut}(F_2)$, we first compute the automorphism $\varphi_{x_1} \in \mathrm{Aut}(F_2)$ by looking at the image of the generators $\omega_1, \omega_2 \in F_2$ under φ_{x_1} to find

$$x_1\omega_1x_1^{-1} = [x_2, x_1] = ([x_1, x_2])^{-1} = \omega_1^{-1}$$

and

$$x_1\omega_2x_1^{-1} = [x_2^2, x_1] = ([x_1, x_2^2])^{-1} = \omega_2^{-1}.$$

Looking at the induced map of φ_{x_1} onto the abelianization $\mathbb{Z} \oplus \mathbb{Z} \cong F_2/[F_2, F_2]$, then

$$\widetilde{\varphi}_{x_1}(\omega_1, \omega_2) = (-\omega_1, -\omega_2)$$

which is given by the matrix

$$[\widetilde{\varphi}_{x_1}] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect the basis $\{\omega_1, \omega_2\}$. Similarly, one can compute $\varphi_{x_2} \in \mathrm{Aut}(F_2)$ by finding

$$x_2\omega_1x_2^{-1} = x_2[x_1, x_2]x_2^{-1} = [x_2, x_1][x_1, x_2^2] = \omega_1^{-1}\omega_2$$

and

$$x_2\omega_2x_2^{-1} = [x_2, x_1] = ([x_1, x_2])^{-1} = \omega_1^{-1}.$$

Looking at the induced map of φ_{x_2} onto the abelianization $\mathbb{Z} \oplus \mathbb{Z} \cong F_2/[F_2, F_2]$, then

$$\widetilde{\varphi}_{x_2}(\omega_1, \omega_2) = (-\omega_1 + \omega_2, -\omega_1)$$

which is given by the matrix

$$[\widetilde{\varphi}_{x_2}] = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect the basis $\{\omega_1, \omega_2\}$. Using properties of group actions, any automorphism $\varphi_g, g \in F_2$ can be found using φ_{x_1} and φ_{x_1}. For example $\varphi_{x_1 x_2} = \varphi_{x_1} \circ \varphi_{x_2}$ and so on. Note that φ_{x_1} and φ_{x_2} are not elements of IA_2 since the functions do not restrict to the identity in the abelianization.

This calculation gives a homomorphism $\mathrm{ab} \circ \Theta : \mathbb{Z}_2 * \mathbb{Z}_3 \to GL_2(\mathbb{Z})$ by composing the homomorphisms

$$\mathbb{Z}_2 * \mathbb{Z}_3 \xrightarrow{\Theta} \mathrm{Aut}(F_2) \xrightarrow{\mathrm{ab}} GL_2(\mathbb{Z}).$$

The map Θ induces a homomorphism $\widetilde{\Theta} : \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow \mathrm{Out}(F_2)$. Moreover, the map $\mathrm{ab} \circ \Theta$ can be considered the same as the composition $p \circ \widetilde{\Theta}$, where p is the projection to the abelianization of $\mathbb{Z}_2 * \mathbb{Z}_3$, since $[\widetilde{\varphi}_{x_1}]$ and $[\widetilde{\varphi}_{x_2}]$ commute.

Example 2.37. Let Σ_3 be the symmetric group on three letters, given by

$$\Sigma_3 = \{1, (12), (13), (23), (123), (132)\}.$$

Let $C_2 = \mathbb{Z}_2 = \{1, x\}$ be the cyclic group with two elements. There is a short exact sequence of groups

$$1 \longrightarrow F_5 \longrightarrow \mathbb{Z}_2 * \Sigma_3 \longrightarrow \mathbb{Z}_2 \times \Sigma_3 \longrightarrow 1,$$

where F_5 is the free group on letters $W = \{[x, g] | x, g \neq 1, x \in \mathbb{Z}_3, g \in \Sigma_2\}$. To calculate the representation $\mathbb{Z}_2 \times \Sigma_3 \to \mathrm{Out}(F_5)$, start with evaluating φ_g for $g \in \mathbb{Z}_2 * \Sigma_3$. Hence, $\varphi_x([x, g]) = [g, x] = [x, g]^{-1}$ for all $g \in \Sigma_3$. After restricting to the abelianization $\widetilde{\varphi}_x([x,g]) = -[x,g]$. Hence, the matrix representation of $\widetilde{\varphi}_x$ is given by $[\widetilde{\varphi}_x] = -I_5$.

Similarly, $\varphi_{(12)}([x,(12)]) = [(12),x]$ and $\varphi_{(12)}([x,g]) = [(12),x][x,(12)\cdot g]$ if $g \neq (12)$. In the abelianization, we get $\widetilde{\varphi}_{(12)}([x,(12)]) = -[x,(12)]$ and $\widetilde{\varphi}_{(12)}([x,g]) = -[x,(12)] + [x,(12)g]$. Order the basis as follows

$$W = \{[x,(12)],[x,(13)],[x,(23)],[x,(123)],[x,(132)]\}.$$

Then the matrix representation for $\widetilde{\varphi}_{(12)}$ is

$$[\widetilde{\varphi}_{(12)}] = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

One can find the other automorphisms similarly, since $\varphi_g([x,h]) = [g,x][x,gh]$. Hence,

$$[\widetilde{\varphi}_{(13)}] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, [\widetilde{\varphi}_{(23)}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[\widetilde{\varphi}_{(123)}] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, [\widetilde{\varphi}_{(132)}] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

Note that these matrices do not commute in general. For example $[\widetilde{\varphi}_{(13)}]\cdot[\widetilde{\varphi}_{(132)}] \neq [\widetilde{\varphi}_{(132)}]\cdot[\widetilde{\varphi}_{(13)}]$. However, $[\widetilde{\varphi}_x]$ commutes with the other matrices. Hence, the map

$$\mathbb{Z}_2 * \Sigma_3 \xrightarrow{\Theta} \mathrm{Aut}(F_5) \xrightarrow{\mathrm{ab}} GL_5(\mathbb{Z})$$

is the same as the composition

$$\mathbb{Z}_2 * \Sigma_3 \xrightarrow{p} \mathbb{Z}_2 \times \Sigma_3 \xrightarrow{\widetilde{\Theta}} \mathrm{Out}(F_5) \xrightarrow{\mathrm{ab}} GL_5(\mathbb{Z}).$$

Therefore, there is a homomorphism $\mathbb{Z}_2 \times \Sigma_3 \to GL_5(\mathbb{Z})$.

2.7.1 Two finite cyclic groups

Consider the general case of two cyclic groups $G_1 \cong \mathbb{Z}/n\mathbb{Z} \cong \langle x_1 | x_1^n = 1\rangle$ and $G_2 \cong \mathbb{Z}/m\mathbb{Z} \cong \langle x_2 | x_2^m = 1\rangle$. There is a short exact sequence of groups

$$1 \longrightarrow F_k \longrightarrow \mathbb{Z}_n * \mathbb{Z}_m \longrightarrow \mathbb{Z}_n \times \mathbb{Z}_m \longrightarrow 1,$$

where F_k is the free group on $k = (n-1)(m-1)$ letters given by the elements of

$$\mathcal{W}_2 = \{\omega_{ij} = [x_1^i, x_2^j] | 1 \leq i \leq n-1, 1 \leq j \leq m-1\}.$$

To compute the map $\Theta : \mathbb{Z}_n * \mathbb{Z}_m \to \mathrm{Aut}(F_k)$, we first compute the automorphism $\varphi_{x_1} \in \mathrm{Aut}(F_k)$ by looking at the image of the generators $\omega_{ij} \in F_k$ under φ_{x_1} to find

$$x_1\omega_{ij}x_1^{-1} = x_1[x_1^i, x_2^j]x_1^{-1} = [x_1^{i+1}, x_2^j][x_2^j, x_1] = \omega_{i+1,j}{\omega_{1,j}}^{-1}.$$

Looking at the induced map of φ_{x_1} onto the abelianization

$$\bigoplus_{(n-1)(m-1)} \mathbb{Z} \cong F_{(n-1)(m-1)}/[F_{(n-1)(m-1)}, F_{(n-1)(m-1)}]$$

then

$$\widetilde{\varphi}_{x_1}(\omega_{11}, ..., \omega_{(n-1)(m-1)}) = (\omega_{2,1} - \omega_{1,1}, \omega_{2,2} - \omega_{1,2}, \omega_{2,3} - \omega_{1,3}, ..., -\omega_{(n-1),(m-1)})$$

which is given by the matrix

$$[\widetilde{\varphi}_{x_1}] = \begin{pmatrix} -I_{m-1} & I_{m-1} & 0 & 0 & \cdots & 0 \\ 0 & -I_{m-1} & I_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & -I_{m-1} & I_{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -I_{m-1} & I_{m-1} \\ 0 & \cdots & 0 & 0 & 0\cdots & -I_{m-1} \end{pmatrix}$$

with respect the basis $\mathcal{W}_2$, where I_{m-1} is the $(m-1)\times(m-1)$ identity matrix. Hence, clearly φ_{x_1} is not an element of IA_k.

For $\varphi_{x_2} \in \mathrm{Aut}(F_k)$:

$$x_2\omega_{ij}x_2^{-1} = x_2[x_1^i, x_2^j]x_2^{-1} = [x_2, x_1^i][x_1^i, x_2^{j+1}] = \omega_{i,1}{}^{-1}\omega_{i,j+1}.$$

Similarly, looking at the induced map of φ_{x_2} onto the abelianization of F_k we get

$$\widetilde{\varphi}_{x_2}(\omega_{11}, ..., \omega_{(n-1)(m-1)}) = (-\omega_{1,1} + \omega_{1,2} - \omega_{1,1} + \omega_{1,3}, ..., -\omega_{(n-1),(m-1)}),$$

which is given by the matrix

$$[\widetilde{\varphi}_{x_2}] = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n-1} \end{pmatrix}$$

with respect to the basis $\mathcal{W}_2$, where

$$A_i = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(m-1)\times(m-1)}$$

for all i. Hence, φ_{x_2} is not an element of IA_k.

In general, Θ maps an element $x_1^i x_2^j$ to $\varphi_{x_i \cdot x_j} \in \mathrm{Aut}(F_k)$, which when restricted to the abelianization $\bigoplus_k \mathbb{Z}$, can be identified with the matrix $[\widetilde{\varphi}_{x_1}]^i[\widetilde{\varphi}_{x_2}]^j$. This matrix is the identity if and only if $i = n$ and $j = m$. Hence, there is a homomorphism

$$\mathbb{Z}_n * \mathbb{Z}_m \xrightarrow{\mathrm{ab} \circ \Theta} GL_k(\mathbb{Z}).$$

Θ induces a homomorphism $\widetilde{\Theta} : \mathbb{Z}_n \times \mathbb{Z}_m \longrightarrow \mathrm{Out}(F_k)$. Hence, there is a homomorphism $\mathbb{Z}_n \times \mathbb{Z}_m \xrightarrow{\mathrm{ab} \circ \widetilde{\Theta}} GL_k(\mathbb{Z})$.

If m is even and n is odd or vice-versa, then

$$det[\widetilde{\varphi}_{x_1}] = (-1)^{(n-1)(m-1)} = 1,$$

$$det[\widetilde{\varphi}_{x_2}] = det(A_1) \cdots det(A_{n-1}) = (det(A_1))^{n-1}.$$

Since $det(A_i) = 1$ if m is odd and -1 if m is even, and if n is odd we get $(-1)^{n-1} = 1$, then $det[\widetilde{\varphi}_{x_2}] = 1$. Hence, there is a homomorphism

$$\mathbb{Z}_n * \mathbb{Z}_m \longrightarrow SL_k(\mathbb{Z})$$

which induces a homomorphism

$$\begin{tikzcd}
\mathbb{Z}_n * \mathbb{Z}_m \arrow[rrr, dotted, "\mathrm{ab} \circ \widetilde{\Theta} \circ \mathrm{ab}"] \arrow[dr, "\mathrm{ab}"] & & & SL_k(\mathbb{Z}) \subset GL_k(\mathbb{Z}) \\
& \mathbb{Z}_n \times \mathbb{Z}_m \arrow[r, "\widetilde{\Theta}"] & \mathrm{Out}(F_k). \arrow[ur, "\mathrm{ab}"] &
\end{tikzcd}$$

That is, there is a representation of $\mathbb{Z}_n \times \mathbb{Z}_m \to SL_k(\mathbb{Z})$. Similarly as before, the map $\mathrm{ab} \circ \Theta$ can be considered the same as the composition $p \circ \widetilde{\Theta}$, where p is the projection to the abelianization of $\mathbb{Z}_n * \mathbb{Z}_m$, since $[\widetilde{\varphi}_{x_1}]$ and $[\widetilde{\varphi}_{x_2}]$ commute.

To show that $[\widetilde{\varphi}_{x_1}]$ and $[\widetilde{\varphi}_{x_2}]$ commute it suffices to show that they commute for $n = m = 3$

$$[\widetilde{\varphi}_{x_1}] \cdot [\widetilde{\varphi}_{x_2}] = [\widetilde{\varphi}_{x_2}] \cdot [\widetilde{\varphi}_{x_1}] = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

In the next section we discuss the implications that these representations have for the monodromy in for any simplicial complex K.

2.7.2 Any two finite groups

Let G and H be any finite groups, not necessarily cyclic or abelian, with cardinality m and n respectively. That is, $G = \{1, g_1, ..., g_{m-1}\}$ and $H = \{1, h_1, ..., h_{n-1}\}$. There is a short exact sequence of groups

$$1 \longrightarrow F_{(m-1)(n-1)} \longrightarrow G * H \longrightarrow G \times H \longrightarrow 1.$$

To calculate the map $G * H \to \mathrm{Aut}(F_{(m-1)(n-1)})$, start with φ_f, where $f \in G$ or $f \in H$. Choose a basis for $F_{(m-1)(n-1)}$ to be

$$W = \{[g_i, h_j] | 1 \le i \le m-1, 1 \le m \le n-1\}.$$

Then,

$$\varphi_{g_k}([g_i, h_j]) = g_k[g_i, h_j]{g_k}^{-1} = [g_k g_i, h_j][h_j, g_k]$$

and

$$\varphi_{h_k}([g_i, h_j]) = h_k[g_i, h_j]{h_k}^{-1} = [h_k, g_i][g_i, h_k h_j].$$

To find the matrix representation of these, it is necessary to know the group structure of G and H.

Hence, we get a composition of homomorphisms

$$G * H \to G \times H \to \mathrm{Out}(F_{(m-1)(n-1)})$$

which is the same as the composition

$$G * H \to \mathrm{Aut}(F_{(m-1)(n-1)}) \to \mathrm{Out}(F_{(m-1)(n-1)}).$$

Remark. In sections 2.7.1 and 2.7.2 as well as in the examples, data is being collected, with the goal of axiomatizing properties of the monodromy.

2.7.3 A collection of finite discrete groups

In this section we list the properties of the monodromy representation that are observed to hold for a finite collection of finite discrete groups. This will conclude the endeavor in describing this representation completely for the 0–simplicial complex K_0, but with an incomplete answer.

Recall that for a group G, there is a sequence of subgroups called the *descending central series* of G given by

$$G = \Gamma^1(G) \trianglerighteq \Gamma^2(G) \trianglerighteq \cdots \trianglerighteq \Gamma^n(G) \trianglerighteq \cdots$$

such that the second stage is $\Gamma^2(G) = [G, G]$ and the $(n+1)$–st stage is given inductively $\Gamma^{n+1}(G) = [\Gamma^n(G), G]$. The Lie algebra associated to the descending central series is given by

$$\mathrm{gr}_*(G) = \bigoplus_{i\geq 1} \Gamma^i(G)/\Gamma^{i+1}(G)$$

with $\mathrm{gr}_p(G) = \Gamma^p(G)/\Gamma^{p+1}(G)$.

Lemma 2.38. *Let $\{G_i\}_{i=1}^n$ be a collection of finite discrete groups and K_0 be the 0–simplicial complex on n vertices. Let $\rho : \prod_{i=1}^n G_i \to \mathrm{Out}(F_N)$ be the monodromy representation where F_N is isomorphic to the kernel of the projection $p : *_{i=1}^n G_i \to \prod_{i=1}^n G_i$. Then the following hold:*

1. *There is a choice of a generating set for F_N that consists of elements of the form*

$$f = [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]] \in \Gamma^k(*_{i=1}^n G_i)$$

such that $g_{i_j} \in G_{i_j}$, for all i_j.

2. *For any $g \in *_{i=1}^n G_i$, the map $\rho(g) \in \mathrm{Aut}(F_N)$ satisfies $\rho(g)(f) = \Delta \cdot f$, such that $\Delta \in \Gamma^{k+1}(*_{i=1}^n G_i)$. That is, $\Delta = 1 \in \mathrm{gr}_p(*_{i=1}^n G_i)$ for $p \leq i$.*

Proof. Part 1: From the homotopy type of $Z_{K_0}(EG_i, G_i) \subset [0,1]^n$ it is clear that all types of paths can be described using commutators of length at most n. It remains to prove that it is sufficient to consider only $g_{i_j} \in G_{i,j}$ and not other elements in $*_{i=1}^n G_i$ to construct these commutators.

