Life in 4-Dimensions

- Some highlights of 4-mfd theory -

• Simply connected smooth 4-mfds

Ex's: S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, s.c. complex surfaces
eg. K3-surface

What was known circa 1980?

1. Alg. topology - H_2, intersection form
$$H_2 \otimes H_2 \longrightarrow \mathbb{Z}$$
b^+, b^-, $\text{sign} = b^+ - b^-$, $e = 2 + b^+ + b^-$
$\text{rank} = b^2 = b^+ + b^-$

<u>odd</u> form, <u>even</u> form $\longleftrightarrow$ spin

$\mathbb{CP}^2$: (1) $\quad$ $\overline{\mathbb{CP}}^2$: (-1) $\quad$ $S^2 \times S^2$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = H$

K3: $2E_8 + 3H$ $\quad$ $E_8 =$ 

$\text{sign}(K3) = -16$ $\quad$ $e(K3) = 24$
$b_2(K3) = 22$

2. Index Thm information eg Rohlin's Thm
$\Rightarrow$ for even form, $16 \mid \text{sign}$, so if $kE_8 + \ell H$ is int. form of s.c. 4-mfd. $\quad$ k even.

3. What could be saved from pf of h-cob. thm

Questions

- More irreducible smooth s.c. 4-mfds than cx surfaces?
  (Each $\cong$ # cx surfaces?)

- Homeo vs. Diffeo - Exotic smooth structures?

- Geography of char. classes for sc smooth 4-mfds
  Plot e vs. sign - Which points are occupied by
  irred 4-mfd?

(Usually coord's used are $\chi_h = \frac{e + \text{sign}}{4}$, $c_1^2 = 3\,\text{sign} + 2e$)

Related question

- Realization of intersection forms

(Alg classif.: Indef (Hasse - Minkowski)
odd ~ $m(1) \oplus n(-1)$ even ~ $kE_8 \oplus \ell H$

Def - No classif. $\exists$ fin # iso classes of given rank
eg 2 of rank 16
24 ... 24
$>10^{51}$ .... 40 ...

- 11/8 conj : For even s.c. 4-mfd $\frac{b_2}{|\text{sign}|} \geq \frac{11}{8}$

- h-cobordism thm?

- Poincaré Conj?

Simon Donaldson

Thm A. (1982)
An intersection form of a smooth sc 4-mfd which is definite is $n(1)$ or $n(-1)$.

Method - study space of connections on principal $SU(2)$-bdle P assoc. to X.

P classified by single # $c_2(P)$.

$\mathcal{A}$ = conn's on P - ∞-dim'l affine space

$\mathcal{G}$ = gauge gp = $SU(2)$-equiv auto's of P.

$\mathcal{B} = \mathcal{A}/\mathcal{G}$ - ∞-dim'l stratified space.

Idea - study special kind of conn's giving abs min for Yang-Mills fn'l : $\frac{1}{2}\int_X |F_A|^2$

ASD conn's - satisfy $*F_A = -F_A$

Moduli space M = ASD conn's$/\mathcal{G}$. (stratified space) w/ sing's

$\dim M = 8c_2(P) - 3(1+b^+)$

Note for $\begin{cases} c_2(P) = 1, & \dim M = 5 \\ b^+ = 0 & (-\text{'ve def}) \end{cases}$

Stratification of $M \leftrightarrow$ formation of instantons

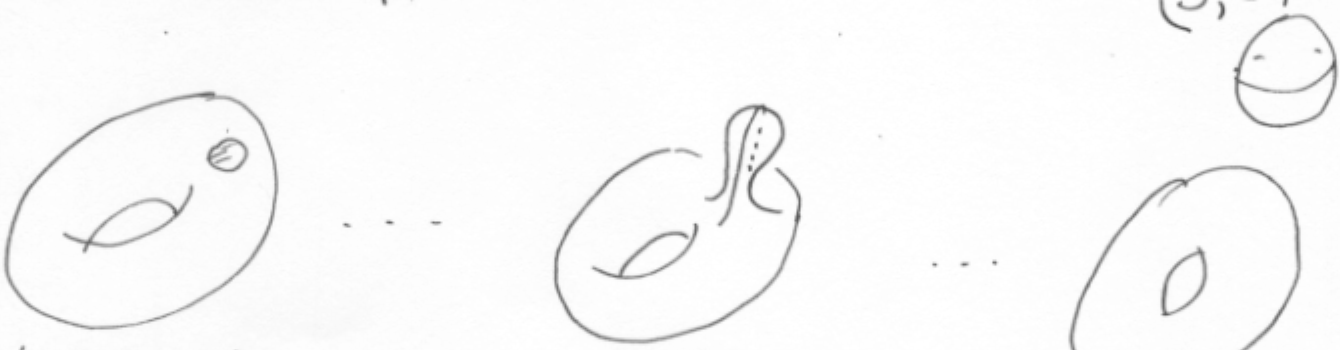
Top. translation of 'instanton':

ASD conn on $SU(2)$-bdle / S^4.

These carry energy $\leftrightarrow$ c_2 (bdle).

