# Double branched covers, Heegaard Floer homology, and applications

National Forum of Young Topologists
Tulane University
November 14, 2009

(Paul Melvin): "Remind yourself that you know more today than you did a year ago."

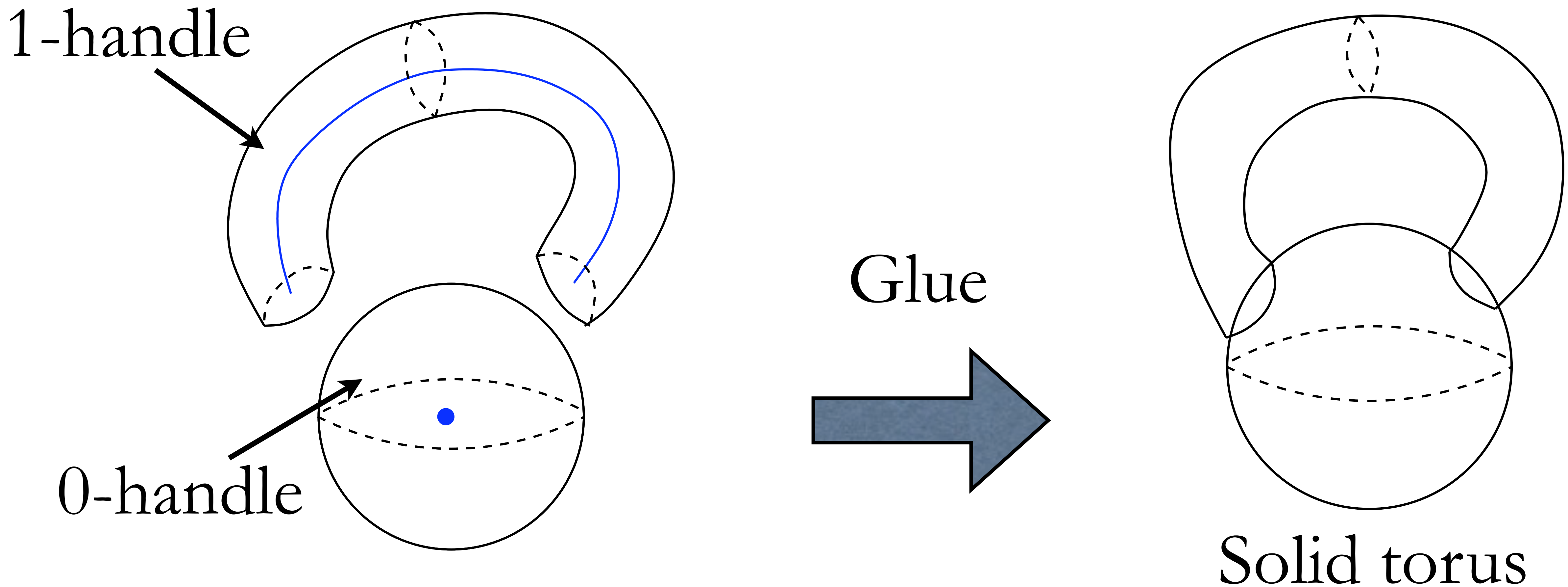
# Outline

I. Background: Handlebody theory & Heegaard Floer homology