Start with $[g_i g_j, g_k] \in \Gamma^3(*_{i=1}^n G_i)$. Then

$$[g_i g_j, g_k] = [g_i, [g_j, g_k]] \cdot [g_j, g_k \cdot][g_i, g_k].$$

Thus for any product, say $g_i = h_1 \cdots h_t$, it follows that

$$[g_i g_j, g_k] = [(h_1 \cdots h_t) g_j, g_k] = [h_1 \cdots h_t, [g_j, g_k]] \cdot [g_j, g_k] \cdot [h_1 \cdots h_t, g_k].$$

Then this product can be reduced to a product of commutators of the form stated in part 1, in finitely many steps by applying the step t more times.

Part 2: If $f = [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]] \in \Gamma^k(*_{i=1}^n G_i)$ is a element in F_N, then

$$\begin{aligned}\rho(g)(f) &= g \cdot [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]] \cdot g^{-1} \\ &= [g, [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]]] \cdot [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]] \\ &= \Delta \cdot f,\end{aligned}$$

where $\Delta = [g, [g_{i_1}, [g_{i_2}, [..., [g_{i_{k-1}}, g_{i_k}]...]]]] = [g, f] \in \Gamma^{k+1}(*_{i=1}^n G_i)$. □

2.8 Monodromy representation for general K

Let $\{G_i\}_{i=1}^n$ be a collection of finite discrete groups and K a simplicial complex on n vertices. Recall the fibration due to Denham and Suciu [Denham and Suciu, 2007],

$$Z_K(EG_i, G_i) \longrightarrow Z_K(BG_i) \longrightarrow \prod_{i=1}^n BG_i$$

and consider the monodromy representation

$$\rho_K : G_1 \times \cdots \times G_n \longrightarrow \mathrm{Out}(\pi_1(Z_K(EG_i, G_i))).$$

The goal of this section is to describe the representation ρ_K using the representation $\rho_{K_0} : G_1 \times \cdots \times G_n \longrightarrow \mathrm{Out}(F_N)$, where F_N is isomorphic to the kernel of the projection $*_i G_i \to \prod_i G_i$.

There is a commutative diagram of fibrations

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & * \arrow[d] \\
Z_{K_0}(EG_i, G_i) \arrow[r] \arrow[d, "p"'] & Z_{K_0}(BG_i) \arrow[r] \arrow[d] & \prod_{i=1}^n BG_i \arrow[d] \\
Z_K(EG_i, G_i) \arrow[r] & Z_K(BG_i) \arrow[r] & \prod_{i=1}^n BG_i,
\end{tikzcd}$$

where F is the homotopy fibre of the map p.

Lemma 2.39. *Let $\pi = \pi_1(Z_K(EG_i, G_i))$. Then π is torsion free.*

Proof. The lemma follows since $Z_K(EG_i, G_i)$ is homotopy equivalent to a finite dimensional CW–complex. $\square$

Hence, since F is connected, it follows from the long exact sequence in homotopy that the map p induces a surjection $p_\# : F_N \to \pi$, on the level of fundamental groups. Thus, the kernel of the projection map is a free group and $\mathrm{Ker}(p_\#) < \pi_1(F)$. From Theorem 2.23 it follows that the fibre F is an Eilenberg–Mac Lane space if and only if K is a flag complex.

Lemma 2.40. *Let K be an abstract flag simplicial complex on n vertices and $G_1, ..., G_n$ be finite discrete groups. Then there is a short exact sequence of groups*

$$1 \to F_q \to G_1 * \cdots * G_n \to \prod_{SK_1} G_i \to 1,$$

where F_q is the free group generated by $\{[g_i, g_j] | \{i, j\} \in SK_1, g_i \in G_i, g_j \in G_j\}$, that is, $\{i, j\}$ is an edge in SK_1.

Proof. There is a fibration $F \to Z_{K_0}(BG_i) \to Z_K(BG_i)$. It follows from Theorem 2.23 that if K is a flag complex, then $Z_K(BG_i)$ is an Eilenberg–Mac Lane space. Hence, F is an Eilenberg–Mac Lane space with fundamental group H. Therefore, there is a short exact sequence of groups

$$1 \to H \to G_1 * \cdots * G_n \to \prod_{SK_1} G_i \to 1,$$

where the equality $\pi_1(Z_K(BG_i)) = \prod_{SK_1} G_i$ follows from [Davis and Okun, 2012]. By definition $\prod_{SK_1} G_i = (*_{i=1}^n G_i)/H$, where N is the normal completion of the group generated by $\{[g_i, g_j] | \{i,j\} \in SK_1, g_i \in G_i, g_j \in G_j\}$. Hence, $H = F_q$ is a free group. This fact can also be deduced by observing that H is a subgroup of $\pi_1(Z_{K_0}(EG_i, G_i))$, which is free. $\square$

Let F_N be the kernel of the projection $G_1 * \cdots * G_n \twoheadrightarrow \prod_{i=1}^n G_i$. Consider the commutative diagram of fibrations above. As mentioned in the proof of Lemma 2.40, if K is a flag complex, then it follows that all the spaces involved in the diagram are Eilenberg-Mac Lane spaces. Hence, there is a commutative diagram of short exact sequences on the level of the fundamental groups

$$\begin{tikzcd}
& 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & F \arrow[r, "\cong"] \arrow[d] & F \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F_N \arrow[r] \arrow[d, "p\#"'] & G_1 * \cdots * G_n \arrow[r] \arrow[d] & \prod_{i=1}^n G_i \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi \arrow[r] \arrow[d] & \prod_{SK_1} G_i \arrow[r] \arrow[d] & \prod_{i=1}^n G_i \arrow[r] \arrow[d] & 1 \\
& 1 & 1 & 1 &
\end{tikzcd}$$

Consider the diagram below, where the dotted homomorphisms are yet to be determined if they exist.

$$\begin{tikzcd}
F_N \arrow[rr] \arrow[dr] \arrow[dd, "p_{\#}"'] & & *_i G_i \arrow[rr] \arrow[dr] \arrow[dd] & & \prod G_i \arrow[dr, "\rho_{K_0}"] \arrow[dd] & \\
& \mathrm{Inn}(F_N) \arrow[rr] \arrow[dd] & & \mathrm{Aut}(F_N) \arrow[rr] \arrow[dd, dotted] & & \mathrm{Out}(F_N) \arrow[dd, dotted] \\
\pi \arrow[rr] \arrow[dr] & & \prod_{SK_1} G_i \arrow[rr] \arrow[dr] & & \prod G_i \arrow[dr, "\rho_K"] & \\
& \mathrm{Inn}(\pi) \arrow[rr] & & \mathrm{Aut}(\pi) \arrow[rr] & & \mathrm{Out}(\pi)
\end{tikzcd}$$

If there is a homomorphism $q : \mathrm{Aut}(F_N) \to \mathrm{Aut}(\pi)$ induced from the projection $p_{\#} : F_N \to \pi$, then the following diagram should commute

$$\begin{tikzcd}
\mathrm{Ker}(p_{\#}) \arrow[r] \arrow[d, "f"] & F_N \arrow[r, "p_{\#}"] \arrow[d, "f"] & \pi \arrow[d, "\bar{f}"] \\
\mathrm{Ker}(p_{\#}) \arrow[r] & F_N \arrow[r, "p_{\#}"] & \pi .
\end{tikzcd}$$

That means, $f \in \mathrm{Aut}(F_N)$ should preserve the kernel of the induced projection $p_{\#}$. Hence, this proves the following lemma

Lemma 2.41. *If $f \in \mathrm{Aut}(F_N)$, then $f(\mathrm{Ker}(p_{\#})) \subseteq \mathrm{Ker}(p_{\#})$.*

Lemma 2.42. *Let K be a flag complex. Then*

$$H_1(Z_K(BG_i); \mathbb{Z}) \cong H_1(Z_{K_0}(BG_i); \mathbb{Z}).$$

Proof. If K is a flag complex, then $Z_K(BG_i)$ is an Eilenberg–Mac Lane space. Hence, $H_1(Z_K(BG_i); \mathbb{Z}) \cong H_1(\pi_1(Z_K(BG_i); \mathbb{Z})) = H_1(\prod_{SK_1} G_i; \mathbb{Z})$. Similarly, it follows that $H_1(Z_{K_0}(BG_i); \mathbb{Z}) \cong H_1(\pi_1(Z_{K_0}(BG_i); \mathbb{Z})) = H_1(*_{i=1}^n G_i; \mathbb{Z})$. Since both $\prod_{SK_1} G_i$ and $*_{i=1}^n G_i$ both project to $G_1 \times \cdots \times G_n$, then they have the same abelianization. $\square$

Remark. The goal is to show that if there is a homomorphism $r : \mathrm{Out}(F_N) \to \mathrm{Out}(\pi)$ induced by $p_{\#}$, then there is a homomorphism $\rho_K : G_1 \times \cdots \times G_n \to \mathrm{Out}(\pi)$ such that the following diagram commutes

$$
\begin{tikzcd}
 & \mathrm{Out}(F_N) \arrow[dd, dotted, "r"] \\
G_1 \times \cdots \times G_n \arrow[ur, "\rho_{K_0}"] \arrow[dr, "\rho_K"'] & \\
 & \mathrm{Out}(\pi).
\end{tikzcd}
$$

That means, $\rho_K = r \circ \rho_{K_0}$. Hence, we want to find such a map r.

2.9 An extension problem

In this section we are interested in investigating a certain extension problem that arises from polyhedral products. Assume $\{G_1, ..., G_n\}$ is a family of subgroups of a finite discrete group G. Then there is a natural map

$$G_1 * ... * G_n \xrightarrow{\varphi} G.$$

In work of A. Adem, F. R. Cohen and E. Torres Giese [Adem *et al.*, 2011], the spaces $B(q, G)$ were introduced, such that the sequence of spaces

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(\infty, G) = BG$$

gives a filtration of BG. $B(2, G)$ is defined to be the geometric realization

$$B(2, G) = \big(\bigsqcup_{k \geq 1} \mathrm{Hom}(\mathbb{Z}^k, G) \times \Delta[k] \big) / \sim,$$

where $\sim$ is generated by the standard face and degeneracy operations.

A proof of the following lemma can be found in [Adem *et al.*, 2011].

Lemma 2.43. *Let G be a nonabelian group. The following are equivalent*

a. If $g \notin Z(G)$, then $C(g)$ is abelian.

b. If $[g, h] = 1$, then $C(g) = C(h)$ whenever $g, h \notin Z(G)$.

c. If $[g,h] = 1 = [h,k]$*, then* $[g,k] = 1$ *whenever* $h \notin Z(G)$*.*

d. If $A, B \leq G$ *and* $Z(G) < C_G(A) \leq C_G(B) < G$*, then* $C_G(A) \leq C_G(B)$*.*

Definition 2.44. Let G be a nonabelian group. If G satisfies any of the equivalent statements in Lemma 2.43, then G is called a *transitively commutative* group, or simply a TC group.

In [Adem *et al.*, 2011] Adem, Cohen and Torres Giese prove the following theorem

Theorem 2.45 (Adem, Cohen & Torres Giese)**.** *If G is a finite TC group with trivial center, then there is a homotopy equivalence*

$$B(2,G) \simeq \bigvee_{1 \leq i \leq k} \Big(\prod_{p || C_G a_i|} BP \Big),$$

where $P \in \mathrm{Syl}_P(G)$*.*

Basically, this theorem states that if G is a TC group, then

$$B(2,G) = BG_1 \vee \cdots \vee BG_n$$

and this is the polyhedral product $Z_{K_0}(BG_i)$. Hence, there is a map

$$Z_{K_0}(BG_i) \longrightarrow BG.$$

On the level of the fundamental groups, this brings the discussion back to the homomorphism

$$G_1 * ... * G_n \xrightarrow{\varphi} G.$$

It is natural to ask for which abstract simplicial complexes K on $[n]$, does the map $BG_1 \vee ... \vee BG_n \xrightarrow{B\varphi} BG$ extend to $Z_K(BG_i)$, that is for which K does the following diagram commute

$$\begin{tikzcd}
BG_1 \vee ... \vee BG_n \arrow[r, "B\varphi"] \arrow[d, hook, "i"] & BG \\
Z_K(BG_i) \arrow[ur] &
\end{tikzcd}$$

The importance of this question is that if the map in question extends, then we detect commuting elements in G. One could also pose the same question algebraically. For what simplicial complexes K does the following diagram commute

$$\begin{tikzcd}
G_1 * \cdots * G_n \arrow[r, "\varphi"] \arrow[d, "i_\#"] & G \\
\prod_{SK_1} G_i \arrow[ur] &
\end{tikzcd}$$

The rest of this section will be devoted to understanding this extension question.

The first example is that of a transitively commutative (TC) group G with trivial center, and the family of subgroups $\{G_1, ..., G_n\}$ is chosen to consist of the maximal abelian subgroups of G.

Proposition 2.46. *Let G be a transitively commutative group and let $\{A_1, ..., A_k\}$ be the distinct maximal abelian subgroups of G. Then the map $B\varphi : BA_1 \vee \cdots \vee BA_k \to BG$ does not extend for any simplicial complex K.*

Proof. If it extends for a simplicial complex K, then K has at least one edge, say $\{i, j\}$. This implies that $[A_i, A_j] = 1$ for some $i \neq j$. One can check that the groups A_i are centralizers of elements in G not in the center, and that they intersect trivially, yielding a contradiction to the assumption. $\square$

Let $\Gamma^k(G)$ denote the k–th stage of the descending central series of the group G. Next we define a similar class of groups with the same property.

Definition 2.47. Let $g_l, k_l \in \Gamma^{l-1}(G)$. We say that G is an *l–transitively commutative group*, or simply l–TC group, if

$$[g_l, h] = 1 = [h, k_l] \Longrightarrow [g_l, k_l] = 1 \text{ for all } h \in G.$$

Note that a 2–TC group is the same as the ordinary TC group. Also that the condition that $l \geq 2$ is required for the definition.

Remark. If G is an m–TC group then it is an n–TC group for all $n \geq m$. This follows from the structure of the descending central series. Hence, if G is a TC group, then it is a k–TC group for all $k \geq 2$. By convention, let finite simple groups be called 1–*TC groups*.

Example 2.48. A group which is 3–TC but not 2–TC is the quaternion group, Q_8. This group has a presentation as follows

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k | i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}.$$

First we note that $[i, -1] = 1 = [-1, j]$ but $[i, j] = -1$, so Q_8 is not 2–TC. Now $[i, j] = [j, k] = [k, i] = -1$ and $[[i, j], h] = 1$ for any $h \in Q_8$. Therefore Q_8 is 3–TC. One can also compute the descending central series for Q_8. Its commutator subgroup is $\Gamma^2(Q_8)[Q_8, Q_8] = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ and $\Gamma^3(Q_8) = [[Q_8, Q_8], Q_8] = 1$. Therefore, the descending central series for Q_8 is

$$1 \lhd \mathbb{Z}/2\mathbb{Z} \lhd Q_8.$$

Lemma 2.49. *If G is nilpotent of nilpotency class m, then G is an m–TC group.*

Proof. G is nilpotent of nilpotency class m means $\Gamma^m(G) = 1$, that is, $[g_m, h] = 1$ for all $g_m \in \Gamma^{m-1}(G), h \in G$. In particular, $[g_m, k_m] = 1$ for all $g_m, k_m \in \Gamma^{m-1}(G)$, hence G is m-TC. □

Definition 2.50. Let $g \in G$. The *l–stage centralizer of $g \in G$*, denoted $C_G^l(g)$, is the subgroup

$$C_G^l(g) := \{g_{l+1} \in \Gamma^l(G) | g_{l+1} g g_{l+1}^{-1} = g\}.$$

We can also write $C_G^l(g) = C_G(g) \cap \Gamma^l(G)$.

Lemma 2.51. *Let G be an $(l+1)$–TC group with trivial center. Then $C_G^l(g)$ are abelian subgroups of G for all $g \in G$. Moreover, if $Z(G) = 1$, then $C_G^l(g)$ are distinct and either intersect trivially or coincide.*

Proof. It is clear that $C_G^l(g)$ are abelian. If $C_G^l(g) \cap C_G^l(h)$ is non-trivial then there is $1 \neq k \in C_G^l(g) \cap C_G^l(h)$ and thus, the two coincide by definition of an $(l+1)$–TC group. □

Lemma 2.52. *Let $\{C_G^l(g)\}_{g\in\Lambda}$ be the family of distinct l-stage centralizers of elements in G. If G is an $(l+1)$–TC group with $Z(G) = 1$, the map*

$$\bigvee_{g\in\Lambda} BC_G^l(g) \longrightarrow BG$$

does not extend for any K on $[m]$ where $m = |\{C_G^l(g)\}_{g\in\Lambda}|$.