What can happen :

$(S^4, Q, \overset{ASD}{B})$



(X^4, P, A_0) . . . (X^4, P, A_n) $(X^4, P_\infty, A_\infty)$

ASD conn. ASD

$$c_2(P_\infty) = c_2(P) - c_2(Q)$$

Assume $b^+ = 0$, $c_2(P) = 1$. X parametrizes the instantons.

$\Rightarrow M$ = 5-dim'l singular space with $\partial = X$.

Sing's $\leftrightarrow$ ASD conn's whose holonomy $= U(1) \subset SU(2)$.

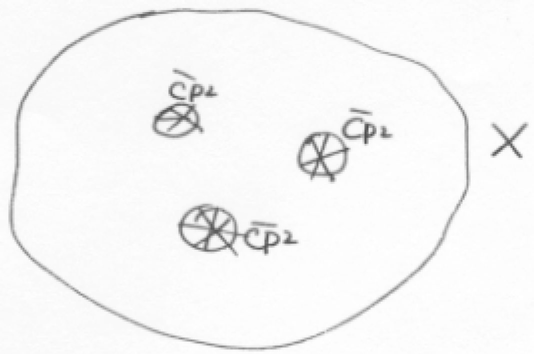
$\leftrightarrow$ Pts in $\mathcal{Q}$ where $\mathcal{G}$ has isotropy

$\leftrightarrow$ $c(\overline{\mathbb{CP}}^2) \subseteq M$.

Index thm computes dim'n if <u>nonempty</u>.

Shown by Cliff Taubes.

M :



$M =$ orientable cob. from X to $\#_n \overline{CP^2}$.
(Donaldson)

$n = \#$ reducible ASD conns on P.

$\underset{\text{can show}}{=} \# \{ a \in H_2 \mid -a^2 = c_2(P) = 1 \}$

See: $\operatorname{sign} X = -n$ & each a splits off int. form

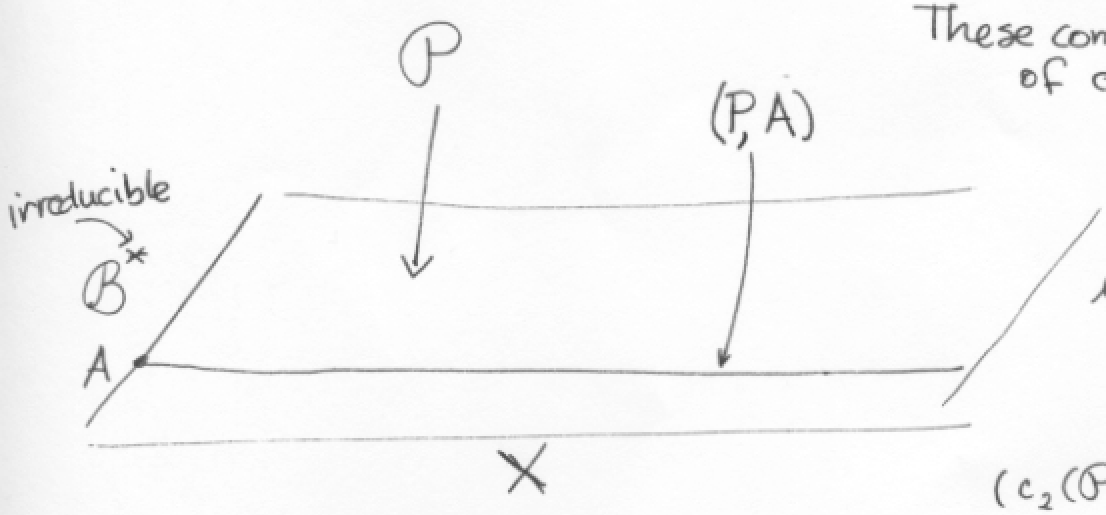
$\Rightarrow$ Int form $= n(-1)$. $\square$

Donaldson's Polynomial Invariant :

Idea: View M as a kind of homology class in $\mathcal{B}$

Get inv'ts by intersecting with <u>other homology classes</u>

These come from families of connections



$\mu : H_2(X) \longrightarrow H^2(\mathcal{B}^*)$

$\mu(a) = c_2(\mathcal{P}) / a$

(integ. along fiber)

$(c_2(\mathcal{P}) \in H^4(\mathcal{B}^* \times X))$

Donaldson inv't: b^+ odd >1, $k = c_2(P)$

$$\dim M = 8k - 3(1+b^+) = 2q$$

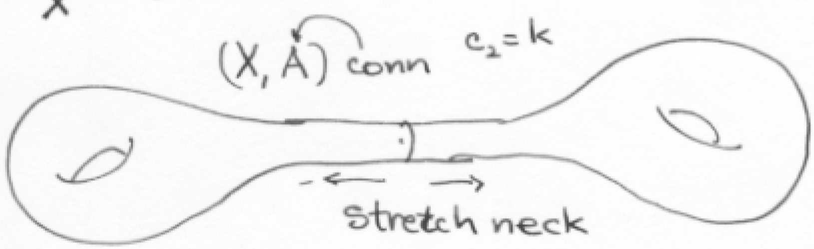
$$D_X(\underbrace{\alpha_1, \dots, \alpha_q}_{\text{classes in } H_2(X)}) = \langle \mu(\alpha_1) \cup \cdots \mu(\alpha_q), [M] \rangle$$

defined via intersections

'Diffeomorphism' inv't of oriented 4-mfd.