II. Branched double covers: Applications to knot concordance

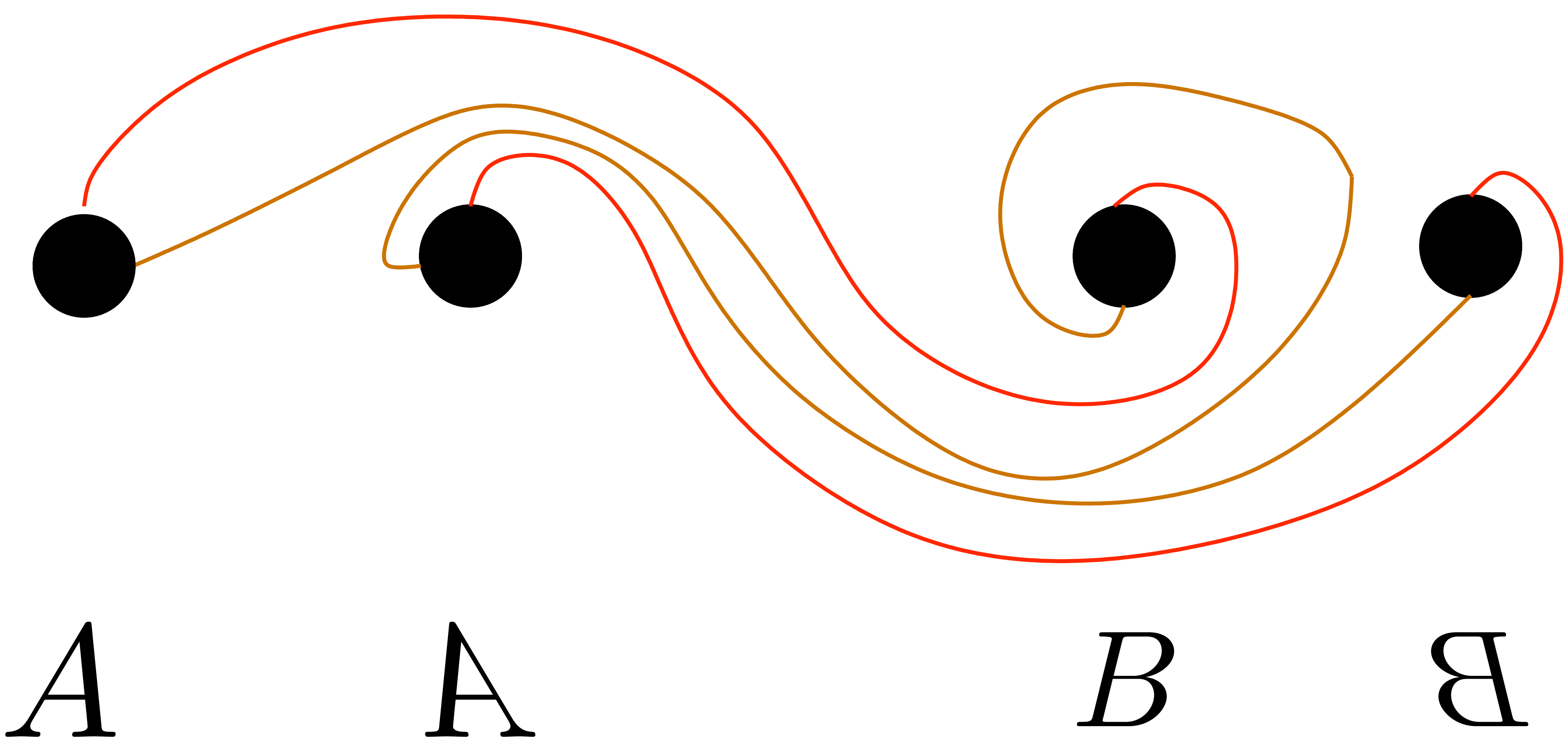
III. Branched double covers: Connections between Heegaard Floer homology & Khovanov homology

# Studying 3- & 4-manifolds: Handlebody theory



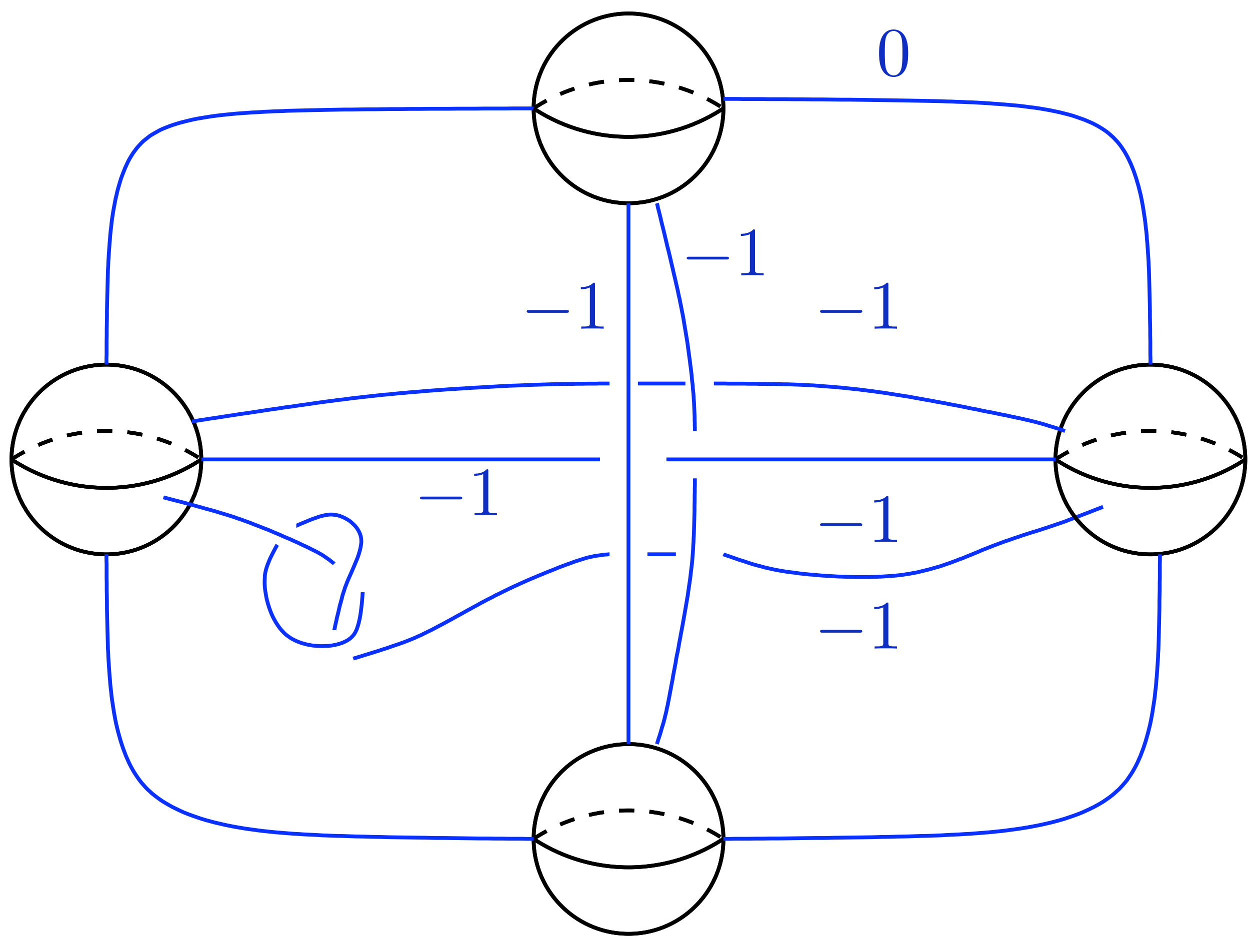
# Studying 3- & 4-manifolds: Handlebody theory