Proof. This follows from the previous lemma. □

Corollary 2.53. *Let G be a finite $(k+1)$–TC group with trivial center. Let G_1, G_2 be two subgroups of G such that $C_G^k(g_1) \leq G_1$ and $C_G^k(g_2) \leq G_2$, where $C_G^k(g_1) \cap C_G^k(g_2) = 1$. Then the map*

$$BG_1 \vee BG_2 \longrightarrow BG$$

does not extend to $BG_1 \times BG_2$.

Proof. The proof follows from Lemma 2.52. □

Using this corollary the following theorem follows easily. This theorem applies to TC groups in particular.

Theorem 2.54. *Let $\{G_i\}_{i=1}^m$ be a family of subgroups of a finite $(k+1)$–TC group G with trivial center. If G_i contain distinct $C_G^l(g_i)$, for $i = 1, 2, ..., m$ and $l \geq k$, then the map*

$$\bigvee_{1\leq i\leq m} BG_i \longrightarrow BG$$

does not extend to $Z_K(BG_i)$ for any K on $[m]$, other than the 0-skeleton.

Proof. From the definition we have $C_G^l(g_i) = C_G(g_i) \cap \Gamma^l(G)$ which gives $C_G^l(g_i) \geq C_G^{l+1}(g_i)$. The rest of the proof follows from Corollary the 2.9. $\square$

Below we give a result for when this extension actually exists.

Definition 2.55. Let K be a simplicial complex. The *flag complex of* K is the simplicial complex $\mathrm{Flag}(K)$ obtained from K by completing the minimal non–faces to faces.

Note that an edge $\{i, j\}$ is in K if and only if it is an edge in $\mathrm{Flag}(K)$. Now let $\{H_1, ..., H_n\}$ be a collection of subgroups of a group G. Let Γ be a graph on n vertices that records the commutativity relations of the set $\{H_1, ..., H_n\}$. That means, $\{i, j\}$ is an edge in Γ if and only if $[H_i, H_j] = 1$. Let $\mathrm{Flag}(\Gamma)$ be flag complex of Γ.

Lemma 2.56. *Let $\{H_1, ..., H_n\}$ be a collection of subgroups of a finite group G with $Z(G) = 1$. If Γ is the graph described above, then the map*

$$BH_1 \vee \cdots \vee BH_n \longrightarrow BG$$

extends to $Z_{\mathrm{Flag}(\Gamma)}(BH_i)$.

Proof. By definition, $\mathrm{Flag}(\Gamma)$ is a flag complex. Hence, the polyhedral product $Z_{\mathrm{Flag}(\Gamma)}(BH_i)$ is an Eilenberg–Mac Lane space with fundamental group $\prod_\Gamma H_i$. So there is a commutative diagram of groups

$$\begin{tikzcd}
H_1 * \cdots * H_n \arrow[r, "\varphi"] \arrow[d, "i_\#"'] & G \arrow[d, "id"] \\
\prod_\Gamma H_i \arrow[r, "\widetilde{\varphi}"] & G.
\end{tikzcd}$$

Hence, there is a commutative diagram of spaces

$$\begin{tikzcd}
BH_1 \vee \cdots \vee BH_n \arrow[r, "B\varphi"] \arrow[d, "Bi_\#"'] & BG \arrow[d, "id"] \\
Z_{\mathrm{Flag}(\Gamma)}(BH_i) \arrow[r, "B\widetilde{\varphi}"] & BG.
\end{tikzcd}$$

□

This lemma actually holds for any simplicial complex K such that $\Gamma \subseteq K \subseteq \mathrm{Flag}(\Gamma)$, even though K is not a flag complex.

Theorem 2.57. *Let $\{H_1, ..., H_n\}$ be a collection of subgroups of a finite group G with $Z(G) = 1$. If Γ is the graph described above and K is a simplicial complex such that $\Gamma \subseteq K \subseteq \mathrm{Flag}(\Gamma)$, then the map*

$$BH_1 \vee \cdots \vee BH_n \longrightarrow BG$$

extends to $Z_K(BH_i)$.

Proof. The proof follows by naturality and 2.56. First, the following diagram of inclusions commutes

$$\begin{tikzcd} BH_1 \vee \cdots \vee BH_n \arrow[rr, hook, "Bi_{\#} = i"] \arrow[dr, hook, "i_1"'] & & Z_{\mathrm{Flag}(\Gamma)}(BH_i) \\ & Z_K(BH_i) \arrow[ur, hook, "i_2"'] & \end{tikzcd} .$$

Combining this diagram with the diagram of spaces in Lemma 2.56, it follows that the diagram

$$\begin{tikzcd} BH_1 \vee \cdots \vee BH_n \arrow[r, "="] \arrow[d, hook, "i_1"'] & BH_1 \vee \cdots \vee BH_n \arrow[r, "B\varphi"] \arrow[d, "Bi_{\#}"'] & BG \arrow[d, "id"'] \\ Z_K(BH_i) \arrow[r, hook, "i_2"] & Z_{\mathrm{Flag}(\Gamma)}(BH_i) \arrow[r, "B\widetilde{\varphi}"] & BG \end{tikzcd}$$

commutes. Hence, the map $BH_1 \vee \cdots \vee HB_n \xrightarrow{B\varphi} BG$ extends to $Z_K(BH_i)$. That is, the following diagram of spaces commutes

$$\begin{tikzcd} BH_1 \vee ... \vee BH_n \arrow[r, "B\varphi"] \arrow[d, hook, "i_1"'] & BG \\ Z_K(BG_i) \arrow[ur, "B\widetilde{\varphi} \circ i_2"'] & \end{tikzcd} .$$

□

We finish this section by posing two questions:

Question 2.58. A natural question to ask about the notion of a k–TC groups is whether every finite non–abelian group G is k–TC, for some integer $k \geq 2$.

Question 2.59. Are these the only obstructions to the extension of the maps $\bigvee_i BG_i \longrightarrow BG$? Equivalently, if G_1 and G_2 do not contain distinct $C^l_G(g_i)$ for any $l \geq k$, does the map $\bigvee_i BG_i \longrightarrow BG$ extend to $Z_K(BG_i)$ for some K other than the 0–skeleton? If the answer to this question is affirmative then we will have found a topological characterization for $(k+1)$–TC groups. In particular, we will have found a topological characterization for TC groups.

3 Commuting Elements in Lie groups

3.1 Introduction

The goal of this chapter is to study the homotopy theoretic properties of the spaces of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ for certain Lie groups G. In particular, we study the homology and cohomology groups of these spaces.

The approach taken here is somewhat different from previous approaches that are present in the literature. Instead of investigating the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ individually, they are assembled into a single space, which is called $X(2, G)$, see Definition 3.23. The first class of Lie groups considered are the compact and connected Lie groups with maximal torus T such that every abelian subgroup of G can be conjugated to a subgroup of T. A torus with this additional property will be called a *strong maximal torus*, see also Definition 3.20. It turns out that the assembly of all the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ into a single space $X(2, G)$, which lies inside the James reduced product $J(G)$, has a cohomology algebra that can be computed explicitly when G has a *strong maximal torus* T and the coefficients are in the ring of integers with the order of the Weyl group inverted. In cases when G does not have a strong maximal torus T, e.g. $SO(3)$, the methods of calculation need some modification, and the answer is known for all such groups, but we need to restrict to the connected component of the identity of the space $X(2, G)$. The main theme of this chapter is to identify properties of the space $X(2, G)$.

Assume that all the spaces considered here are of the homotopy type of $CW-$

complexes with non–degenerate basepoints. Together with the main theorem, the following are the main results in this chapter. Let Y be a G-space such that the projection $Y \longrightarrow Y/G$ is a locally trivial fibre bundle. Let X be a G-space such that the basepoint $* \in X$ is fixed by the action of G. The James construction $J(X)$ is also a G-space with fixed basepoint.

Theorem 3.1. *Let Y be a G-space such that the projection $Y \longrightarrow Y/G$ is a locally trivial fibre bundle. Let X be a G-space with fixed basepoint $*$. Then there is a stable homotopy equivalence*

$$\Sigma(Y \times_G J(X)) \simeq \Sigma(Y/G \vee (\bigvee_{n\geq 1} Y \times_G \widehat{X}^n / Y \times_G *)).$$

As an application of this theorem, let Y be a Lie group G with basepoint 1 and X be its maximal torus T. Both Y and X are $NT-$spaces, where NT is the normalizer of T, acting on G by right translation and on T by conjugation. Given this data the following corollary holds,

Corollary 3.2. *Let G, T and NT be as above. Then there is a stable homotopy equivalence*

$$\Sigma(G \times_{NT} J(T)) \simeq \Sigma(G/NT \vee (\bigvee_{n\geq 1} G \times_{NT} \widehat{T}^n / G \times_{NT} 1)).$$

Given the space $X(2, G)$, if G has a strong maximal torus T, then there is a surjection

$$\Theta : G \times_{NT} J(T) \twoheadrightarrow X(2, G).$$

Let $\mathbb{Z}[|W|^{-1}]$ denote the ring of integers with the order of the Weyl group W inverted. Then the fibers of θ have the following property,

Proposition 3.3. *Let G be a compact and connected Lie group with strong maximal torus T and Weyl group W. Then the fibres of Θ*

$$\Theta : G \times_{NT} J(T) \twoheadrightarrow X(2, G)$$

have trivial reduced homology with coefficients in $\mathbb{Z}[|W|^{-1}]$.

Therefore, using this proposition, the next theorem holds. For the remaining of this chapter let R denote the ring $\mathbb{Z}[|W|^{-1}]$.

Theorem 3.4. *If G is a compact and connected Lie group with strong maximal torus T, then Θ induces a homology isomorphism with coefficients in R. That is, there are isomorphisms*

$$H_*(G \times_{NT} J(T), R) \cong H_*(X(2, G), R).$$

Finally, the case when the compact and connected Lie group G does not necessarily have a strong maximal torus T, is considered. Let G be the Lie group $SO(3)$, which does not have a strong maximal torus. The following proposition about $SO(3)$ holds,

Proposition 3.5. *There is a homotopy equivalence*

$$X(2, SO(3)) \simeq X(2, SO(3))_1 \coprod \big(\coprod_{\infty} S^3/Q_8\big),$$

where $X(2, SO(3))_1$ is the connected component of the identity.

This proposition is an instance of a more general construction that works here. For any compact and connected Lie group G, there is a surjection

$$\Theta_1 : G \times_{NT} J(T) \twoheadrightarrow X(2, G)_1.$$

Similar to the map Θ, the following proposition holds for the fibres of Θ_1, which gives explicit answers for the homotopy theoretic invariants of $X(2, SO(3))$, such as homology or cohomology.

Proposition 3.6. *Let G be a compact and connected Lie group with maximal torus T and Weyl group W. Then the fibres of Θ_1*

$$\Theta_1 : G \times_{NT} J(T) \twoheadrightarrow X(2, G)_1$$

have trivial reduced homology with coefficients in the ring R.

Therefore, using Proposition 3.6 the next theorem follows.

Theorem 3.7. *If G is a compact and connected Lie group with maximal torus T, then Θ_1 induces a homology isomorphism with coefficients in R. That is, there are isomorphisms*

$$H_*(G \times_{NT} J(T), R) \cong H_*(X(2,G)_1, R).$$

It has been proved that if G has a strong maximal torus, then $\mathrm{Hom}(\mathbb{Z}^n, G)$ is path–connected for all $n \geq 1$, see [Adem and Cohen, 2007]. Hence, Theorem 3.7 implies Theorem 3.4.

Remark. Another fact to note is that Theorem 3.7 applies to compact and connected exceptional Lie groups, namely the groups G_2, F_4, E_6, E_7 and E_8.

The homology groups are given by the following coinvariants.

Theorem 3.8. *Let G be a connected and compact Lie group with maximal torus T and Weyl group W. Then there is an isomorphism in homology*

$$H_*(X(2,G)_1; R) \cong \big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

If grading of homology is disregarded, that is, if ungraded homology is considered, then the homology groups are given as follows. Let H_*^U denote the ungraded homology, $\mathcal{T}_U[V]$ denote the ungraded tensor algebra over V and R denote the ring R. Then the following theorem holds.

Theorem 3.9. *Let G be a connected and compact Lie group with maximal torus T and Weyl group W. Then there is an isomorphism in ungraded homology*

$$H_*^U(X(2,G)_1; R) \cong \mathcal{T}_U[V].$$

The space $X(2,G)$ also admits a stable decomposition as wedge sum of components as follows,

Proposition 3.10. *Let G be a compact and connected Lie group. There is a stable homotopy equivalence*

$$\Sigma X(2,G) \simeq \Sigma \bigvee_{n\geq 1} \widehat{\mathrm{Hom}}(\mathbb{Z}^n, G).$$

This proposition can be used to recover information about the spaces of commuting elements $\mathrm{Hom}(\mathbb{Z}^n, G)$, see Section 3.3. If G is a closed subgroup of $GL_n(\mathbb{C})$, the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ admit a stable decomposition as well, see Theorem 1.1, which was given by A. Adem and F. Cohen [Adem and Cohen, 2007] to be

$$\Sigma(\mathrm{Hom}(\mathbb{Z}^n, G)) \simeq \bigvee_{1\leq k\leq n} \Sigma(\bigvee^{\binom{n}{k}} \mathrm{Hom}(\mathbb{Z}^k, G)/S_k(G)),$$

where $\mathrm{Hom}(\mathbb{Z}^k, G)/S_k(G) = \widehat{\mathrm{Hom}}(\mathbb{Z}^k, G)$ and $S_k(G)$ is the subspace of $\mathrm{Hom}(\mathbb{Z}^k, G)$ consisting of k-tuples with at least one coordinate being the identity. So the stable sums for $\mathrm{Hom}(\mathbb{Z}^k, G)/S_k(G)$ can be obtained from the stable decomposition of $X(2, G)$.

The structure of this chapter is as follows. We start by proving a decomposition theorem in Section 3.2. A few applications of the decomposition are given in the same section. The space $X(2, G)$ is defined in Section 3.3 and its homology is computed in Section 3.4 for Lie groups with strong maximal tori. In Section 3.6 the homology of the connected component of the identity representation is calculated for compact and connected Lie groups in general.

3.2 A decomposition theorem

Let $(X, *)$ be a CW–pair with non-degenerate basepoint and let G be a topological group. What follows is the definition of the James reduced product or James construction, which is a free associative monoid with generators the elements of the pointed space X, where $*$ is the identity element.

Definition 3.11. The *James reduced product* on the pointed CW–complex $(X,*)$ is defined to be

$$J(X) := \bigsqcup_{n\geq 0} X^n/\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(x_1,...,x_n) \sim (x_1,...,x_{i-1},\widehat{x_i},x_{i+1},...,x_n) \text{ if } x_i = *$$

and X^0 is the basepoint $*$.

The space $J(X)$ has the structure of an associative monoid with elements being the reduced words of the form $x_1\cdots x_k$, where $* \neq x_i \in X$ for all i, or all of them are $*$, and the multiplication being concatenation of words. If X is a G–space with fixed basepoint, then $J(X)$ is also a $G-$space with the action being $g\cdot(x_1\cdots x_k) = (g\cdot x_1)\cdots(g\cdot x_k)$, where the class of the basepoint is fixed by G. The James reduced product has a natural filtration as follows,

Definition 3.12. Let $J_q(X) \subset J(X)$ be the image of X^q in $J(X)$. Then

$$J_q(X) = \bigsqcup_{1\leq i\leq q} X^i/\sim,$$

where $\sim$ is the same relation as in 3.11. So $J_q(X)$ is the subspace consisting of words with length at most q.

So there is a filtration

$$J_0(X) = \bar{*} \subset J_1(X) = X \subset J_2(X) \subset \cdots \subset J_q(X) \subset \cdots \subset J(X),$$

where G acts on each $J_q(X)$. Moreover, the inclusions $J_{q-1}(X) \hookrightarrow J_q(X)$ are cofibrations with cofiber $J_q(X)/J_{q-1}(X) = \widehat{X}^q$, where $\widehat{X}^q = X\wedge\cdots\wedge X$, the q–fold smash product of X with itself. This follows since the diagram

$$\begin{tikzcd}
\bigsqcup_{1\leq i\leq q-1} X^i \arrow[r, hook, "i"] \arrow[d, "p_1"] & \bigsqcup_{1\leq i\leq q} X^i \arrow[d, "p_2"'] \\
J_{q-1}(X) \arrow[r, hook, "\bar{i}"] & J_q(X)
\end{tikzcd}$$

commutes, where p_1 and p_2 are the quotient maps by the relation $\sim$.