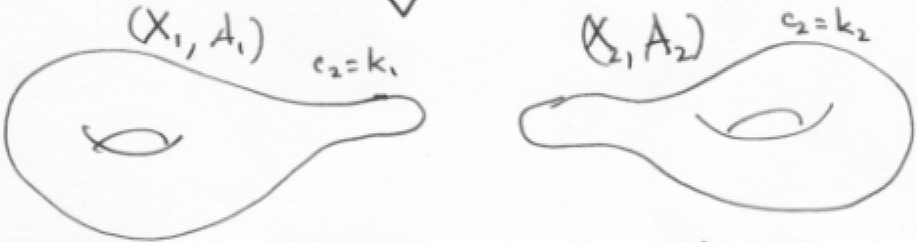
<u>Donaldson's # Th'm</u> If $X \cong X_1 \# X_2$ & $b^+_{X_1} > 0$, $b^+_{X_2} > 0$ then $D_X \equiv 0$.

<u>Pf</u>.



$a_1, \dots, a_{q_1}$ $\qquad$ $b_1, \dots, b_{q_2}$

$\downarrow$



$k = k_1 + k_2$

$\dim M_1 = 8k_1 - 3(1 + b^+_{X_1})$ $\qquad$ $\dim M_2 = 8k_2 - 3(1 + b^+_{X_2})$

$$2(q_1 + q_2) = \dim M_1 + \dim M_2 = 8k_1 - 3(1 + b^+_{X_1}) + 8k_2 - 3(1 + b^+_{X_2})$$

$$2(q_1 + q_2) \overset{\#}{=} \dim M = 8k - 3(1 + b^+_X)$$

'$\square$'

<u>Cor</u> $D_{mCP^2 \# n\overline{CP}^2} \equiv 0$ if $m > 1$.

Ex. $D_{K3} \neq 0$.

$\Rightarrow$ K3 cannot be split as a # with $b^+ > 0$ on both sides

C/E to h-cob. conj.

$K3 \# \overline{\mathbb{CP}}^2$ odd, int form is $2E_8 \oplus 3H \oplus (-1)$

$$\cong 3(1) \oplus 20(-1)$$

Nontriviality of D persists after blowups

$$\Rightarrow \quad D_{K3\#\overline{CP}^2} \neq 0 \qquad D_{3CP^2 \# 20\overline{CP}^2} \equiv 0.$$

(Freedman (1982) $\Rightarrow$ Smooth sc 4-mfds class. up to homeo by int. form.)

Log transforms : If torus T of self-$\cap$ # 0 $\subset X$

$Nbd = T^2 \times D^2 = S^1 \times (S^1 \times D^2)$ cut & reglue so that

$S^1 \times S^1 \times \{0\}$ is covered p-fold by tori on $\partial(X \setminus T \times D^2)$

'Homotopy K3-surfaces' One or two log transforms of rel prime multiplicities on K3. [All cx surfaces homeo to K3 are these.

Friedman-Morgan showed these depend up to diffeo on multiplicities

Gompf-Mrowka Example (1990)

Int form of K3 $= 2E_8 \oplus 3H$ Each H has T^2, square 0.

Log transf's on all 3 tori.

Mrowka's Thesis → Calculations ⇒ not diffeo, K3, $K3_p$, $K3_{p,q}$

⇒ NO COMPLEX STRUCTURE

Structure of Donaldson Inv'ts

Kronheimer & Mrowka, key realization

Importance of p1-class

Recall $c_2(\mathbb{P}) \in H^4(\mathcal{B}^* \times X)$

For $a \in H_2(X)$ $\mu(a) = c_2(\mathbb{P}) / a \in H^2(\mathcal{B}^*)$

Let $x \in H_0(X)$, gen. $\nu = c_2(\mathbb{P})/x \in H^4(\mathcal{B}^*)$

$$D_X(a_1, \dots, a_q, x^b) = \langle \mu(a_1) \cup \cdots \cup \mu(a_q) \cup \nu^b, [M] \rangle$$

where $\dim M = 2q + 4b$.

Simple type condition: $D_X(\vec{a}, x^2) = 4 D_X(\vec{a})$

Holds ∀ known sc smooth 4-mfds, $b^+ \geq 3$.

Simple-type hypoth: Kronheimer & Mrowka proved

$$D_X = \exp(\underset{\substack{\uparrow \\ \text{int form of } X}}{Q/2}) \sum_{s=1}^{p} \underset{\substack{\Uparrow \\ \mathbb{Q}}}{a_s} e^{K_s \cdot -}$$ ← basic classes in $H_2(X)$.

Power series. When applied to $a_1, \ldots, a_q, x^b$ only look at $q+2b$ term in series.

Basic classes K_s characteristic in $H_2(X)$.