3 dimensions: (Screen is $S^2$)



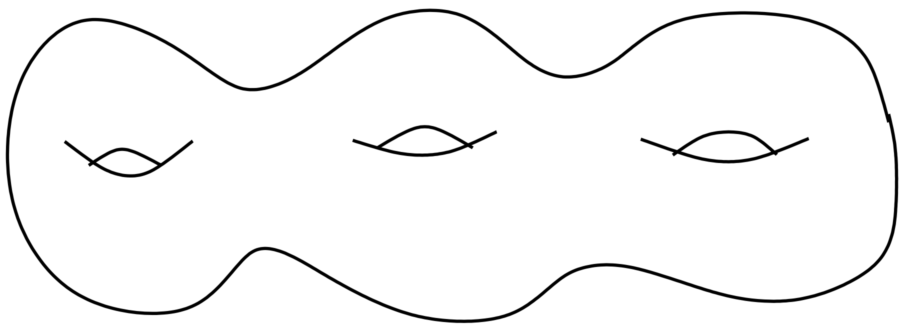
# Studying 3- & 4-manifolds: Handlebody theory

4 dimensions: (Screen is a projection of $S^3$ )



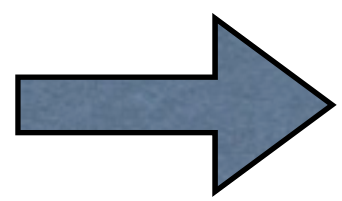
# Studying 3- & 4-manifolds: Gauge theory & Floer homology

Y



+

geometric structures "g"
(e.g., complex structure on an imbedded surface)

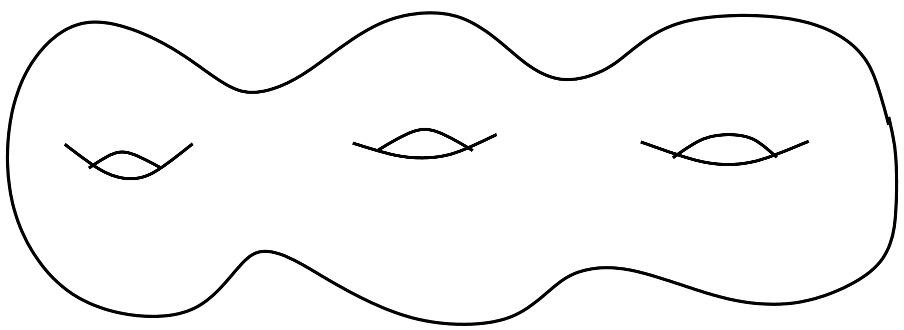


*some other topological space constructed from data (Y,g):*

$$\mathcal{M}(Y, g)$$

infinite-dimensional moduli space
(e.g., space of flat connections modulo gauge on principal SU(2) bundle on Y)

# Studying 3- & 4-manifolds: Gauge theory & Floer homology

Y 

$+$

geometric structures "g"
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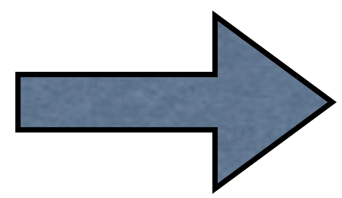