Let X be of the homotopy type of a connected CW-complex. Consider the following theorem which is due to I. James [James, 1955].

Theorem 3.13 (James)**.** *If X has the homotopy type of a connected $CW-$complex, there is a homotopy equivalence $J(X) \simeq \Omega\Sigma X$.*

One result is stated next.

Theorem 3.14. *Let X be a path-connected $G-$space with non-degenerate fixed basepoint. There is a G-equivariant homotopy equivalence $\Sigma J(X) \simeq \Sigma(\bigvee_{n\geq 1} \widehat{X}^n)$.*

The following classical technical lemma is used in the proof of Theorem 3.14.

Lemma 3.15. *A map $f : A \to \Omega B$ extends to $J(A) \simeq \Omega\Sigma A$, that is, there is Ωf such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & \Omega B \\ J(A) = \Omega\Sigma A. \arrow[ur, dotted, "\exists\Omega f"'] & \end{tikzcd}$$

Proof. See [Cohen *et al.*, 1987]. $\square$

Proof of Theorem 3.14. Define a map H as follows

$$H : J(X) \longrightarrow J(\bigvee_{q\geq 1} \widehat{X}^q)$$

$$(x_1, ..., x_n) \mapsto \prod_{I\subset[n]} x_I$$

where $x_I = x_{i_1} \wedge \cdots \wedge x_{i_q}$ for $I = (i_1, ..., i_q)$ running over all admissible sequences in $[n]$, that is all sequences of the form $(i_1 < \cdots < i_q)$, and x_I having the left lexicographic order. From Theorem 3.13 it follows that there is a homotopy equivalence $J(\bigvee_{q\geq 1} \widehat{X}^q) \simeq \Omega\Sigma(\bigvee_{q\geq 1} \widehat{X}^q)$. Now, it follows from Lemma 3.15 that there is a map ΩH, such that the following diagram commutes

$$\begin{tikzcd} J(X) \arrow[r, "H"] \arrow[d] & \Omega\Sigma(\bigvee_{n\geq 1}\widehat{X}^n) \\ J(J(X)) = \Omega\Sigma J(X). \arrow[ur, "\Omega H"'] & \end{tikzcd}$$

The claim is that ΩH is a homotopy equivalence, and therefore there is a homotopy equivalence

$$\Sigma J(X) \simeq \Sigma(\bigvee_{n\geq 1}\widehat{X}^n).$$

To prove the claim we use induction and the filtration in definition 3.12.

For $n = 1$ there is a map of spaces

$$\Sigma J_1(X) = \Sigma X \longrightarrow \Sigma(\bigvee_{1\leq n\leq 1}\widehat{X}^n) = \Sigma X$$

which is a homotopy equivalence. Since the suspension of cofibrations is a cofibration, there are cofibrations $\Sigma J_1(X) \hookrightarrow \Sigma J_2(X)$ and $\Sigma X \hookrightarrow \Sigma(X \vee \widehat{X}^2)$ both with cofiber $\widehat{X}^2$ as follows

$$\Sigma J_1(X) \hookrightarrow \Sigma J_2(X) \to \Sigma(J_2(X)/J_1(X)) = \Sigma\widehat{X}^2$$

$$\Sigma X \hookrightarrow \Sigma(X \vee \widehat{X}^2) \to \Sigma((X \vee \widehat{X}^2)/X) = \Sigma\widehat{X}^2.$$

So there is a commutative diagram

$$\begin{tikzcd} \Sigma J_1(X) \arrow[r, "\simeq"] \arrow[d] & \Sigma X \arrow[d] \\ \Sigma J_2(X) \arrow[r, dotted, "h_1"] \arrow[d] & \Sigma(X \vee \widehat{X}^2) \arrow[d] \\ \Sigma\widehat{X}^2 \arrow[r, "\simeq"] & \Sigma\widehat{X}^2. \end{tikzcd}$$

Hence, there is a map h_1 which is a homotopy equivalence. Now, assume there is a homotopy equivalence $\Sigma J_q(X)\simeq\Sigma(\bigvee_{1\leq n\leq q}\widehat{X}^n)$. There are cofibrations

$$\Sigma J_q(X) \longrightarrow \Sigma J_{q+1}(X) \longrightarrow \Sigma(J_{q+1}(X)/J_q(X)) = \Sigma\widehat{X}^{q+1},$$

$$\Sigma(\bigvee_{1\le n\le q}\widehat{X}^n) \longrightarrow \Sigma(\bigvee_{1\le n\le q+1}\widehat{X}^n) \longrightarrow \Sigma((\bigvee_{1\le n\le q+1}\widehat{X}^n)/(\bigvee_{1\le n\le q}\widehat{X}^n)) = \Sigma\widehat{X}^{q+1}.$$

So there is a homotopy commutative diagram of spaces

$$\begin{tikzcd}
\Sigma J_q(X) \arrow[r, "\simeq"] \arrow[d] & \Sigma(\bigvee_{1\le n\le q}\widehat{X}^n) \arrow[d] \\
\Sigma J_{q+1}(X) \arrow[r, dotted, "h_q"] \arrow[d] & \Sigma(\bigvee_{1\le n\le q+1}\widehat{X}^n) \arrow[d] \\
\Sigma\widehat{X}^{q+1} \arrow[r, "\simeq"] & \Sigma\widehat{X}^{q+1}.
\end{tikzcd}$$

Hence h_q is a homotopy equivalence. Therefore, by induction there is a homotopy equivalence

$$\Sigma J(X) \simeq \Sigma(\bigvee_{n\ge 1}\widehat{X}^n).$$

The group G acts on the product X^n by

$$g\cdot(x_1,\dots,x_n) = (g\cdot x_1,\dots,g\cdot x_n).$$

Hence, G acts on the n–fold smash product $\widehat{X}^n$ by

$$g\cdot(x_1\wedge\cdots\wedge x_n) = (g\cdot x_1\wedge\cdots\wedge g\cdot x_n).$$

These two actions induce actions of G on $J(X)$ and $J(\bigvee_{n\ge1}\widehat{X}^n)$, respectively. Note that, by hypothesis, the action satisfies $g\cdot * = *$ for all $g\in G$. The map $H : J(X)\to J(\bigvee_{n\ge1}\widehat{X}^n)$ satisfies

$$\begin{aligned}
H(g\cdot(x_1,\dots,x_n)) &= H(g\cdot x_1,\dots,g\cdot x_n)\\
&= \prod_{\{i_1,\dots,i_q\}=I\subset[n]} (g\cdot x_{i_1}\wedge\cdots\wedge g\cdot x_{i_q})\\
&= g\cdot\big(\prod_{I\subset[n]} x_I\big) = g\cdot H(x_1,\dots,x_n)
\end{aligned}$$

so it is G–equivariant. Similarly, it follows that the map $\Omega H : J(J(X)) \to J(\bigvee_{n\ge1}\widehat{X}^n)$ is also G–equivariant, by extending H multiplicatively. $\square$

Using a similar method of induction, the following decomposition theorem holds.

Theorem 3.16. *Let Y be a G-space such that the projection $Y \longrightarrow Y/G$ is a locally trivial fibre bundle. Let X be a G-space with fixed basepoint $*$. Then there is a homotopy equivalence*

$$\Sigma(Y \times_G J(X)) \simeq \Sigma\big(Y/G \vee (\bigvee_{n\geq 1}(Y \times_G \widehat{X}^n)/(Y \times_G *))\big).$$

Before proving the theorem, we state a technical lemma.

Lemma 3.17. *Let Y be a G-space such that the projection $Y \longrightarrow Y/G$ is a locally trivial fibre bundle. Let X be a G-space with fixed basepoint $*$. Then there is a fibre bundle*

$$X \longrightarrow Y \times_G X \longrightarrow Y/G$$

such that $Y/G \longrightarrow Y \times_G X$ is a cofibration with cofibre $(Y \times_G X)/(Y/G)$ and there is a homotopy equivalence

$$\Sigma(Y \times_G X) \simeq \Sigma(Y/G) \vee \Sigma\big((Y \times_G X)/(Y/G)\big).$$

A version of this lemma can be found in [Adem and Cohen, 2007]. Note that the orbit space Y/G can be rewritten as $Y \times_G *$. Now we are ready to prove Theorem 3.16.

Proof of Theorem 3.16. Define a map

$$H : Y \times J(X) \longrightarrow J(\bigvee_{q\geq 1} Y \times \widehat{X}^q)$$

$$(y, (x_1, ..., x_n)) \mapsto \prod_{I\subset[n]} y \times x_I$$

where $x_I = x_{i_1} \wedge \cdots \wedge x_{i_q}$ for $I = (i_1, ..., i_q)$ running over all admissible sequences in $[n]$, that is all sequences of the form $(i_1 < \cdots < i_q)$, and x_I having the left lexicographic order. G acts on $Y \times J(X)$ diagonally by

$$g \cdot (y, (x_1 \cdots x_n)) = (g \cdot y, ((g \cdot x_1) \cdots (g \cdot x_n))).$$

Hence, G acts on $J(\bigvee_{q\geq 1} Y \times \widehat{X}^q)$ by

$$g \cdot \big(\prod_{I \subset [n]} y \times x_I \big) = \prod_{\{i_1,\dots,i_q\}=I\subset[n]} (g\cdot y) \times (g \cdot x_{i_1} \wedge \cdots \wedge g \cdot x_{i_q}).$$

Note that the basepoint $*$ is fixed by the action of G. It follows that the map H satisfies

$$\begin{aligned} H(g \cdot (y, (x_1, ..., x_n)) &= \prod_{\{i_1,\dots,i_q\}=I\subset[n]} (g\cdot y) \times (g \cdot x_{i_1} \wedge \cdots \wedge g \cdot x_{i_q}) \\ &= g \cdot \big(\prod_{I \subset [n]} y \times x_I \big) = g \cdot H((y, (x_1, ..., x_n)), \end{aligned}$$

so H is G–equivariant. Taking the quotient by the diagonal G-action we get an induced map H_G

$$\begin{tikzcd} Y \times J(X) \arrow[r, "H"] \arrow[d] & J(\bigvee_{q\geq 1} Y \times \widehat{X}^q) \arrow[d] \\ Y \times_G J(X) \arrow[r, "H_G"] & J(Y/G \vee (\bigvee_{q\geq 1} Y \times \widehat{X}^q/(Y \times_G *))). \end{tikzcd}$$

By Lemma 3.15 there is a map ΩH_G, such that the following diagram commutes

$$\begin{tikzcd} Y \times_G J(X) \arrow[r, "H_G"] \arrow[d] & J(Y/G \vee (\bigvee_{q\geq 1} Y \times \widehat{X}^q/(Y \times_G *))) \\ J(Y \times_G J(X)) = \Omega\Sigma(Y \times_G J(X)). \arrow[ur, "\Omega H_G"'] & \end{tikzcd}$$

The claim is that ΩH_G is a homotopy equivalence, and the theorem follows.

To prove the claim, similarly as in the case of Theorem 3.14, induction on James filtration will be used. On one side q will mean the q–th stage of the James filtration, and on the other side it will be the q–fold smash product of X. Hence, we will restrict ΩH_G to those spaces and compare them. Recall that the filtration is given by

$$J_0(X) = \bar{*} \subset J_1(X) = X \subset J_2(X) \subset \cdots \subset J_q(X) \subset \cdots \subset J(X).$$

Let f_q be the map of ΩH_G restricted to the q–th stage. For the case $q = 1$, there is a map

$$\Sigma(Y \times_G J_1(X)) \xrightarrow{f_1} \Sigma(Y/G \vee (Y \times_G X)/Y \times_G *).$$

From Lemma 3.17, it follows that there is a homotopy equivalence

$$\Sigma(Y \times_G J_1(X)) \simeq \Sigma(Y/G \vee (Y \times_G X)/Y \times_G *).$$

Now assume there are maps f_2, f_3 such that the following diagram commutes

$$\begin{tikzcd}
\Sigma(Y \times_G J_1(X)) \arrow[r, "\simeq"] \arrow[d] & \Sigma(Y/G \vee (\bigvee_{1\leq n\leq 1} Y \times_G X/Y \times_G *)) \arrow[d] \\
\Sigma(Y \times_G J_2(X)) \arrow[r, "f_2"] \arrow[d] & \Sigma(Y/G \vee (\bigvee_{1\leq n\leq 2} Y \times_G X^n/Y \times_G *)) \arrow[d] \\
\Sigma(Y \times_G J_2(X)/Y \times_G J_1(X)) \arrow[r, "f_3"] & \Sigma((Y \times_G \widehat{X}^2)/Y \times_G *).
\end{tikzcd}$$

Our aim is to prove that either f_2 or f_3 is a homotopy equivalence. In the proof of Theorem 3.14 we got that there are homotopy equivalences

$$\Sigma J_n(X) \simeq \Sigma(\bigvee_{1\leq q\leq n} \widehat{X}^n).$$

Also there is a fibration

$$J_2(X) \longrightarrow Y \times_G J_2(X) \longrightarrow Y/G$$

and hence there is a first quadrant spectral sequence

$$E^2_{p,q} = H_p(Y/G; H_q(J_2(X); \mathbb{Z}))$$

converging to $H_*(Y \times_G J_2(X); \mathbb{Z})$.

For $q \geq 1$ we have that

$$H_q(J_2(X); \mathbb{Z}) \cong \bigoplus_{1\leq i\leq 2} H_q(\widehat{X}^i; \mathbb{Z}).$$

Hence, for $q \geq 1$ we get

$$E^2_{p,q} = H_p(Y/G; \bigoplus_{1\leq i\leq 2} H_q(\widehat{X}^i;\mathbb{Z})) \cong \bigoplus_{1\leq i\leq 2} H_p(Y/G; H_q(\widehat{X}^i;\mathbb{Z}))$$

and for $q = 0$ we get

$$E^2_{p,0} = H_p(Y/G;\mathbb{Z}).$$

Therefore we get that the same spectral sequence converges to

$$H_*(Y/G \vee (\bigvee_{1\leq n\leq 2} Y \times_G X^n/Y \times_G *);\mathbb{Z}).$$

Hence, f_2 is a homotopy equivalence and so is f_3.

Using the induction step we get

$$\begin{tikzcd}
\Sigma(Y \times_G J_q(X)) \arrow[r, "\simeq"] \arrow[d] & \Sigma(Y/G \vee (\bigvee_{1\leq n\leq q} Y \times_G X/Y \times_G *)) \arrow[d] \\
\Sigma(Y \times_G J_{q+1}(X)) \arrow[r, "g_2"] \arrow[d] & \Sigma(Y/G \vee (\bigvee_{1\leq n\leq q+1}(Y \times_G X^n)/Y \times_G *)) \arrow[d] \\
\Sigma\big((Y \times_G J_{q+1}(X))/(Y \times_G J_q(X))\big) \arrow[r, "g_3"] & \Sigma\big((Y \times_G \widehat{X}^{q+1})/(Y \times_G *)\big).
\end{tikzcd}$$

The same argument works for the spectral sequence getting

$$E^2_{p,q} = H_p(Y/G; \bigoplus_{1\leq i\leq q+1} H_q(\widehat{X}^i;\mathbb{Z})) \cong \bigoplus_{1\leq i\leq q+1} H_p(Y/G; H_q(\widehat{X}^i;\mathbb{Z}))$$

for $q \geq 1$, and

$$E^2_{p,0} = H_p(Y/G;\mathbb{Z})$$

which gives a spectral sequence that converges to

$$H_*(Y/G \vee (\bigvee_{1\leq n\leq q+1} Y \times_G X^n/Y \times_G *);\mathbb{Z})$$

and the theorem follows. $\square$

3.2.1 Applications of Theorem 3.16

Let G be a compact Lie group with maximal torus T. Consider the quotient space $G \times_{NT} J(T)$, where NT is the normalizer of T, acting diagonally on the product $G \times J(T)$. More precisely NT acts on on T by conjugation and on G by group multiplication on the left. Then the following are immediate corollaries of Theorem 3.16.

Corollary 3.18. *Let G be a compact and connected Lie group with maximal torus T and NT acting on T by conjugation and on G by group multiplication. There is a homotopy equivalence*

$$\Sigma(G \times_{NT} J(T)) \simeq \Sigma(G/NT \vee (\bigvee_{n\geq 1} G \times_{NT} \widehat{T}^n / G \times_{NT} \{1\})).$$

Corollary 3.19. *With the same assumptions of Corollary 3.18, there is a homotopy equivalence*

$$\Sigma(G \times_{NT} J(T)/(G \times \{1\})) \simeq \Sigma(G/NT \vee (\bigvee_{n\geq 1} G \times_{NT} \widehat{T}^n / G \times_{NT} \{1\}))/(G \times \{1\}).$$

3.3 The space $X(2, G)$

Let G be a Lie group and $1 \in G$ be the identity element and basepoint of G. In this section we construct a space called $X(2, G)$, which assembles all the spaces of commuting elements $\mathrm{Hom}(\mathbb{Z}^n, G)$ into a single space inside the James reduced product of G. This space will be used to study the homology and cohomology groups of the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$. For the remaining of the chapter assume that G is compact and connected, unless otherwise stated.