Important consequence of proof

Adjunction Inequality Surface $\Sigma \subset X$ of $g \geq 1$ & $\Sigma \cdot \Sigma > 0$

$$2g-2 \geq \Sigma \cdot \Sigma + |K_s \cdot \Sigma| \quad \forall \text{ basic classes } K_s$$

K-M showed many cx surfaces (e.g. p_g = odd) have their can. class K as one of basic classes.

$\Rightarrow$ If Σ rep's cx curve then for any rep of $[\Sigma]$:

$2g-2 \geq \Sigma \cdot \Sigma + K \cdot \Sigma$ for cx curve itself $2g-2 = \Sigma \cdot \Sigma + K \cdot \Sigma$

$\Rightarrow$ genus minimized by cx curve.

Ex. $D_{K3} = \exp(Q/2)$ (Only basic class = 0)

$D_{E(n)} = \exp(Q/2) \sinh^{n-2}(f)$ $\quad$ f = fiber class

Log transform formula: (F- Stern)

If T = ess. torus of square 0 in X, $b^+_X > 1$ ...

and if X_p = result of p-log transform on T

then $D_{X_p} = D_X \cdot \dfrac{\sinh(t)}{\sinh(t_p)}$

t_p = class of multiple fiber $= \frac{1}{p}[T]$, $t = [T] = p\,t_p$.

Accomplished via rational blowdowns (next time)

Consequence Complete calculation of Donaldson invts of elliptic surfaces

$$D_{E(n)_{p,q}} = \exp(Q/2)\,\frac{\sinh^n(f)}{\sinh(f_p)\sinh(f_q)}$$

$\Rightarrow$ Smooth classif of elliptic surfaces (first done by Morgan, Mrowka by calculating first 2 coeffs of D)

Blowup Formula

Given D_X , find $D_{X \# \overline{\mathbb{CP}}^2}$

for simple-type (Kronheimer-Mrowka)

$$D_{X\#\overline{\mathbb{CP}}^2} = D_X \cdot \exp(-E^2/2) \cosh(E)$$

In general $D_{X\#\overline{\mathbb{CP}}^2}(E^k \vec{z}) = D_X(B_k(x)\vec{z})$

$\vec{z} = (a_1, \dots, a_q, x^b)$, classes in X — blowup poly's

Roll up into power series $B(x,t) = \sum_{k=0}^{\infty} B_k(x) t^k/k!$

Stern & I calculated $B(x,t)$ in terms of elliptic fns related to the elliptic curve

$$z^2 = 4y^3 - 4\left(\frac{x^2}{3} - 1\right) y - \frac{8x^3 - 36x}{27} \qquad (x = \text{pt class})$$

Seiberg-Witten Theory

New equs - simplified leading to new invt

$SW_X = \sum_{s=1}^{p} SW_X(k_s)\, e^{k_s}$ or simply $\sum_{s=1}^{p} SW_X(k_s)\, t_{k_s} \in \mathbb{Z} H_2(X)$

S-W basic classes.

Conj. $= D_X / \exp(Q/2)$ up to $2^{(\text{power dep only on hty type of } X)}$

- u-plane : parametrizes family of elliptic curves.

Lecture 2 - The Seiberg-Witten Era

Basic facts about SW (for s.c. 4-mfolds)

SW_X : char elts of $H_2(X) \longrightarrow \mathbb{Z}$

Arise from solving SW-eqns for which moduli space has formal dim'n $(M_X(k)) = \frac{1}{4}(k^2 - (3\,\mathrm{sign}(X) + 2e(X)))$

char in $H_2(X)$

Condition for X to admit almost-cx str. with $c_1 = k$ is equiv. to $\dim M_X(k) = 0$.

Homology orientation for X, i.e. orientation of $H_2^+(X)$, orients each moduli space.

$\Rightarrow$ In case $\dim M_X(k) = 0$ can define

$SW_X(k) =$ count of pts with sign.

SW Simple Type: If only SW invts come from dim $= 0$ moduli spaces.

Thm. (Witten) X minimal Kahler surface, $|SW_X(\pm K)| = 1$ (can. class) if $c_1^2(X) > 0$, then $SW_X(k) = 0$ all other k.

Generally, $SW_X(-k) = (-1)^{\frac{e+\mathrm{sign}}{4}} SW_X(k)$

- $SW_X = 0$, if X admits +'ve scalar curv. metric
- $SW_{X_1 \# X_2} = 0$ if $b^+_{X_1} > 0$, $b^+_{X_2} > 0$.
- Taubes : $SW_X(\overset{\text{can. class}}{K}) = \pm 1$ if X is a sympl 4-mfd, $b^+ > 1$.

& Geometric content to SW : $SW_X = Gr_X$ for sympl X.

The Thom Conj. Minimal genus rep in $\mathbb{CP}^2$ of a degree d curve is that curve itself.