*If lucky...*

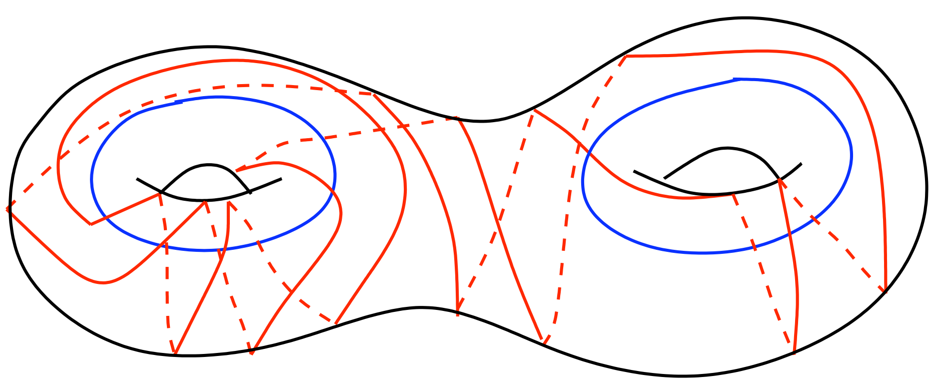
$$H_*(\mathcal{M}(Y,g))$$

an invariant of
diffeomorphism class of Y
(independent of choice of
extra geometric structures)

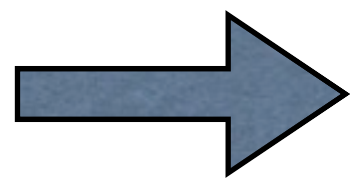
# Handlebodies + Gauge theory = Heegaard-Floer homology

Ozsváth-Szabó

3-manifold Y



(+ extra geometric structure)



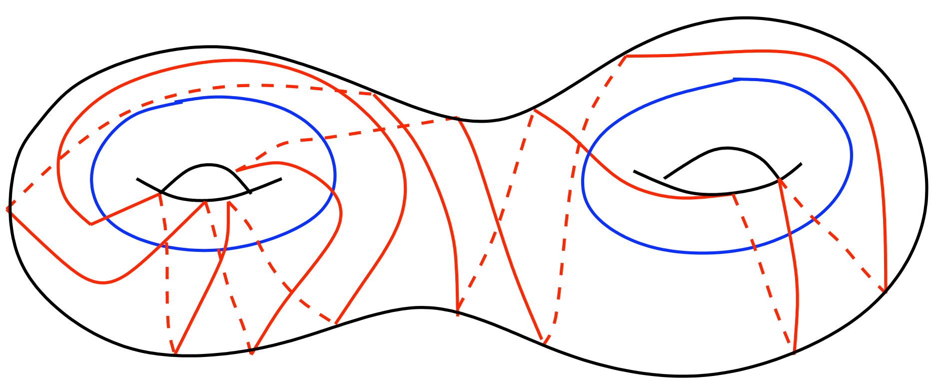
$(\mathcal{C}, \partial)$

abstract chain complex

# Handlebodies + Gauge theory = Heegaard-Floer homology

Ozsváth-Szabó

3-manifold Y



(+ extra geometric structure)

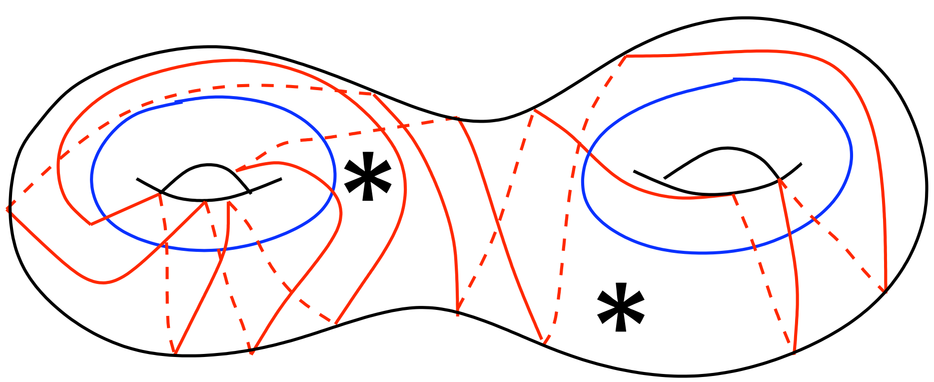
$$H_*(\mathcal{C}, \partial)$$

is an invariant of Y

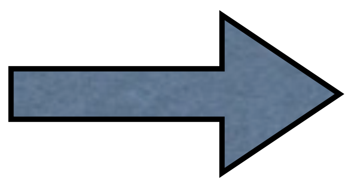
# Handlebodies + Gauge theory = Heegaard-Floer homology

Ozsváth-Szabó, Rasmussen

Knot in Y




(+ extra geometric structure)