Definition 3.20. A subgroup T of G is called a *torus* if it is a isomorphic to a k–torus for some k. It is a *maximal torus* if it is not contained in any larger torus subgroup of G.

In this discussion we will consider a maximal torus T of a compact and connected Lie group G which has the additional property that every abelian subgroup of G is conjugate to a subgroup of T. Every compact and connected G has a maximal torus. However, this additional property of the maximal torus does not hold for all G. For example $SO(3)$ is compact and connected and has a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which is not conjugate to a subgroup of its maximal torus T, which is isomorphic to the circle S^1. To distinguish between these types of tori and the standard maximal torus, the following definition is introduced.

Definition 3.21. Let G be a Lie group. T is called a *strong maximal torus* if it is a maximal torus of G such that every abelian subgroup of G is conjugate to a subgroup of T.

Definition 3.22. Let G be a Lie group with maximal torus T and NT the normalizer of T. Then the *Weyl group of G*, denoted by W, is the group $W = NT/T$.

It is a classical result that if G is a compact Lie group, then the Weyl group of G is finite, see [Adams, 1969] for a proof.

Let G be a compact Lie group with maximal torus T. Recall that the James reduced product on the pointed space $(G, 1 = *)$ is defined as

$$J(G) = \bigsqcup_{n \geq 1} G^n / \sim$$

where $\sim$ is the equivalence relation generated by the identifications

$$(g_1, ..., g_{i-1}, 1, g_{i+1}, ..., g_n) \sim (g_1, ..., g_{i-1}, \widehat{1}, g_{i+1}, ..., g_n).$$

Recall that the space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be realized as a subspace of G^n by identifying all $f \in \mathrm{Hom}(\mathbb{Z}^n, G)$ with $f \sim (g_1, ..., g_n) \in G^n$, where $f(t_1, ..., t_n) = (g_1^{t_1}, ..., g_n^{t_n})$. Define the space $X(2, G)$ as follows

Definition 3.23. Let G be a Lie group. The space $X(2, G)$ is defined to be

$$X(2, G) := \bigsqcup_{n \geq 1} \mathrm{Hom}(\mathbb{Z}^n, G)/\sim$$

where $\sim$ is the equivalence relation generated by

$$(g_1, ..., g_{i-1}, 1, g_{i+1}, ..., g_n) \sim (g_1, ..., g_{i-1}, \widehat{1}, g_{i+1}, ..., g_n).$$

Note that since each $\mathrm{Hom}(\mathbb{Z}^n, G) \subseteq G^n$, there is an inclusion $X(2, G) \hookrightarrow J(G)$.

Remark. As a set, $X(2, G)$ is the set of words $g_1 \cdots g_n$, such that the letters satisfy $g_i g_j = g_j g_i$ in G, for all $1 \leq i, j \leq n$ and all $n \geq 1$.

This remark follows directly from the definition.

The remaining of this chapter will devoted to studying the space $X(2, G)$ and how it can be used to study the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$.

Remark. The spaces $X(2, G)$ are a special case of a more general construction on the Lie group G. Let F_n be a free group on n letters. Consider the descending central series of F_n. Then $\mathbb{Z}^n = F_n/\Gamma^2(F_n) = F_n/[F_n, F_n]$. Now consider the quotients $F_n/\Gamma^q(F_n)$. There are spaces of homomorphisms $\mathrm{Hom}(F_n/\Gamma^q(F_n), G)$ which can be realized as subspaces of G^n by identifying $f \sim (g_1, ..., g_n)$, where $f(x_1, .., x_n) = (g_1, ..., g_n)$ and $x_1, ..., x_n$ are the generators of $F_n/\Gamma^q(F_n)$. The following definition justifies the notation of $X(2, G)$.

Definition 3.24. Define spaces $X(q, G)$ for $q \geq 2$ as follows

$$X(q, G) := \bigsqcup_{n \geq 1} \mathrm{Hom}(F_n/\Gamma^q(F_n), G)/\sim$$

where $\sim$ is the single relation of Definition 3.23.

The maps that will be introduced next are commonly used when studying the spaces of commuting elements in G. Start with a map

$$\theta_n : G \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

defined by $\theta_n(g, t_1, ..., t_n) = (gt_1g^{-1}, ..., gt_ng^{-1})$. The maximal torus T acts on the product $G \times T^n$ by

$$t \cdot (g, t_1, ..., t_n) = (gt, t^{-1}t_1t, ..., t^{-1}t_nt) = (gt, t_1, ..., t_n).$$

Hence T acts trivially on itself. So the map θ_n is T–invariant and factors through $(G \times T^n)/T = G \times_T T^n$

$$\begin{tikzcd} G \times T^n \arrow[r, "\theta_n"] \arrow[d, "q"] & \mathrm{Hom}(\mathbb{Z}^n, G) \\ G \times_T T^n = G/T \times T^n. \arrow[ur, "\widetilde{\theta_n}"] & \end{tikzcd}$$

Let NT be the normalizer of the maximal torus T and let $W = NT/T$ be the Weyl group of G. Then W acts on $G/T \times T^n$ by

$$w \cdot (gT, t_1, ..., t_n) = (gwT, w^{-1}t_1w, ..., w^{-1}t_nw).$$

where gT is a coset in G/T. It follows that the map $\widetilde{\theta_n}$ is W–invariant since

$$\begin{aligned}(gw, w^{-1}t_1w, ..., w^{-1}t_nw) &= ((gw)w^{-1}t_1w(gw)^{-1}, ..., (gw)w^{-1}t_nw(gw)^{-1}) \\ &= (gt_1g^{-1}, ..., gt_ng^{-1}).\end{aligned}$$

Therefore, the map $\widetilde{\theta_n}$ factors through $G \times_{NT} T^n$

$$\begin{tikzcd} G \times_T T^n \arrow[r, "\widetilde{\theta_n}"] \arrow[d, "q'"] & \mathrm{Hom}(\mathbb{Z}^n, G) \\ G \times_{NT} T^n. \arrow[ur, "\widehat{\theta_n}"] & \end{tikzcd}$$

The following proposition is due to A. Adem and F. Cohen [Adem and Cohen, 2007].

Proposition 3.25 (Adem & Cohen)**.** *If every abelian subgroup of G is contained in a path–connected abelian subgroup, then the space* $\mathrm{Hom}(\mathbb{Z}^n, G)$ *is path–connected.*

Hence, if G has a strong maximal torus T, then $\mathrm{Hom}(\mathbb{Z}^n, G)$ is path connected. Let G be a compact and connected Lie group that has a strong maximal torus T. Consider the action map

$$\begin{aligned} G \times \mathrm{Hom}(\mathbb{Z}^n, G) &\longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G) \\ g \cdot (g_1, ..., g_n) &\longmapsto (g_1^g, ..., g_n^g). \end{aligned}$$

Then the fixed set of the strong maximal torus T is $(\mathrm{Hom}(\mathbb{Z}^n, G))^T = T^n$ and the map

$$G \times (\mathrm{Hom}(\mathbb{Z}^n, G))^T \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

is surjective since every abelian subgroup of G is conjugate to a subgroup of T. We get surjective maps

$$\theta_n : G \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G).$$

Therefore, the map $\widehat{\theta_n} : G \times_{NT} T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$ is surjective.

Lemma 3.26. *Let G be a compact and connected Lie group with strong maximal torus T. Then the maps $\widehat{\theta_n}$ induce a map Θ*

$$G \times_{NT} J(T) \xrightarrow{\Theta} X(2, G)$$

which is surjective.

Proof. This map is obtained by considering the James reduced product on the second coordinate of the spaces $G \times T^n$ and then factoring through the quotient $G \times_{NT} T^n$ as all the maps are NT–invariant. On the other hand the construction of $X(2, G)$ is applied on the spaces of commuting elements. Surjectivity follows from the comments preceding this lemma. $\square$

We study the map Θ to understand the homology of $X(2, G)$. Note that in general a compact Lie group does not have a strong maximal torus, for example $SO(3)$ and G_2. Lie groups with a strong maximal torus include the groups $U(n), SU(n)$ and $Sp(n)$. These cases will be treated separately.

Definition 3.27. Let G be a Lie group. The space $S(\mathrm{Hom}(\mathbb{Z}^n, G))$ denotes the subspace of $\mathrm{Hom}(\mathbb{Z}^n, G)$ consisting of $n-$tuples, such that at least one coordinate is the basepoint of G, that is, the identity $1 \in G$. Sometimes the notation $S_n(G)$ is used for $S(\mathrm{Hom}(\mathbb{Z}^n, G))$. Denote the quotient of $\mathrm{Hom}(\mathbb{Z}^n, G)$ by $S(\mathrm{Hom}(\mathbb{Z}^n, G))$ as follows

$$\widehat{\mathrm{Hom}}(\mathbb{Z}^n, G) = \mathrm{Hom}(\mathbb{Z}^n, G)/S(\mathrm{Hom}(\mathbb{Z}^n, G)).$$

Before giving explicit homology computations of $X(2, G)$, it is useful to note the following stable decomposition of this space for any Lie group G.

Proposition 3.28. *Let G be a Lie group. There is a stable homotopy equivalence*

$$\Sigma X(2, G) \simeq \Sigma \bigvee_{n\geq 1} \widehat{\mathrm{Hom}}(\mathbb{Z}^n, G).$$

Proof. The proof here is similar to the proof of theorem 3.16. Define a filtration of $X(2, G)$ as follows

$$X_1(2, G) \subseteq X_2(2, G) \subseteq \cdots \subseteq X_q(2, G) \subseteq X_{q+1}(2, G) \subseteq \cdots$$

where each stage of the filtration is defined to be

$$X_q(2, G) := \big(\coprod_{i\leq q} \mathrm{Hom}(\mathbb{Z}^i, G)\big)/\sim .$$

Hence, $X_q(2, G)$ is the space of words of length at most q, such that the letters of the words pairwise commute in G. Then it follows that the quotient of two consecutive stages of the filtration is

$$X_{q+1}(2, G)/X_q(2, G) \simeq \mathrm{Hom}(\mathbb{Z}^{q+1}, G)/S(\mathrm{Hom}(\mathbb{Z}^{q+1}, G))$$

where $S(\mathrm{Hom}(\mathbb{Z}^{q+1}, G))$ is the same as in Definition 3.27. Therefore, after taking the suspension, it follows that

$$\Sigma\big(X_{q+1}(2, G)/X_q(2, G)\big) \simeq \Sigma\big(\mathrm{Hom}(\mathbb{Z}^{q+1}, G)/S(\mathrm{Hom}(\mathbb{Z}^{q+1}, G))\big).$$

To prove the decomposition in the theorem, we use induction on the filtration of $X(2, G)$. First, for $q = 1$ it follows from definitions that $X_1(2, G) = G$ and $\mathrm{Hom}(\mathbb{Z}, G)/S(\mathrm{Hom}(\mathbb{Z}, G)) = G$. Hence, there is a homotopy equivalence

$$\Sigma X_1(2, G) \simeq \Sigma\mathrm{Hom}(\mathbb{Z}, G)/S(\mathrm{Hom}(\mathbb{Z}, G)).$$

There is morphism of cofibrations as follows

$$\begin{tikzcd}
\Sigma X_1(2,G) \arrow[r, "\simeq"] \arrow[d] & \Sigma \bigvee_{1\le i\le 1} \widehat{\mathrm{Hom}}(\mathbb{Z}^i,G) \arrow[d] \\
\Sigma X_2(2,G) \arrow[r, dotted, "h_2"] \arrow[d] & \Sigma \bigvee_{1\le i\le 2} \widehat{\mathrm{Hom}}(\mathbb{Z}^i,G) \arrow[d] \\
\Sigma\big(X_2(2,G)/X_1(2,G)\big) \arrow[r, "\simeq"] & \Sigma\widehat{\mathrm{Hom}}(\mathbb{Z}^2,G).
\end{tikzcd}$$

Thus, h_2 is homotopy equivalence equivalence. Now assume for q to get the following diagram

$$\begin{tikzcd}
\Sigma X_q(2,G) \arrow[r, "\simeq"] \arrow[d] & \Sigma \bigvee_{1\le i\le q} \widehat{\mathrm{Hom}}(\mathbb{Z}^i,G) \arrow[d] \\
\Sigma X_{q+1}(2,G) \arrow[r, dotted, "h_{q+1}"] \arrow[d] & \Sigma \bigvee_{1\le i\le q+1} \widehat{\mathrm{Hom}}(\mathbb{Z}^i,G) \arrow[d] \\
\Sigma\big(X_{q+1}(2,G)/X_q(2,G)\big) \arrow[r, "\simeq"] & \Sigma\widehat{\mathrm{Hom}}(\mathbb{Z}^{q+1},G).
\end{tikzcd}$$

Similarly, h_{q+1} is an equivalence and the theorem follows. $\square$

Recall the stable decomposition of the spaces $\mathrm{Hom}(\mathbb{Z}^n,G)$ for G a closed subgroup of $GL_n(\mathbb{C})$, due to A. Adem and F. Cohen, stated here as Theorem 1.1. There is a stable homotopy equivalence

$$\Sigma(\mathrm{Hom}(\mathbb{Z}^n,G)) \simeq \bigvee_{1\le k\le n} \Sigma\Big(\bigvee^{\binom{n}{k}} \mathrm{Hom}(\mathbb{Z}^k,G)/S_k(G)\Big)$$

where $\mathrm{Hom}(\mathbb{Z}^n,G)/S_n(G) = \widehat{\mathrm{Hom}}(\mathbb{Z}^n,G)$. Therefore, it is possible to obtain all the stable summands of $\Sigma(\mathrm{Hom}(\mathbb{Z}^n,G))$ from the stable summands of $\Sigma X(2,G)$. It follows that if certain homotopy theoretic invariants, such as the homology or cohomology groups, of the space $X(2,G)$ can be determined, then information about the spaces $\mathrm{Hom}(\mathbb{Z}^n,G)$ can be obtained.

3.4 The homology of $X(2,G)$

Let G be a compact and connected Lie group with strong maximal torus T. Consider the surjective maps

$$\theta_n : G \times T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G)$$

$$\widehat{\theta}_n : G \times_{NT} T^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, G).$$

To find the homology of the space $X(2,G)$ we will use a list of technical lemmas which are stated next. More general versions of these lemmas appear in a paper by T. Baird [Baird, 2007].

Let F be a field and $\chi(F)$ be the characteristic of F. For the remaining of this chapter let R denote the ring $\mathbb{Z}[|W|^{-1}]$.

Lemma 3.29. *Let G be a compact and connected Lie group with strong maximal torus T. Then $H_*(G/NT, F) \cong H_*(pt, F)$ for $(\chi(F), |W|) = 1$.*

By an analogue of Lemma 3.1 and 3.2 in [Baird, 2007], we have the following lemma

Lemma 3.30. *If G is a compact and connected Lie group with strong maximal torus T, then $H_*(\widehat{\theta}_n^{-1}(g_1, ..., g_n); R)$ is isomorphic to $H_*(pt, R)$.*

Proof. Let $X = T^n$ and let T act on X by conjugation. Then the fixed point set of the action is $X^T = X$. Thus, the fibre of the action map $\phi : G \times X^T \to X$ is $\phi^{-1}(x) = G_x^0 NT$, where G_x^0 is the connected component of the stabilizer of x in G, and

$$\widehat{\theta}_n^{-1}(x) \equiv \phi^{-1}(x)/NT = G_x^0 NT/NT \cong G_x^0/N_{G_x^0}T.$$

If W_G is the Weyl group of G and $W_{G_x^0}$ is the Weyl group of G_x^0, it follows that $|W_{G_x^0}| \mid |W_G|$. Hence, by $A.4$ in Appendix A in [Baird, 2007] it follows that there are isomorphisms $H_*(\widehat{\theta}_n^{-1}(x), R) = H_*(pt, R)$. Equivalently, the reduced homology groups of the fibres with coefficients in R are trivial. $\square$

The following theorem will be used to state the next result. A version of the following theorem also appears in [Baird, 2007].