- Kronheimer & Mrowka proved this by establishing Adjunction Inequality for SW in a context that works for $\mathbb{CP}^2$ ($b^+ = 1$)

11/8 - Conjecture (i.e. that $\frac{b_2}{|\text{sign}|} \geq \frac{11}{8}$ for s.c. even X)

Donaldson had shown true, in case $b^+ = 1$ or 2 (& 0)
(i.e. if even & $b^+ = 1$ then $Q = H$
if even & $b^+ = 2$ then $Q = H \oplus H$)

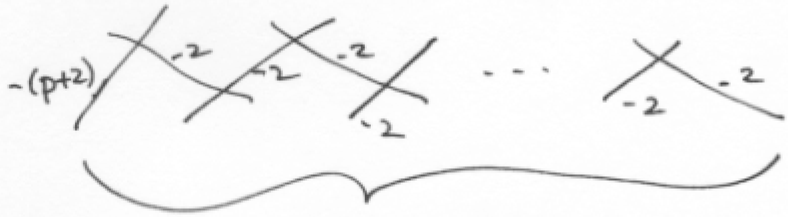
Furuta used version of SW to prove
$$b_2 \geq \frac{5}{4}|\text{sign}| + 2 \quad \text{for } X \text{ sc even.}$$

basically, the 10/8 - Thm.

Construction Techniques -

1. Rational blowdown (F-Stern)

Suppose X contains config. C_p of spheres



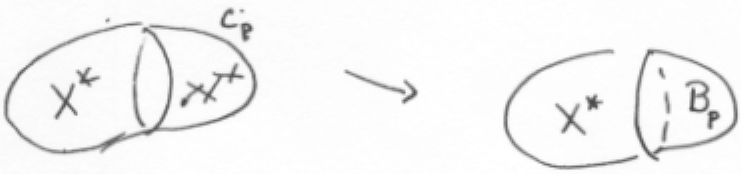
$= C_p$

p-1 spheres.

$\partial C_p = L(p^2, 1-p)$. This lens space bds rational ball B_p.

Rational blowdown: Remove C_p, glue in B_p.

(reduces b^- by $p-1$, leaves b^+ unchanged)



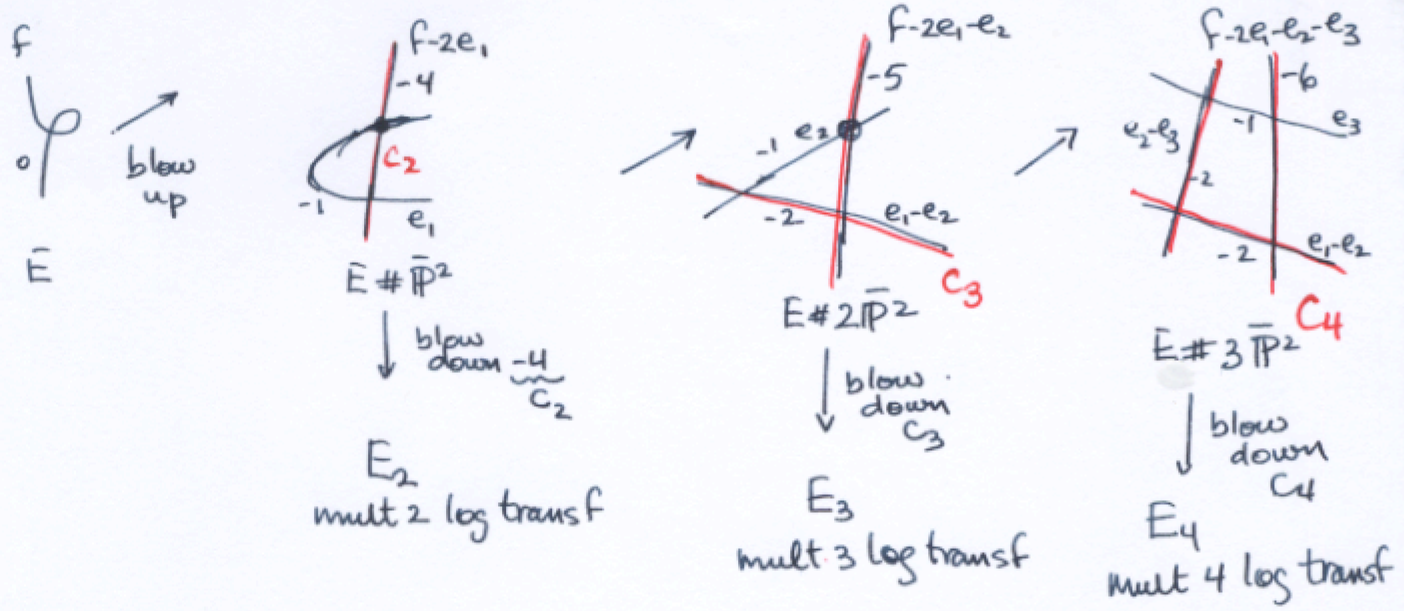
X $\qquad$ X_p = rational blowdown

If k char in $H^2(X_p)$

lift = class $\tilde{k} \in H^2(X) \ni \tilde{k}|_{X^*} = k|_{X^*}$

Thm (F-S) $SW_{X_p}(k) = SW_X(\tilde{k})$

Ex. Log transform   E: elliptic surface
f: nodal fiber



etc...