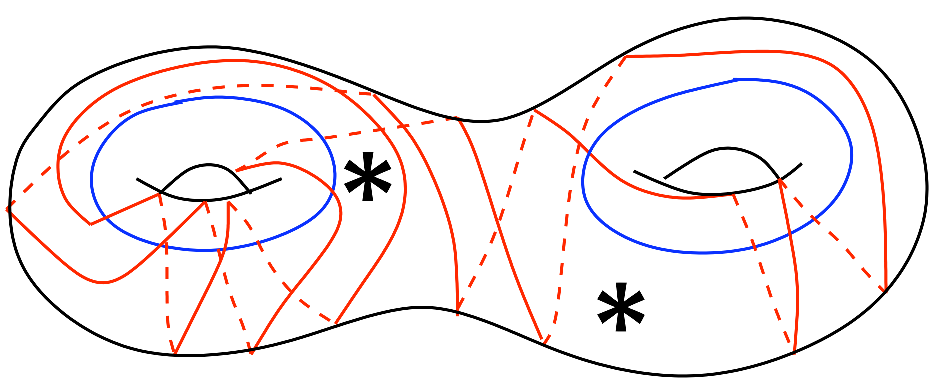
$$(\mathcal{C}, \partial)$$

abstract chain complex
(+ extra algebraic structure)

# Handlebodies + Gauge theory = Heegaard-Floer homology

Ozsváth-Szabó, Rasmussen

Knot in Y



(+ extra geometric structure)

$$H_*(\mathcal{C}, \partial)$$

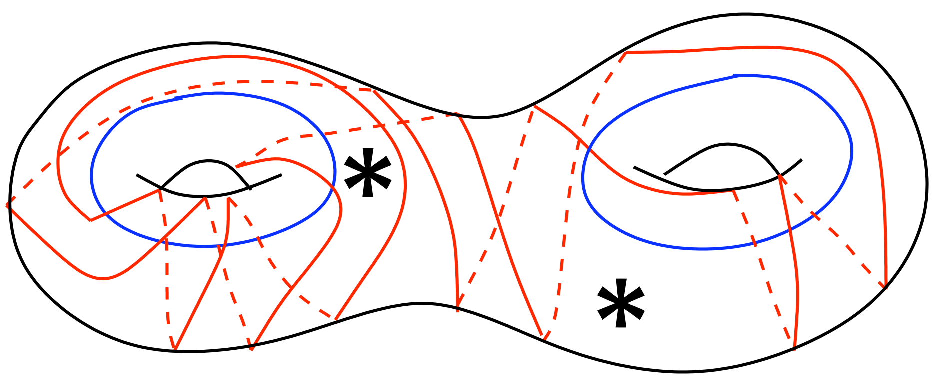
(+ extra algebraic structure)

is an invariant of K

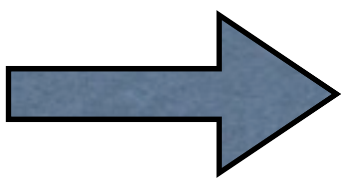
# Handlebodies + Gauge theory = Heegaard-Floer homology

Ozsváth-Szabó, Rasmussen

Knot in Y



(+ extra geometric structure)



$$H_*(\mathcal{C}, \partial)$$

(Ozsváth-Szabó)

- Detects knot genus

(Ghiggini, Ni)

- Detects knot fiberedness

# Heegaard-Floer homology

Question (Kirby, Melvin, 2003): Does the Heegaard Floer chain complex have a combinatorial description?

Hard problem...

Trying to prove that there was a combinatorial description for the Heegaard Floer chain complex for *any* Heegaard diagram.

# Heegaard-Floer homology

Question (Kirby, Melvin, 2003): Does the Heegaard Floer chain complex have a combinatorial description?

Answer (Sarkar, 2006): Yes, for special types of Heegaard diagrams.

# Lessons

*“You gotta know know when to fold ‘em.”*

*-Kenny Rogers*

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I. Background: Handlebody theory & Heegaard Floer homology

II. Branched double covers: Applications to knot concordance

III. Branched double covers: Connections between Heegaard Floer homology & Khovanov homology

# Double-branched covers

(Ozsváth-Szabó, 2003)

There is an algebraic relationship between certain invariants (of links) defined by Khovanov and Heegaard-Floer invariants (of their double-branched covers).