Theorem 3.31 (Vietoris & Begle)**.** *Let $h : X \longrightarrow Y$ be a closed surjection, where X is a paracompact Hausdorff space. Suppose that for all $y \in Y$, $H_*(h^{-1}(y), R) = H_*(pt, R)$. Then the induced maps in homology $h_* : H_*(X, R) \longrightarrow H_*(Y, R)$ are isomorphisms.*

Now as a consequence of 3.30 and 3.31, the following theorem holds,

Theorem 3.32. *If G is a compact and connected Lie group with strong maximal torus T, then the map $\Theta : G \times_{NT} J(T) \longrightarrow X(2, G)$ induces a homology isomorphism with coefficients in $R = \mathbb{Z}[|W|^{-1}]$, where $|W|$ is the order of the Weyl group. That is, there are isomorphisms*

$$\Theta_* : H_*(G \times_{NT} J(T); R) \cong H_*(X(2, G); R).$$

This theorem, as stated, does not yet give explicit computations for the homology of $X(2, G)$. However, it will suffice to compute the homology groups of the space $G \times_{NT} J(T)$. Before computing these homology groups note that using Theorem 3.16 and Proposition 3.28 the following proposition holds.

Proposition 3.33. *Let G be a compact and connected Lie group with strong maximal torus T. There are isomorphisms*

$$\begin{aligned} H_*\big(X(2, G); R\big) &\cong H_*\big(\bigvee_{n \geq 1} \widehat{\mathrm{Hom}}(\mathbb{Z}^n, G); R\big) \\ &\cong H_*\big(G/NT \vee \big(\bigvee_{n \geq 1} (G \times_{NT} \widehat{T}^n)/(G \times_{NT} \{1\})\big); R\big). \end{aligned}$$

Proof. This follows immediately from the stable decompositions in Theorem 3.16 and Proposition 3.28. $\square$

However, this proposition will not be used directly in computations here. This proposition only shows a different way to compute the homology. Here we will use a spectral sequence argument to find the homology groups.

Consider the following classical result. A version with coefficients being the complex numbers can be found in [Baird, 2007].

Theorem 3.34. *Let G be a Lie group with maximal torus T. As an ungraded left $W-$module,*

$$H_*(G/T; R) \cong R[W]$$

where $R[W]$ is the group ring of W over the ring R.

We first compute the homology groups of the space $G \times_{NT} J(T)$. Let $\mathcal{T}[M]$ denote the tensor algebra on an R–module M.

Proposition 3.35. *Let G be a compact and connected Lie group with strong maximal torus T. Then the homology groups of $G \times_{NT} J(T)$ with coefficients in R are given by*

$$H_*(G \times_{NT} J(T); R) \cong \big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

Proof. There is a short exact sequence of groups $1 \to T \to NT \to W \to 1$, and associated to it, there is a fibration sequence

$$\big(G \times J(T)\big)/T \to \big(G \times J(T)\big)/NT \to BW,$$

which is equivalent to the fibration

$$G \times_T J(T) \longrightarrow G \times_{NT} J(T) \longrightarrow BW.$$

The Leray spectral sequence has second page given by the groups

$$E^2_{p,q} = H_p(BW; H_q(G \times_T J(T); P))$$

which converges to $H_{p+q}(G \times_{NT} J(T); P)$, where P is a ring. If $|W|^{-1} \in P$ it follows that $E^2_{s>0,t} = 0$ and the groups on the vertical axis are given by

$$E^2_{0,t} = H_0\big(BW; H_t(G \times_T J(T); P)\big).$$

As a special case, if we let $P = R = \mathbb{Z}[|W|^{-1}]$, then

$$E^2_{0,t} = E^{0,t}_{\infty} = H_0\big(BW; H_t(G \times_T J(T); R)\big)$$

Recall that homology in degree 0 is given by the coinvariants

$$H_0\big(BW; H_t(G \times_T J(T); R)\big) = \big(H_t(G \times_T J(T); R)\big)_W.$$

Also T acts by conjugation and thus trivially on T^n, so it acts trivially on $J(T)$. Hence, $G \times_T J(T) = G/T \times J(T)$.

The *flag variety* G/T has torsion free integer homology, see [Bott, 1954], and so does $J(T)$. So the homology of $G/T \times J(T)$ with coefficients in R is given by the following tensor product

$$H_t(G \times_T J(T); R) = \bigoplus_{i+j=t} \big[H_i(G/T; R) \bigotimes_R H_j(J(T); R)\big].$$

The spectral sequence collapses at the E^2 term as stated above, hence,

$$\begin{aligned} H_t(G \times_T J(T); R) &\cong \big(H_t(G \times_T J(T); R)\big)_W \\ &\cong \left(\bigoplus_{i+j=t} \big[H_i(G/T; R) \bigotimes_R H_j(J(T); R)\big]\right)_W. \end{aligned}$$

Using theorem 3.34, it follows that

$$H_*(G \times_{NT} J(T); R) = \big(R[W] \otimes_R H_*(J(T); R)\big)_W.$$

Recall that the homology of $J(T)$ is the tensor algebra on the reduced homology of T. Let $\mathcal{T}[V]$ denote the tensor algebra on the reduced homology of T, denoted by V. Then there is an isomorphism

$$H_*(G \times_{NT} J(T); R) = \big(R[W] \otimes_R \mathcal{T}[V]\big)_W$$

□

The results proved so far will suffice to finally state the following theorem.

Theorem 3.36. *Let G be a compact Lie group with strong maximal torus T. Then the homology of $X(2,G)$ with coefficients in R is*

$$H_*(X(2,G);R) \cong \big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

Proof. This theorem is an immediate corollary of Theorem 3.32 and Proposition 3.35. □

This theorem concludes the section. Examples are given in the next section.

3.5 Examples

Let G be a compact and connected Lie group. Assume that G has a strong maximal torus T. Examples of such Lie group are the groups $U(n), SU(n)$ and $Sp(n)$. To determine the homology of $X(2,G)$ for a particular Lie group G as stated in Theorem 3.32, it is necessary to determine the action of the Weyl group of G on the tensor product. More precisely, it is necessary to determine the coinvariants

$$\big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

For the unitary group $U(n)$ it is well-known that the Weyl group is isomorphic to the symmetric group on n letters Σ_n. Hence, it follows from Theorem 3.32 that

$$H_*(X(2,U(n));\mathbb{Z}[|\Sigma_n|^{-1}]) \cong \big(\mathbb{Z}[|\Sigma_n|^{-1}][\Sigma_n] \otimes_{\mathbb{Z}[|\Sigma_n|^{-1}]} \mathcal{T}[V]\big)_{\Sigma_n}$$

that is,

$$H_*(X(2,U(n));\mathbb{Z}[\frac{1}{n!}]) \cong \big(\mathbb{Z}[\frac{1}{n!}][\Sigma_n] \otimes_{\mathbb{Z}[\frac{1}{n!}]} \mathcal{T}[V]\big)_{\Sigma_n}.$$

The table below shows the Weyl groups for some connected and compact Lie groups with strong maximal tori. A reference for Lie groups in general and for the Weyl groups in particular is the book by J. F. Adams, "Lectures on Lie groups"

Lie group G	Weyl group W	$\|W\|$	Rank
$U(n)$	Σ_n	$n!$	n
$SU(n)$	Σ_n	$n!$	$n-1$
$Sp(2n)$	$\Sigma_2 \wr \Sigma_n$	$2^n n!$	n

Table 3.1: G with a strong maximal torus

[Adams, 1969]. Recall that the group $\Sigma_2 \wr \Sigma_n$ is the wreath product of Σ_2 and Σ_n where Σ_n acts on the n–fold product $\Sigma_2 \times \cdots \times \Sigma_2$ by permuting coordinates. In Appendix A the case of the unitary group $U(n)$ will be investigated.

Recall that not all compact and connected Lie groups have a *strong maximal torus*. A basic example is the special orthogonal group $SO(3)$. The maximal torus of $SO(3)$ has rank 1 and is isomorphic to S^1. However, $SO(3)$ has a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and clearly it can not be conjugated to a subgroup of S^1. Therefore, $T = S^1$ is not a strong maximal torus. It turns out that the Lie groups $SO(2n)$ and $SO(2n+1)$ have Weyl groups as shown in Table 3.2 below. Recall

Lie group G	Weyl group W	$\|W\|$	Rank
$SO(2n+1)$	$\Sigma_2 \wr \Sigma_n$	$2^n n!$	n
$SO(2n)$	$\Sigma_2 \wr A_n$	$2^n(\frac{n!}{2})$	n

Table 3.2: G with no strong maximal torus

that the group A_n is the subgroup of even permutations in Σ_n. Tables 3.1 and 3.2 will suffice for this chapter, even though not very instructive, as we are not momentarily investigating the module structure of the homology. For more details see [King and Al-Qubanchi, 1981].

Next we give the homotopy type of $X(2, SO(3))$. It was proven by E. Torres-Giese and D. Sjerve in [Torres Giese and Sjerve, 2008] that there is a homeomorphism of spaces

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3)) \approx \big(\bigsqcup_{x_n} S^3/Q_8\big) \bigsqcup \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$$

where x_n is given in Theorem 2.4 in [Torres Giese and Sjerve, 2008] and the space $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$ is the connected component of the identity representation. The integer x_n is given by

$$x_k = \begin{cases} \frac{1}{6}(4^n - 3 \cdot 2^n + 2) & : \text{ if } n \text{ is even} \\ \frac{2}{3}(4^{n-1} - 1) - 2^{n-1} + 1 & : \text{ if } n \text{ is odd.} \end{cases}$$

The quotient S^3/Q_8 can be obtained by taking the quotient $SO(3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is group multiplication if we consider it as the subgroup of $SO(3)$ generated by $\{diag(-1, -1, 1), diag(-1, 1, -1)\}$.

Proposition 3.37. *There is a homotopy equivalence*

$$X(2, SO(3)) = X(2, SO(3))_1 \bigsqcup \big(\bigsqcup_{\infty} S^3/Q_8\big).$$

Proof. The proof of this proposition will be given in the form of a discussion.

For every connected component S^3/Q_8, there is an n–tuple $(A_1, ..., A_n)$, where there are two matrices A_i, A_j which are involutions about orthogonal axes, and all the other A_k's are one of $I, A_i, A_j, A_iA_j = A_jA_i$. There is a natural inclusion

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3)) \hookrightarrow \mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))$$

given by $(A_1, ..., A_n) \hookrightarrow (A_1, ..., A_n, I)$. Therefore, there is an inclusion

$$\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1 \hookrightarrow \mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))_1$$

because of the property described above. It follows from [Torres Giese and Sjerve, 2008] that $x_n < x_{n+1}$, that is the number of connected components in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))$ is strictly larger than the number of connected components in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$, so the inclusion above misses some copies of S^3/Q_8 in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))$.

Recall that for any Lie group G, the space $X(2,G)$ is defined by

$$X(2,G) = \big(\bigsqcup_{n\geq 1} \mathrm{Hom}(\mathbb{Z}^n, G)\big)/\sim$$

with the single relation being $(y_1, ...y_{i-1}, 1, y_{i+1}, ..., y_n) \sim (y_1, ...y_{i-1}, y_{i+1}, ..., y_n)$. Therefore, it follows by definition that

$$X(2, SO(3)) = \big(\bigsqcup_{n\geq 1} \big(\big(\bigsqcup_{x_n} S^3/Q_8\big) \bigsqcup \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1\big)\big)/\sim .$$

Note that the n–tuple $(A_1, ..., A_n)$ is in he component $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$ if and only if $(A_1, ..., A_n, I)$ is in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))_1$. Similarly, $(A_1, ..., A_{i-1}, I, A_{i+1}, ..., A_n)$ is in the component $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))_1$, if and only if $(A_1, ..., A_{i-1}, A_{i+1}, ..., A_n)$ is in the space $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$. The same fact rephrased states that the n–tuple $(A_1, ..., A_n)$ is not in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$ if and only if $(A_1, ..., A_n, I)$ is not in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))_1$. And the $(n+1)$–tuple $(A_1, ..., A_{i-1}, I, A_{i+1}, ..., A_n)$ is not in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))_1$, if and only if $(A_1, ..., A_{i-1}, A_{i+1}, ..., A_n)$ is not in the space $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$. These arguments suffice to show that

$$X(2, SO(3)) = \big(\bigsqcup_{n\geq 1} \big(\bigsqcup_{x_n} S^3/Q_8\big)\big)/\sim \bigsqcup \big(\bigsqcup_{n\geq 1} \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1\big)/\sim .$$

Define a space

$$X(2, SO(3))_1 = \big(\bigsqcup_{n\geq 1} \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1\big)/\sim .$$

Clearly, the space $X(2, SO(3))_1$ is path–connected. It remains to identify the path components of the space

$$\big(\bigsqcup_{n\geq 1} \big(\bigsqcup_{x_n} S^3/Q_8\big)\big)/\sim .$$

Consider the following properties from [Torres Giese and Sjerve, 2008], of paths not in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$. Let $p(t) = (A_1(t), ..., A_n(t))$ be a path not in the component $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$. Then the following statements hold:

1. If some $A_i(0) = I$, then $A_i(t) = I$ for all t.

2. If $A_i(0) = A_j(0)$, then $A_i(t) = A_j(t)$ for all t.

3. If $A_i(0)$ and $A_j(0)$ are distinct involutions then so are $A_i(t), A_j(t)$ for all t.

4. If $A_k(0) = A_i(0)A_j(0)$, then $A_k(t) = A_i(t)A_j(t)$ for all t.

Therefore, if $(A_1, ..., A_n)$ is an n-tuple not in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$, for which none of the components is I, and $p(t) = (A_1(t), ..., A_n(t))$ is a path containing $(A_1, ..., A_n)$, then none of the $A_i(t)$ is I, for any i and any t. Clearly, such an n–tuple can be found for any positive integer n. Therefore there are infinitely many path components in the space

$$(\bigsqcup_{n\geq 1}(\bigsqcup_{x_n} S^3/Q_8))/\sim .$$

By definition it follows that

$$(A_1, ..., A_{i-1}, I, A_{i+1}, ..., A_n) \sim (A_1, ..., A_{i-1}, A_{i+1}, ..., A_n).$$

Hence, if none o the m-tuples is in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$, from the statements above it follows that the component S^3/Q_8 corresponding to $(A_1, ..., A_{i-1}, I, A_{i+1}, ..., A_n)$ in $\mathrm{Hom}(\mathbb{Z}^{n+1}, SO(3))$ is identified with the component S^3/Q_8 corresponding to $(A_1, ..., A_{i-1}, A_{i+1}, ..., A_n)$ in $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$. Therefore, the connected components of

$$(\bigsqcup_{n\geq 1}(\bigsqcup_{x_n} S^3/Q_8))/\sim$$

are all homotopy equivalent to S^3/Q_8. Moreover, the number of these components is countably infinite. $\square$

Note that the space $SO(3) \times_{NS^1} (S^1)^n$ is path–connected, but the on the other hand the space $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$ is not path–connected. Hence, the map $\theta_n : SO(3) \times_{NS^1} (S^1)^n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, SO(3))$ is not a surjection. However, the image of the map θ_n surjects onto the connected component $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_1$. Therefore, there are surjections

$$\theta_n^1 : SO(3) \times_{NS^1} (S^1)^n \twoheadrightarrow \mathrm{Hom}(\mathbb{Z}^n, SO(3))_1,$$

where θ_n^1 is the same as θ_n and the only change is the space it lands in. The maps θ_n^1 induce a surjection

$$\Theta_1 : SO(3) \times_{NS^1} J(S^1) \twoheadrightarrow X(2, SO(3))_1.$$

Proposition 3.38. *The fibres of θ_n^1 have trivial reduced homology when the coefficients are taken to be $\mathbb{Z}[\frac{1}{2}]$.*

Proof. The Weyl group of $SO(3)$ is isomorphic to Σ_2, which has order 2. Hence, the proposition follows from Lemma 3.30 since the only condition required in that same setting, is that θ_n^1 surjects. □

Therefore, the homology groups of $X(2, SO(3))$ with coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$ is given as follows

Theorem 3.39. *There is an isomorphism of homology groups with coefficients in $\mathbb{Z}[\frac{1}{2}]$ given by*

$$H_*(X(2, SO(3))_1; \mathbb{Z}[\frac{1}{2}]) \cong \big(\mathbb{Z}[\frac{1}{2}]\Sigma_2 \otimes_{\mathbb{Z}[\frac{1}{2}]} \mathcal{T}[V]\big)_{\Sigma_2}.$$

Proof. The Weyl group W of $SO(3)$ is Σ_2, hence, the ring $\mathbb{Z}[|W|^{-1}]$ is equal to $\mathbb{Z}[\frac{1}{2}]$. The map Θ_1

$$\Theta_1 : SO(3) \times_{NS^1} J(S^1) \twoheadrightarrow X(2, SO(3))_1$$

is a surjection. Hence, it follows from Proposition 3.38 that there is an isomorphism

$$H_*(SO(3) \times_{NS^1} J(S^1); \mathbb{Z}[\frac{1}{2}]) \cong H_*(X(2, SO(3))_1; \mathbb{Z}[\frac{1}{2}]).$$

With the same calculations as in Theorem 3.36, it follows that the homology of $X(2, SO(3))$ with coefficients in $\mathbb{Z}[\frac{1}{2}]$ is given by

$$H_*(X(2, SO(3))_1; \mathbb{Z}[\frac{1}{2}]) \cong \big(\mathbb{Z}[\frac{1}{2}]\Sigma_2 \otimes_{\mathbb{Z}[\frac{1}{2}]} \mathcal{T}[V]\big)_{\Sigma_2},$$

where V is the reduced homology of S^1 as a Σ_2–module and $\mathcal{T}[V]$ is the tensor algebra over V, again as a Σ_2–module. $\square$

3.6 Homology in general

In this section we state and prove the main theorem of this chapter. The main observation is that the construction in the proof of Theorem 3.39 holds in general and is used to prove the main theorem.