Using above formula prove log transf formula

$$SW_{X(T,p)} = SW_K \cdot (t^{p-1} + t^{p-3} + \cdots + t^{3-p} + t^{1-p})$$

p-log transf
on ess. torus T of square = 0

$t \longleftrightarrow \frac{1}{p}[T]$ in $\mathbb{Z}H_2(X)$.

2. Knot Surgery (F-Stern)

Again T = ess. torus, square $= 0 \subset X$.

K = knot in S^3.

$$X_K = (X \setminus T^2 \times D^2) \cup S^1 \times (S^3 \setminus N(K))$$

glued $\ni$ longitude of $K \longleftrightarrow \partial D^2$.

X_K : same homology & int. form as X.

If also, $\pi_1 X = 1 = \pi_1(X \setminus T)$ then X_K s.c. so

- X_K homeo X.

<u>Knot Surgery Thm</u> : $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$

Δ_K = sym. Alex poly of K. t = class in $\mathbb{Z}H_2(X) \leftrightarrow [T]$

<u>Ex</u>. $X = K3$ surface. Witten $\Rightarrow SW_X = 1$.

Do knot surgery on elliptic fiber. X_K homeo X.

$$SW_{X_K} = SW_X \cdot \Delta_K(t^2) = \Delta_K(t^2)$$

$\Rightarrow$ • ∞'ly many distinct smooth mfds homeo K3

- K with nonmonic $\Delta_K(t)$ give <u>nonsymplectic</u> mfds. i.e. do not admit sympl. str.

Proof of knot surgery thm –

Show SW_{X_K}/SW_X satisfies crossing change formula for Alex poly.

[<u>Conj</u>. If X = sc smooth 4-mfd with $SW_X \neq 0$ then $\exists$ ∞'ly many distinct smooth 4-mfds homeo X.]

Important fact :

- Knot surgery achieved by surgery (log transf) on nullhomologous tori. eg:

SC 4-Mfds with $b^+ = 1$.

Rational surface $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$, $S^2 \times S^2$

All have $SW = 0$.

$E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ elliptic surface

(p,q)-Dolgachev surface $= E(1)_{p,q}$ (log transf's) homeo to $E(1)$

Donaldson (1985) $E(1)_{2,3}$ not diffeo $E(1)$

(first C/E h-cob thm in dim'n 4)

Kotschick (1988) Barlow surface homeo not diffeo to $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$.

No progress until 2004

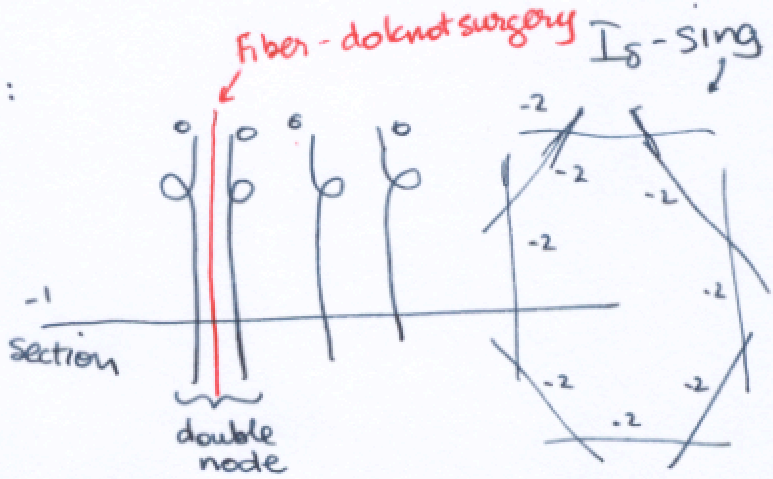
Jongil Park.

Park mfd P homeo to $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$ & sympl. not diffeo to.

Park's idea: blow up until seeing C_p then blow down.

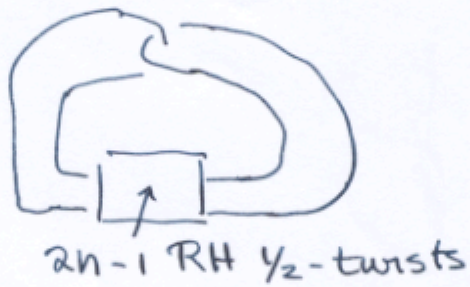
Version of his construction souped-up with knot surgery (F-Stern):

Start with E(1) with:

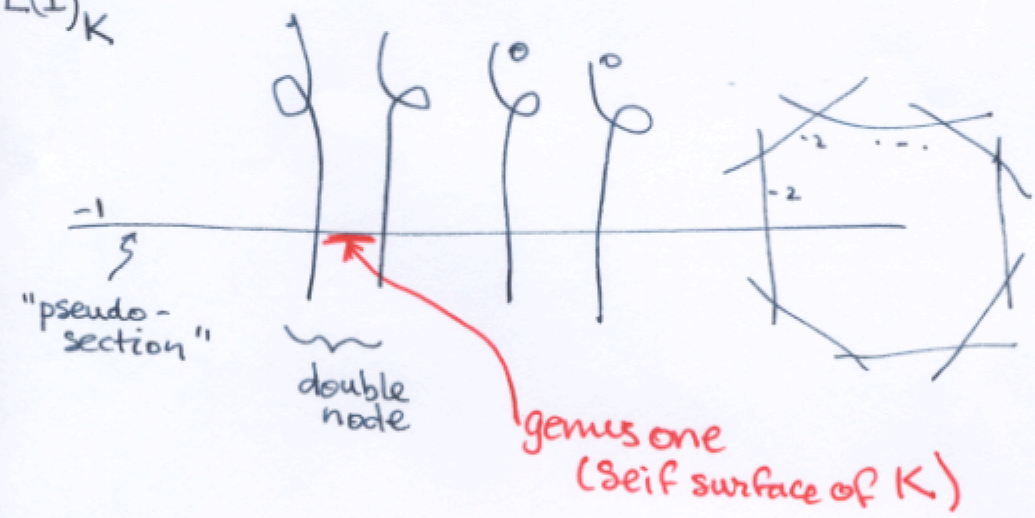


Knot surgery where K = twist knot

Alex poly $= nt - (2n-1) + nt^{-1}$

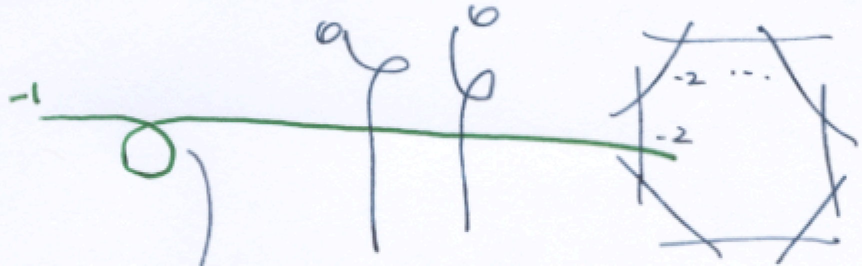


$E(1)_K$



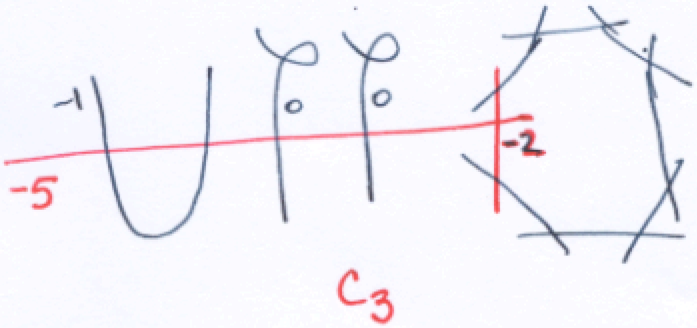
Double node trick trades the genus 1 pseudosection for immersed sphere

$E(1)_K$



blow up double pt

$E(1)_K \# \overline{\mathbb{CP}}^2$

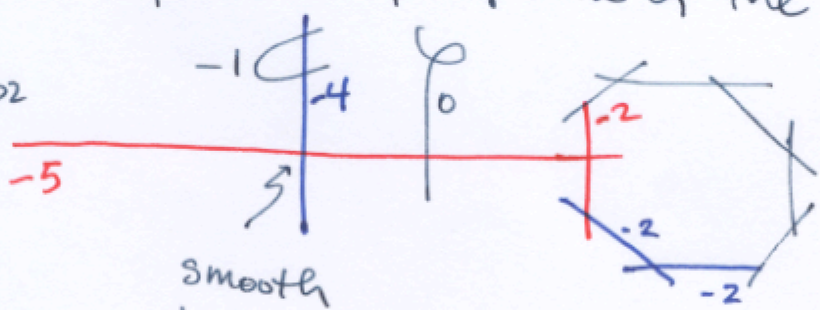


rat blow down

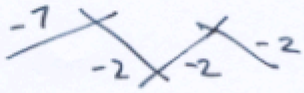
Get ∞ family of smooth mflds homeo $\mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2$ by varying K (and computing SW)

or blow up double pt of one of the nodal fibers

$E(1)_K \# 2\overline{\mathbb{CP}}^2$



smooth becomes -7

& Rationally blow down C_5 = 

get ∞-family homeo $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$

(probably $n=1$ knot (trefoil) is Park's ex.)

$b^- = 6, 5$ as well. – Also due in part to Stipsicz-Szabo & Park-Stipsicz-Szabo.

<u>Smaller mfd's</u>

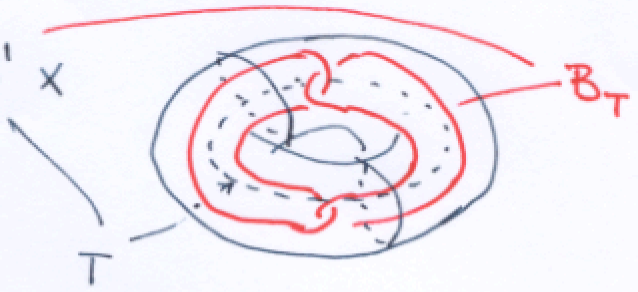
Exotic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ due to Baldridge-Kirk and to Akhmedov – D. Park.