gauge theory, symplectic geometry

geometric representation theory

*Connection between two seemingly unrelated areas.*

# Double-branched covers

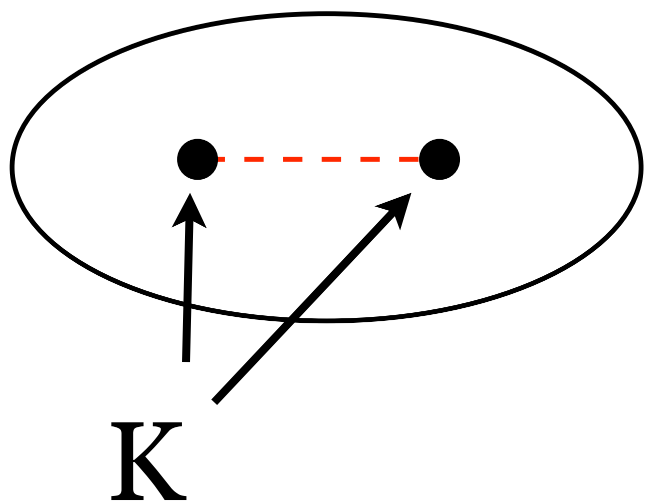
(Ozsváth-Szabó, 2003)

There is an algebraic relationship between certain invariants (of links) defined by Khovanov and Heegaard-Floer invariants (of their double-branched covers).

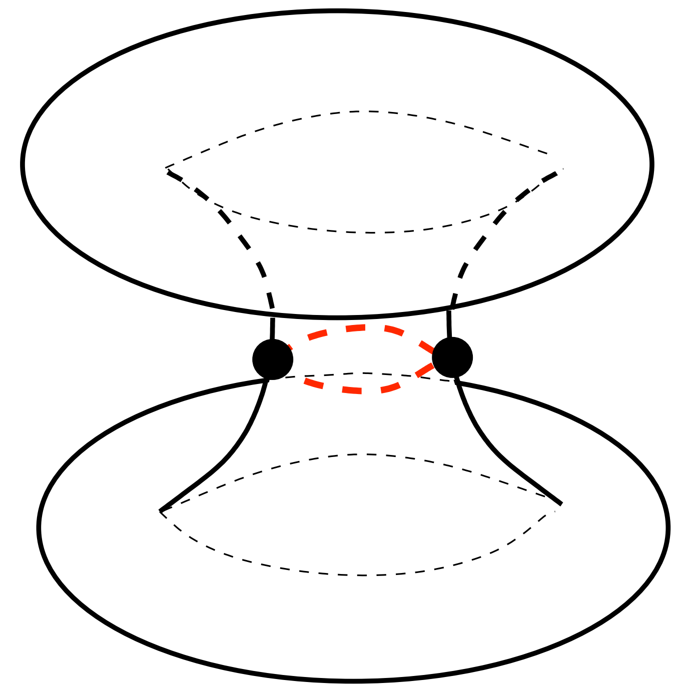
topological operation

*Connection between two seemingly unrelated areas.*

# Double-branched covers

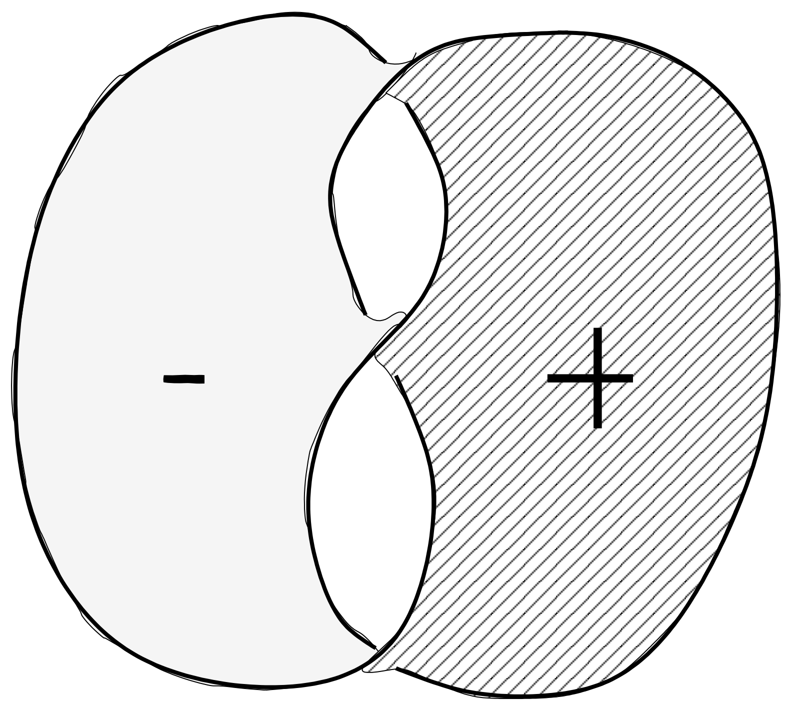



codim 2
“branch set”