Let G be a compact and connected Lie group and T be a maximal torus of G. Let $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ be the connected component of the trivial representation $(1, ..., 1)$ in $\mathrm{Hom}(\mathbb{Z}^n, G)$. Since T^n consists of commuting $n-$tuples and is path-connected, it is a subspace of $\mathrm{Hom}(\mathbb{Z}^n, G)_1 \subseteq G^n$. G acts on the space $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ by conjugation, that is, there is an action given by

$$G \times \mathrm{Hom}(\mathbb{Z}^n, G)_1 \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$$

$$g \times (t_1, ..., t_n) = (t_1^g, ..., t_n^g).$$

The fixed point set of the action of the maximal torus T is $(\mathrm{Hom}(\mathbb{Z}^n, G)_1)^T = T^n$. Therefore, there is a map

$$G \times (\mathrm{Hom}(\mathbb{Z}^n, G)_1)^T \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$$

which is a surjection since every point in $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ is fixed by a maximal torus in G and all the maximal tori in G are conjugate, hence every G orbit must intersect the T fixed point set $(\mathrm{Hom}(\mathbb{Z}^n, G)_1)^T$. This same argument appears in section 3 in [Baird, 2007] and is valid for a more general setting where the space of commuting elements is replaced by another $G-$space X.

Therefore, there are surjections

$$\theta_n^1 : G \times T^n \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1$$

where the maps θ_n^1 are $NT-$invariant for all n. Note that if T has a strong maximal torus, then this follows also from definitions.

Similar to Section 3.3, it can be shown that the maps θ_n^1 are NT–equivariant. Hence, there are surjections

$$\hat{\theta}_n^1 : G \times_{NT} T^n \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Hom}(\mathbb{Z}^n, G)_1.$$

Similar to lemma 3.30, the following holds. Note that this result can be also proved by a direct inspection of the definitions if a strong maximal torus is assumed. Let $R = \mathbb{Z}[|W|^{-1}]$.

Lemma 3.40. *If G is a connected and compact Lie group with maximal torus T and Weyl group W, then $H_*((\hat{\theta}_n^1)^{-1}(g_1, ..., g_n); R)$ is isomorphic to the homology of a point $H_*(pt, R)$.*

Proof. The lemma follows from Lemma 3.30 since the only condition required in that same setting, is that $\hat{\theta}_n^1$ surjects. □

Taking the James reduced product on the second coordinate of the spaces $G \times T^n$ to get $G \times_{NT} J(T)$, and the similar construction to get the space $X(2, G)_1$, there is a surjection

$$G \times_{NT} J(T) \longrightarrow\!\!\!\!\!\rightarrow X(2, G)_1$$

where $X(2, G)_1$ is defined to be the space

$$X(2, G)_1 := \big(\coprod_{n \geq 1} \mathrm{Hom}(\mathbb{Z}^n, G)_1\big)/ \sim$$

and $\sim$ is the single relation that deletes the basepoint. Using Lemma 3.40 and Theorem 3.31 it follows that

Theorem 3.41. *Let G be a connected and compact Lie group with maximal torus T and Weyl group W. Then there is an isomorphism in homology*

$$H_*(G \times_{NT} J(T); R) \cong H_*(X(2,G)_1; R).$$

The theorem below summarizes this section.

Theorem 3.42. *Let G be a connected and compact Lie group with maximal torus T and Weyl group W. Then there is an isomorphism in homology*

$$H_*(X(2,G)_1; R) \cong \big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

Proof. The homology of $G \times_{NT} J(T)$ with coefficients in the ring R can be computed in exactly the same way as for Theorem 3.36 which is isomorphic to

$$\big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

Hence, using Theorem 3.41 the result follows. $\square$

This theorem applies to all groups considered previously. For example if $G = U(n)$, $SU(n)$ or $Sp(n)$ it follows from Proposition 3.25 that $\mathrm{Hom}(\mathbb{Z}^n, G)$ is path-connected for these Lie groups. Thus, $X(2,G)$ is also path-connected and it follows that $X(2,G)_1 = X(2,G)$. Therefore, theorem 3.32 is recovered by theorem 3.42.

Note that as mentioned before, to find the homology groups of $X(2,G)_1$ explicitly for G a connected and compact Lie group, it is necessary to know the action of the Weyl group on the tensor product

$$R[W] \otimes_R \mathcal{T}[V].$$

More precisely, it is necessary to find the coinvariants.

$$\big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

This again leads the subject to representation theory, which will be discussed in the next Sections and in Appendix A. Theorem 3.42 can be used to also study the cases of the compact and connected simple exceptional Lie groups G_2, F_4, E_6, E_7 and E_8.

Let R denote the ring R. Let H_*^U denote ungraded homology and $\mathcal{T}_U[V]$ denote the ungraded tensor algebra over V. It is important to note that if the ungraded tensor algebra is considered the following theorem holds.

Theorem 3.43. *Let G be a connected and compact Lie group with maximal torus T and Weyl group W. Then there is an isomorphism in ungraded homology*

$$H_*^U(X(2,G)_1;R) \cong \mathcal{T}_U[V].$$

Proof. From Theorem 3.42 there is an isomorphism in homology given by

$$H_*(X(2,G)_1;R) \cong \big(H_*(G/T;R) \otimes_R \mathcal{T}[V]\big)_W.$$

If all homology is ungraded, then there are isomorphisms in ungraded homology given by

$$H_*^U(X(2,G)_1;R) \cong \big(RW \otimes_R \mathcal{T}_U[V]\big)_W \cong RW \otimes_{RW} \mathcal{T}_U[V] \cong \mathcal{T}_U[V].$$

□

This shows that as an abelian group, without the grading, the homology of $X(2,G)_1$ with coefficients in R is just the ungraded tensor algebra $\mathcal{T}_U[V]$. The following is an immediate corollary of Theorem 3.43.

Corollary 3.44. *Let G be a connected and compact Lie group with strong maximal torus T and Weyl group W. Then there is an isomorphism in ungraded homology*

$$H_*^U(X(2,G);R) \cong \mathcal{T}_U[V].$$

3.7 Classical Representations

Let G be a compact and connected Lie group. The homology of the space of commuting $n-$tuples $\mathrm{Hom}(\mathbb{Z}^n, G)$ is not well–understood. Homology of the spaces $X(2,G)$ and $X(2,G)_1$ was studied in Sections 3.4 and 3.6, respectively, with the goal to understand the homology of $\mathrm{Hom}(\mathbb{Z}^n, G)$. Now let T be the maximal torus of G and W be the Weyl group. One of the implications of Theorem 3.42 is that the homology of $X(2,G)_1$, and thus the homology of $\mathrm{Hom}(\mathbb{Z}^n, G)$, is given purely in terms of classical representation theory, described by the coinvariants

$$H_*(X(2,G)_1; R) \cong \big(R[W] \otimes_R \mathcal{T}[V]\big)_W.$$

This section is devoted to studying the module structure of the tensor algebra $\mathcal{T}[V]$, which comes from V having a W–module structure. Here we lay out only the theoretical methods. Some computations are done in Appendix A.

3.7.1 Poincaré–Birkhoff–Witt theorem

Recall that the Weyl group W acts on $J(T)$ by conjugation, which induces an action of W on the tensor algebra $H_*(J(T);\mathbb{Z}) \cong \mathcal{T}[V]$, where $V = \widetilde{H}_*(T)$. Let R denote the ring $\mathbb{Z}[|W|^{-1}]$. If G has rank n, then the tensor algebra is given by

$$\mathcal{T}[V] = \mathcal{T}[\bigoplus_{1\le k\le n} H_k(T;R)]$$

which has the structure of a RW–module. Also recall that

$$\mathcal{T}[V] = \bigoplus_{d\ge 1} V^{\otimes d}$$

where the summands $V^{\otimes d}$ are given by

$$V^{\otimes d} = \bigoplus_{1\le i_j\le n} \big(H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_d}\big).$$

Identifications of the tensor products $H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_d}$ as representations gives the structure of $V^{\otimes d}$ as a RW–module, where W acts diagonally.

Definition 3.45. Let V be a graded free abelian group. The graded *free Lie algebra* generated by V is the smallest graded sub–Lie algebra in $\mathcal{T}[V]$ generated by V. Denote the graded free Lie algebra by $L[V]$. The Lie tensors of weight q are given by $L_q[V] = L[V] \cap V^{\otimes q}$.

Therefore, it follows that $L[V] = \bigoplus_{q \geq 1} L_q[V]$. The following theorem is due to J. Milnor and J. Moore, see [Milnor and Moore, 1965].

Theorem 3.46 (Milnor & Moore)**.** *Let V be a free R–module. There is an isomorphism of graded abelian groups*

$$S(L[V]) \cong \mathcal{T}[V]$$

where $S(L[V])$ is the symmetric algebra on the generators of $L[V]$.

Assume that V is concentrated in even degrees and that V is a free R–module. A special case of Theorem 3.46 for the ungraded version is called the Poincaré–Birkhoff–Witt theorem and is a classical result. A proof of the Poincaré–Birkhoff–Witt theorem can also be found in [Cohn, 1963].

It is possible to use Theorem 3.46 to compute tensor products as follows. Recall that $S(L[V])$ can be rewritten as

$$S\big(\bigoplus_{q \geq 1} L_q[V]\big) = \bigotimes_{q \geq 1} S(L_q[V]).$$

The information to keep track of in $\mathcal{T}[V]$ is the number of times each of the representations appears, as well as the tensor and homological degrees. If V is a free R–module, then the Poincaré series of the tensor algebra is given by

$$\chi(\mathcal{T}[V]) = \frac{1}{1 - \chi(V)} = \chi\left(S\big(\bigoplus_{q \geq 1} L_q[V]\big)\right) = \prod_{q=1}^{\infty} \chi\left(S(L_q[V])\right).$$

Detailed calculations for the unitary groups are given in Appendix A.

A Tensor Algebra Computations

A.1 The unitary group $U(2)$

The first example we discuss is the unitary group $G = U(2)$ which has maximal torus $T = S^1 \times S^1$ and $W = \Sigma_2$, the symmetric group on two letters. The calculations here are done over the ring $\mathbb{Z}$ instead of the ring R. Let $\mathbb{Z}_{sgn}$ denote the sign representation and $\mathbb{Z}_{triv}$ denote the trivial representation. The reduced homology of T is given by

$$H_1(T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}\Sigma_2$$

and

$$H_2(T;\mathbb{Z}) \cong \mathbb{Z} \cong \mathbb{Z}_{sgn}.$$

Hence, $V = H_1 \oplus H_2 \cong \mathbb{Z}\Sigma_2 \oplus \mathbb{Z}_{sgn}$. In an attempt to understand the representation theory and combinatorics of $\mathcal{T}[V]$ for the group $U(2)$, we compute the first few tensor products $V^{\otimes d}$ explicitly.

The first tensor product, $V \otimes V$ equals

$$(H_1 \oplus H_2) \otimes (H_1 \oplus H_2) = (H_1 \otimes H_1) \oplus (H_1 \otimes H_2) \oplus (H_2 \otimes H_1) \oplus (H_2 \otimes H_2).$$

Let $\Sigma_2 = \{1, \sigma\}$ and H_1 have a basis $\{1, \sigma\}$ such that $\sigma^2 = 1$, which makes $H_1 = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma$ into a $\mathbb{Z}\Sigma_2-$module. Therefore, the first few tensor products are given as follows

$$
\begin{aligned}
H_1 \otimes H_1 &\cong (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma) \otimes (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma) \\
&\cong [\mathbb{Z} \cdot 1 \otimes (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma)] \oplus [\mathbb{Z} \cdot \sigma \otimes (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma)] \\
&\cong (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma) \oplus (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \sigma) \otimes \mathbb{Z}_{sgn} \\
&\cong \mathbb{Z}\Sigma_2 \oplus (\mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn}) \\
&\cong \oplus_2 \mathbb{Z}\Sigma_2 \\
&\cong \oplus_{2^{2-1}} \mathbb{Z}\Sigma_2
\end{aligned}
$$

which is two copies of the group ring. Similarly, the other tensor products are given by

$$
\begin{aligned}
H_1 \otimes H_2 &\cong \mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn} \\
H_2 \otimes H_1 &\cong \mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn} \\
H_2 \otimes H_2 &\cong \mathbb{Z}_{sgn} \otimes \mathbb{Z}_{sgn} \cong \mathbb{Z}_{triv}.
\end{aligned}
$$

Therefore, the 2–fold tensor product $V \otimes V$ is given by

$$
\begin{aligned}
V \otimes V &= (H_1 \otimes H_1) \oplus (H_1 \otimes H_2) \oplus (H_2 \otimes H_1) \oplus (H_2 \otimes H_2) \\
&\cong \mathbb{Z}\Sigma_2 \oplus (\mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn}) \oplus \mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn} \oplus \mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn} \oplus \mathbb{Z}_{triv} \\
&\cong (\oplus_2 \mathbb{Z}\Sigma_2) \oplus \mathbb{Z}\Sigma_2 \oplus \mathbb{Z}\Sigma_2 \oplus \mathbb{Z}_{triv} \\
&\cong (\oplus_4 \mathbb{Z}\Sigma_2) \oplus \mathbb{Z}_{triv}
\end{aligned}
$$

In the next step we compute $V^{\otimes 3} = V \otimes V \otimes V$. In this tensor product we have summands of the form $H_1^{\otimes i} \otimes H_2^{\otimes j}$ such that $i + j = 3$. Since the order of the tensor product will matter, there are $\binom{3}{j}$ terms for each such product, that is for each such i, j. First note that for any $i \geq 1$, the i–fold tensor product of H_1 equals

$$H_1^{\otimes i} \cong \bigoplus_{2^{i-1}} \mathbb{Z}\Sigma_2$$

and for any $j \geq 1$, the j–fold tensor product of H_2 equals

$$H_2^{\otimes j} \cong \mathbb{Z}_{triv} \text{ or } \mathbb{Z}_{sgn}$$

depending on whether j is even or odd. Adopt the convention that $(\mathbb{Z}G)^0 = \mathbb{Z}_{triv}$, for G a finite group. Then, the 3–fold tensor product $V^{\otimes 3}$ splits as a sum of Σ_2 representations as follows

$$\begin{aligned} V^{\otimes 3} &\cong [\oplus_{\binom{3}{0}} H_1^{\otimes 3}] \oplus [\oplus_{\binom{3}{1}} H_1^{\otimes 2} \otimes H_2] \oplus [\oplus_{\binom{3}{2}} H_1 \otimes H_2^{\otimes 2}] \oplus [\oplus_{\binom{3}{3}} H_2^{\otimes 3}] \\ &\cong [\oplus_{\binom{3}{0}} \oplus_4 \mathbb{Z}\Sigma_2] \oplus [\oplus_{\binom{3}{1}} \oplus_2 \mathbb{Z}\Sigma_2 \otimes \mathbb{Z}_{sgn}] \oplus [\oplus_{\binom{3}{2}} \mathbb{Z}\Sigma_2] \oplus [\oplus_{\binom{3}{3}} \mathbb{Z}_{sgn}] \\ &\cong [\oplus_{\binom{3}{0}2^2+\binom{3}{1}2^1+\binom{3}{2}2^0} \mathbb{Z}\Sigma_2] \oplus \mathbb{Z}_{sgn}. \end{aligned}$$

Similarly, for any $d \geq 3$ the summands of the d–fold tensor product $V^{\otimes d}$ are isomorphic as Σ_2 representation to representations of the form $H_1^{\otimes k} \otimes H_2^{\otimes(d-k)}$, for $0 \leq k \leq d$. Each of these terms appears $\binom{d}{k}$ times in $V^{\otimes d}$, that is, there are $\binom{d}{k}$ isomorphic copies of $H_1^{\otimes k} \otimes H_2^{\otimes(d-k)}$ in $V^{\otimes d}$. Therefore, we get the following decomposition of $V^{\otimes d}$ as a direct sum of Σ_2 representations

$$\begin{aligned} V^{\otimes d} &\cong \bigoplus_{0 \leq k \leq d} \Big[\bigoplus_{\binom{d}{k}} H_1^{\otimes k} \otimes H_2^{\otimes(d-k)} \Big] \\ &\cong \bigoplus_{0 \leq k \leq d} \Big[\bigoplus_{\binom{d}{k}} \Big(\bigoplus_{2^{k-1}} \mathbb{Z}\Sigma_2 \Big) \otimes H_2^{\otimes(d-k)} \Big] \\ &\cong \Big(\bigoplus_{1 \leq k \leq d} \Big[\bigoplus_{\binom{d}{k}} \Big(\bigoplus_{2^{k-1}} \mathbb{Z}\Sigma_2 \Big) \otimes H_2^{\otimes(d-k)} \Big] \Big) \oplus [(\mathbb{Z}\Sigma_2)^0 \otimes H_2^{\otimes d}] \\ &\cong \Big(\bigoplus_{1 \leq k \leq d} \Big[\bigoplus_{\binom{d}{k}} \Big(\bigoplus_{2^{k-1}} \mathbb{Z}\Sigma_2 \Big) \Big] \Big) \oplus [H_2^{\otimes d}] \end{aligned}$$

where $H_2^{\otimes d}$ is either $\mathbb{Z}_{sgn}$ or $\mathbb{Z}_{triv}$ depending on whether d is even or odd. Therefore, in tensor degree d there are

$$f(d) = \sum_{1 \leq k \leq d} \binom{d}{k} 2^{k-1}$$

copies of the group ring $\mathbb{Z}\Sigma_2$, and one copy of $\mathbb{Z}_{sgn}$ if d is odd or a copy of $\mathbb{Z}_{triv}$ if d is even.