$\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ to Akhmedov – D. Park (not yet published)

- Different approach to these examples: (F-Stern)

- Santeria surgery -

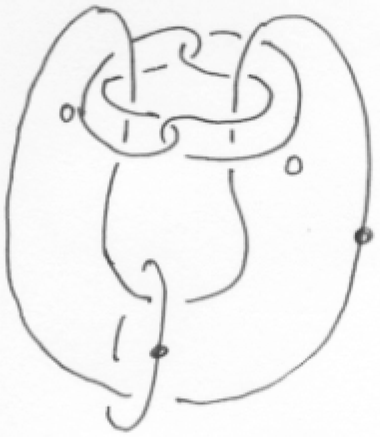
Bing double of torus in $T^2 \times D^2$

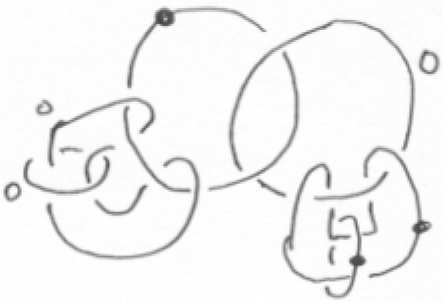
$= S^1 \times$ 

$T_0 = T \setminus D^2$

What is smallest mfd $\supseteq T_0 \times D^2$ & B_T ?

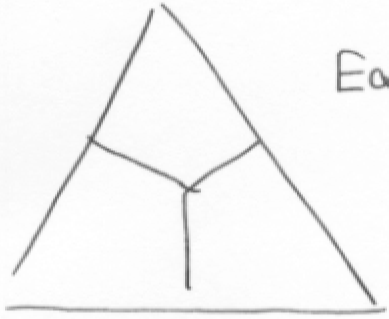
Answer

$A =$ 

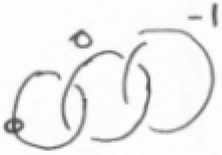
$\cong$ 

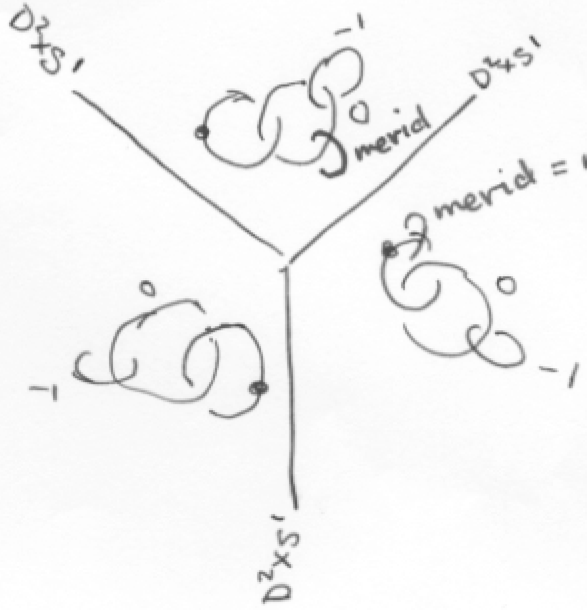
$$B_T \subset A \subset T^2 \times D^2$$

Idea: Embed A when $T^2 \times D^2$ not there.

Ex. $\mathbb{CP}^2 =$ 

Each compartment $\cong B^4$
$=$ compl. of $H \cup E$ in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$
$=$ 

Blow up in each compartment to get 

$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 =$ 

∴ bds disk of square 0 in top compartment

Bing double (merid) also bds disjoint 2-disks in top compartment

Use this to trade handles.

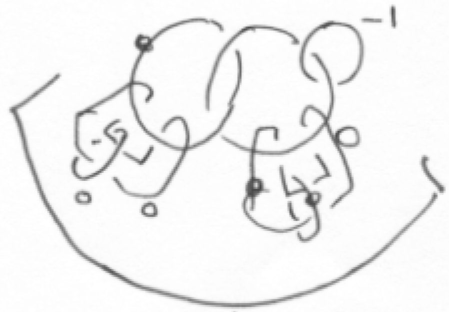
Add 2-hdles to Bing double (merid) in LR.

Subtract . . . . (merid) on Top.

= Add 1 hdles : Get

$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$

Each compartment now



$A \supset B_T$

Get 6 tori (all nullhomologous) in $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Easy to find surgeries on these that change SW & give $\pi_1 = 1$.

Result ∞-family homeo to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ not diffeo

Similar techniques work for $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$

- See notes from GA TOP CONF talk on my website.

Some Important Remaining Questions

- BMY line $c_1^2 = 3c_2$. Can we find symplectic but noncx mfds there? (Not s.c. now)
- Exotic $\mathbb{P}^2 \# \overline{\mathbb{P}}^2$, $S^2 \times S^2$, $\mathbb{P}^2$, S^4?
- Horikawa surfaces $c_1^2 = 2\chi_h - 6$. Smooth classification?
- Exotic, minimal, $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $k > 9$?
- New construction methods
- If K is a nontrivial knot with $\Delta_K(t) = 1$, is $K3_K$ diffeo $K3$?
- What impact will Floer homology, esp. HM_* play in dim 4?