Note that the Euler-Poincaré series for the tensor algebra $\mathcal{T}[V]$ is

$$\chi(\mathcal{T}[V]) = \frac{1}{1-\chi(V)} = \sum_{n\geq 0}(\chi(V))^n = \sum_{n\geq 0}(2t+t^2)^n$$

therefore by definition, the rank of the homology concentrated in degree j is the coefficient of t^j in the formal power series. That is given by the sum of the coefficients of $\binom{j}{i}(2t)^i(t^2)^k$, such that $i+2k=j$, which is $\binom{j}{i}2^i$. Therefore, one can find the rank of degree j homology in tensor degree d. Recall that $V^{\otimes d}$ has summands of the form $H_1^{\otimes k}\otimes H_2^{\otimes(d-k)}$. One gets homology in degree j only if $1\cdot k+2\cdot(d-k) = 2d-k = j$, that is, $k = 2d-j$. As a remark, since $j = 2d-k$ and $0\leq k\leq d$ by definition, one has homology in degree j only in tensor degrees less than or equal to j, that is $d\leq j$.

Now fix the homological degree j and tensor degree d, such that $d\leq j$. The rank of degree j homology in $V^{\otimes d}$ is the rank of $H_1^{\otimes k}\otimes H_2^{\otimes(d-k)}$, where $k=2d-j$, times the number of its occurrences in $V^{\otimes d}$.

$$\begin{aligned}
h_d(j) &= \binom{d}{k}\mathrm{rank}_{\mathbb{Z}}(H_1^{\otimes 2d-j}\otimes H_2^{\otimes(d-(2d-j))})\\
&= \binom{d}{k}\mathrm{rank}_{\mathbb{Z}}(H_1^{\otimes 2d-j}\otimes H_2^{\otimes(j-d)})\\
&= \binom{d}{k}\mathrm{rank}_{\mathbb{Z}}([\bigoplus_{2^{2d-j-1}}\mathbb{Z}\Sigma_2]\otimes H_2^{\otimes(d-(2d-j))})\\
&= \binom{d}{k}\mathrm{rank}_{\mathbb{Z}}[\bigoplus_{2^{2d-j-1}}\mathbb{Z}\Sigma_2]\\
&= \binom{d}{k}\mathrm{rank}_{\mathbb{Z}}[\bigoplus_{2\cdot 2^{2d-j-1}}\mathbb{Z}]\\
&= \binom{d}{2d-j}2^{2d-j}
\end{aligned}$$

Using this calculation, then one can say that the rank of the $j-$th homology in $\mathcal{T}[V]$ is given by $\sum_{d\leq j} h_d(j)$. Therefore we get the following formula

$$h(j) = \sum_{0\leq i\leq j} \binom{j}{i} 2^i = \sum_{d\leq j} h_d(j)$$

Now, a lower bound for d should be a number such that there is k with $0 \leq k \leq d$ and $2d - k = j$. Therefore the lower bound for d is the least upper integer of $\frac{j}{2}$, that is $\lceil \frac{j}{2} \rceil$. Hence, we get that for fixed j

$$h(j) = \sum_{0\leq i\leq j} \binom{j}{i} 2^i = \sum_{\lceil \frac{j}{2} \rceil \leq d\leq 2d-j} \binom{d}{k} 2^{2d-j}$$

We want to have a tri–graded power series such that it encodes the information about homological degree, tensor degree and the number of copies of the group ring $\mathbb{Z}\Sigma_2$,

$$\sum_{\substack{p,r\geq 0\\ q\geq 1}} a(p,q,r)x^p y^q z^r$$

with p, q, r telling the above quantities respectively, that is, the coefficient $a(p, q, r)$ records the records the number r of copies of the group ring $\mathbb{Z}\Sigma_2$ concentrated in tensor degree q and homological degree p. For example, in fixed tensor degree q and fixed homological degree $p = 2q - i$ there are $\binom{q}{i}2^{i-1}$ copies of the group ring $\mathbb{Z}\Sigma_2$, since in $V^{\otimes q}$ there are $\binom{q}{i}$ copies of $H_1^{\otimes i} \otimes H_2^{\otimes(q-i)}$ which has $2^{(i-1)}$ copies of the group ring. So one term of the power series should be $\binom{q}{i}2^{i-1}x^{2q-i}y^q z^{\binom{q}{i}2^{i-1}}$. Therefore, in fixed tensor degree q, we have homological degree i only for $q \leq i \leq 2q$. Therefore, the term $V^{\otimes q}$ can be described by the sum

$$\sum_{q\leq i\leq 2q} \binom{q}{i} 2^{i-1}x^{2q-i}y^q z^{\binom{q}{i}2^{i-1}}.$$

Summing over all values of q, it follows that the tri–graded series is equal to

$$\sum_{\substack{p,r\geq 0\\ q\geq 1}} a(p,q,r)x^p y^q z^r = \sum_{q\geq 1}\left(\sum_{q\leq i\leq 2q} \binom{q}{i} 2^{i-1}x^{2q-i}y^q z^{\binom{q}{i}2^{i-1}}\right).$$

Proposition A.1. *Let $G = U(2)$ and V be the reduced homology of the maximal torus T as a module over the group ring of the Weyl group of G. The coefficient of the term $x^{2q-i}y^q z^{\binom{q}{i}2^{i-1}}$ in the power series*

$$f(x,y,z) = \sum_{q\geq 1}\left(\sum_{q\leq i\leq 2q}\binom{q}{i}2^{i-1}x^{2q-i}y^q z^{\binom{q}{i}2^{i-1}}\right).$$

is the number of copies of the group ring $\mathbb{Z}\Sigma_2$ in tensor degree q and homological degree $2q-i$.

A.2 The unitary group $U(3)$

Let G be the unitary group $U(3)$. The maximal torus of $U(2)$ is $T = S^1 \times S^1 \times S^1$ and the Weyl group is Σ_3. The reduced homology of the maximal torus is as follows

$$\begin{aligned}
H_1(T;\mathbb{Z}) &\cong \oplus_{\binom{3}{1}}\mathbb{Z} \cong \mathbb{Z}\big[\Sigma_3/(\Sigma_1\times\Sigma_2)\big] \cong \mathbb{Z}[\mathbb{Z}/3]\\
H_2(T;\mathbb{Z}) &\cong \oplus_{\binom{3}{2}}\mathbb{Z} \cong \mathbb{Z}\big[\Sigma_3/(\Sigma_2\times\Sigma_1)\big] \cong \mathbb{Z}[\mathbb{Z}/3]\otimes\mathbb{Z}_{sgn}\\
H_3(T;\mathbb{Z}) &\cong \oplus_{\binom{3}{3}}\mathbb{Z} \cong \mathbb{Z}\big[\Sigma_3/(\Sigma_3\times\Sigma_0)\big] \cong \mathbb{Z}_{sgn}
\end{aligned}$$

Thus $V = H_1 \oplus H_2 \oplus H_3$ and

$$\mathcal{T}[V] = \bigoplus_{d\geq 1}(H_1\oplus H_2\oplus H_3)^{\otimes d}.$$

The Euler-Poincaré series for $\mathcal{T}[V]$ for the unitary group $U(3)$ is

$$\chi(\mathcal{T}[V]) = \frac{1}{1-\chi(V)} = \sum_{n\geq 0}(\chi(V))^n = \sum_{n\geq 0}(3t+3t^2+t^3)^n.$$

The various tensor powers of the homology groups can be calculated to be

$$H_1^{\otimes i} = \mathbb{Z}[\Sigma_3/\Sigma_2] \bigoplus \left(\bigoplus_{\frac{3^{i-1}-1}{2}} \mathbb{Z}\Sigma_3 \right)$$

$$H_2^{\otimes j} = \mathbb{Z}[\Sigma_3/\Sigma_2] \bigoplus \left(\bigoplus_{\frac{3^{j-1}-1}{2}} \mathbb{Z}\Sigma_3 \right)$$

$$H_3^{\otimes k} = \text{ either } \mathbb{Z}_{triv} \text{ if } k \text{ is even, or } \mathbb{Z}_{sgn} \text{ if } k \text{ is odd.}$$

Then it follows that the d–fold tensor $V^{\otimes d}$, up to tensoring with the sign representation, is given by

$$V^{\otimes d} \cong \bigoplus_{\substack{0 \le i+j \le d \\ 0 \le i,j \le d}} \left(\bigoplus_{\binom{d}{i,j}} H_1^{\otimes i} \otimes H_2^{\otimes j} \otimes H_3^{\otimes (d-(i+j))} \right)$$

where $\begin{pmatrix} d \\ i,j \end{pmatrix} = \dfrac{d!}{i!j!(d-(i+j))!}$. Hence, the tensor algebra $\mathcal{T}[V]$ is given by

$$\mathcal{T}[V] \cong \bigoplus_{d \ge 1} \left(\bigoplus_{\substack{0 \le i+j \le d \\ 0 \le i,j \le d}} \left(\bigoplus_{\binom{d}{i,j}} H_1^{\otimes i} \otimes H_2^{\otimes j} \otimes H_3^{\otimes (d-(i+j))} \right) \right).$$

For $i + j + k = d$, the tensor products equal

$$H_1^{\otimes i} \otimes H_2^{\otimes j} \otimes H_3^{\otimes (d-(i+j))} \cong \mathbb{Z}[\Sigma_3/\Sigma_2] \bigoplus \left(\bigoplus_{\frac{3^{i+j}-1}{2}} \mathbb{Z}\Sigma_3 \right).$$

Hence,

$$V^{\otimes d} \cong \bigoplus_{\substack{0 \le i+j \le d \\ 0 \le i,j \le d}} \left(\bigoplus_{\binom{d}{i,j}} \left(\mathbb{Z}[\Sigma_3/\Sigma_2] \bigoplus \left(\bigoplus_{\frac{3^{i+j}-1}{2}} \mathbb{Z}\Sigma_3 \right) \right) \right).$$

The goal now is to find a tri–grades series as in Proposition A.1.

The number of copies of the group ring $\mathbb{Z}\Sigma_3$ in tensor degree q and homological degree t is given by the number of copies $\mathbb{Z}\Sigma_3$ in the term $H_1^{\otimes i} \otimes H_2^{\otimes j} \otimes H_3^{\otimes (q-(i+j))}$ such that $i + 2j + 3(q - i - j) = 3q - 2i - j = t$, multiplied by the number of times each term appears (tensors can be rearranged). That is, for fixed tensor degree q and fixed homological degree $3q-2i-j$, there are

$$\binom{q}{i,j}\frac{3^{i+j}-1}{2}$$

copies of group ring $\mathbb{Z}\Sigma_3$. Therefore the tri–graded series should have a term of the form

$$\binom{q}{i,j}\Big(\frac{3^{i+j}-1}{2}\Big)x^{3q-2i-j}y^q z^{\binom{q}{i,j}\frac{3^{i+j}-1}{2}}$$

For fixed tensor degree q there are only homological degrees t that satisfy $d \leq t \leq 3d$. Hence, the sum

$$\sum_{\substack{0\leq i+j\leq q\\ 0\leq i,j\leq q}}\binom{q}{i,j}\Big(\frac{3^{i+j}-1}{2}\Big)x^{3q-2i-j}y^q z^{\binom{q}{i,j}\frac{3^{i+j}-1}{2}}$$

describes the tensor q–fold product $V^{\otimes q}$ concerning the group ring $\mathbb{Z}\Sigma_3$. Therefore,

Lemma A.2. *Let $G = U(3)$ and V be the reduced homology of the maximal torus T as a module over the group ring of the Weyl group of G. The coefficient of the term*

$$\binom{q}{i,j}\Big(\frac{3^{i+j}-1}{2}\Big)x^{3q-2i-j}y^q z^{\binom{q}{i,j}\frac{3^{i+j}-1}{2}}$$

in the power series

$$f(x,y,z) = \sum_{q\geq 1}\left(\sum_{\substack{0\leq i+j\leq q\\ 0\leq i,j\leq q}}\binom{q}{i,j}\Big(\frac{3^{i+j}-1}{2}\Big)x^{3q-2i-j}y^q z^{\binom{q}{i,j}\frac{3^{i+j}-1}{2}}\right)$$

gives the number of copies of the group ring $\mathbb{Z}\Sigma_3$ in tensor degree q and homological degree $3q-2i-j$.

We can do a similar computation for the number of $\mathbb{Z}[\Sigma_3/\Sigma_2]$. Clearly, the tensor product $H_1^{\otimes i}\otimes H_2^{\otimes j}\otimes H_3^{\otimes(d-(i+j))}$ appears $\binom{d}{i,j}$ times, that is, there are $\binom{d}{i,j}$ isomorphic copies of $H_1^{\otimes i}\otimes H_2^{\otimes j}\otimes H_3^{\otimes(d-(i+j))}$ in $V^{\otimes d}$, and each of them has a single copy of the representation $\mathbb{Z}[\Sigma_3/\Sigma_2]$. Hence, there are $\binom{d}{i,j}$ copies of $\mathbb{Z}[\Sigma_3/\Sigma_2]$ in tensor degree d and homological degree $3d - 2i - j$ and in the tri–graded series there must be a term of the form

$$\binom{q}{i,j} x^{3q-2i-j} y^q w^{\binom{q}{i,j}}.$$

Therefore,

Lemma A.3. *Assume same conditions as in Lemma A.2. The coefficient of the term*

$$\binom{q}{i,j} x^{3q-2i-j} y^q w^{\binom{q}{i,j}}$$

in the power series

$$g(x,y,w) = \sum_{q \geq 1} \left(\sum_{\substack{0 \leq i+j \leq q \\ 0 \leq i,j \leq q}} \binom{q}{i,j} x^{3q-2i-j} y^q w^{\binom{q}{i,j}} \right)$$

gives the number of copies of the group ring $\mathbb{Z}[\Sigma_3/\Sigma_2]$ in tensor degree q and homological degree $3q - 2i - j$.

Proposition A.4. *Assume same conditions as in Lemma A.2. Then the power series*

$$p(x,y,z,w) = f(x,y,z) + g(x,y,w)$$

keeps track of the tensor degree, homological degree and number of copies of the representations $\mathbb{Z}\Sigma_3$ and $\mathbb{Z}[\Sigma_3/\Sigma_2]$, where $f(x,y,z)$ is the series in Lemma A.2 and $g(x,y,w)$ is the power series in Lemma A.3.

In general, for any unitary group $U(n)$, the maximal torus is given by $(S^1)^n$ and the Weyl group is Σ_n. The reduced homology of the maximal torus has the following W–module structure

$$H_1 \cong \mathbb{Z}[\Sigma_n/\Sigma_{n-1}]$$
$$H_i \cong \mathbb{Z}[\Sigma_n/(\Sigma_{n-i} \times \Sigma_i)] \otimes \mathbb{Z}_{sgn}, \text{ for } 2 \leq i \leq n.$$

It remains to determine the decomposition of the tensor products

$$H_{i_1}^{\otimes d_1} \otimes \cdots \otimes H_{i_k}^{\otimes d_k}$$

for $1 \leq i_1 < \cdots < i_k \leq n$ and $d_1, \ldots, d_k \geq 0$, as a direct sum of permutation representations of the Weyl group Σ_n. We conclude here this appendix with the remark that more information will be given later. Counting of these representations is complicated because of counting of partitions.

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