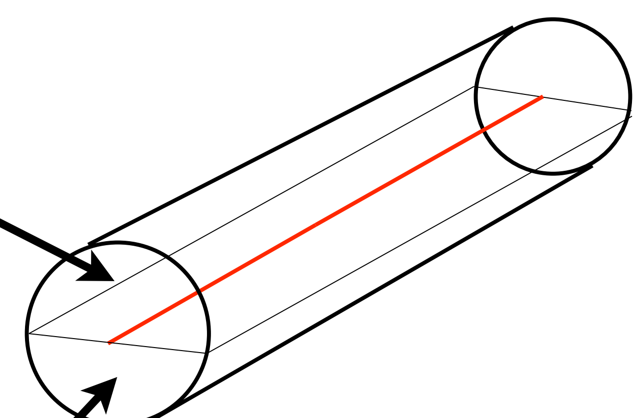
double-branched
cover
$\Sigma(K)$

# Double-branched covers

$S^3 -$ 


(2 copies)

$\Sigma(K)$



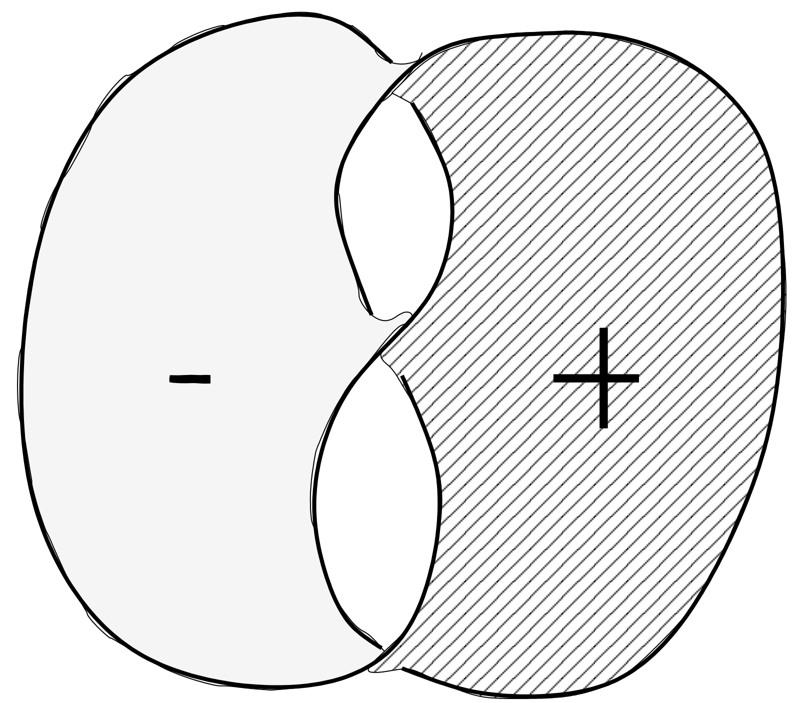
# Double-branched covers

Question (Ozsváth):

What is the relationship between the Heegaard-Floer invariants of a knot and the Heegaard-Floer invariants of the preimage of a knot in its double-branched cover?

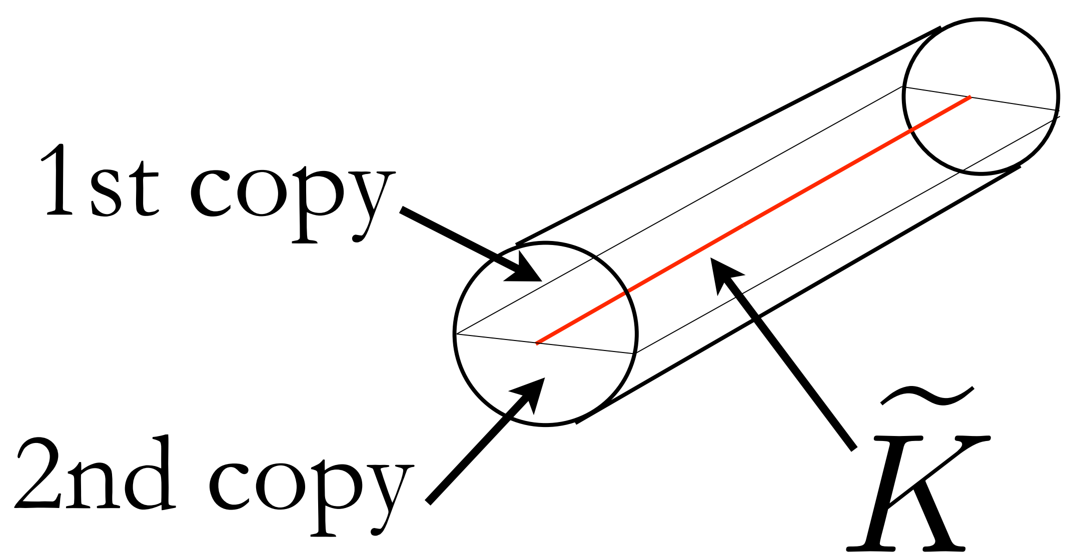
# Double-branched covers

$S^3 -$




(2 copies)

$\Sigma(K)$



# Double-branched covers

Partial answer:

When $K \subset S^3$ a two-bridge knot,

$$\widehat{HFK}(S^3, K) \subset \widehat{HFK}(\Sigma(K), \widetilde{K}) = \bigoplus_{i \in \mathbb{Z}_n} \widehat{HFK}(\Sigma(K), \widetilde{K}; i)$$

$$\widehat{HFK}(S^3, K) = \widehat{HFK}(\Sigma(K), \widetilde{K}; i = 0)$$

# Double-branched covers

Better answer: No nice general relationship

There exist pairs of knots for which

$$\widehat{HFK}(S^3, K_1) \cong \widehat{HFK}(S^3, K_2)$$

but

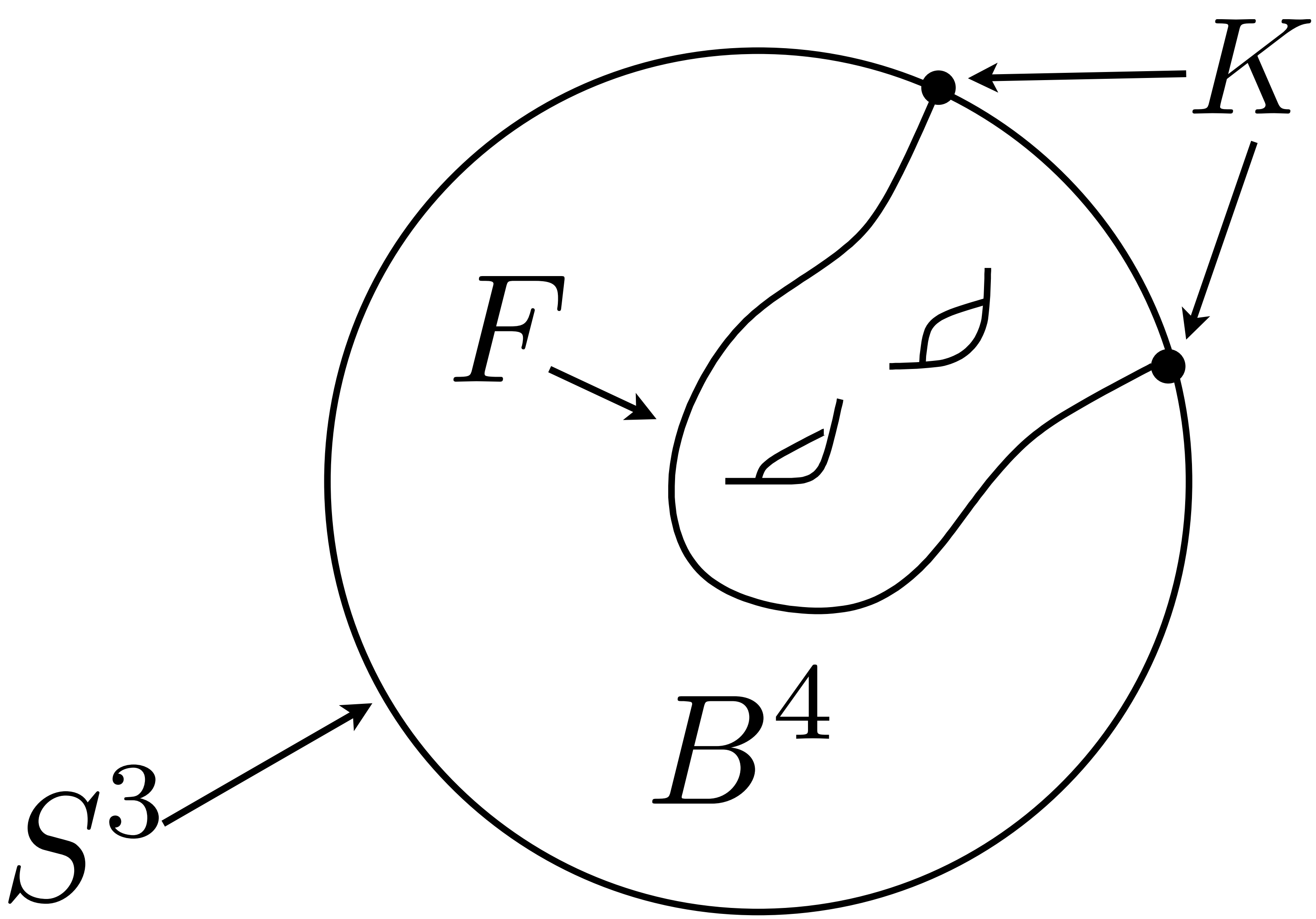
$$\widehat{HFK}(\Sigma(K_1), \widetilde{K}_1) \ncong \widehat{HFK}(\Sigma(K_2), \widetilde{K}_2)$$

# Double-branched covers

Conclusion: The Heegaard-Floer invariants of a knot and the Heegaard-Floer invariants of its preimage in its double-branched cover carry different topological information.

*What good is this???*

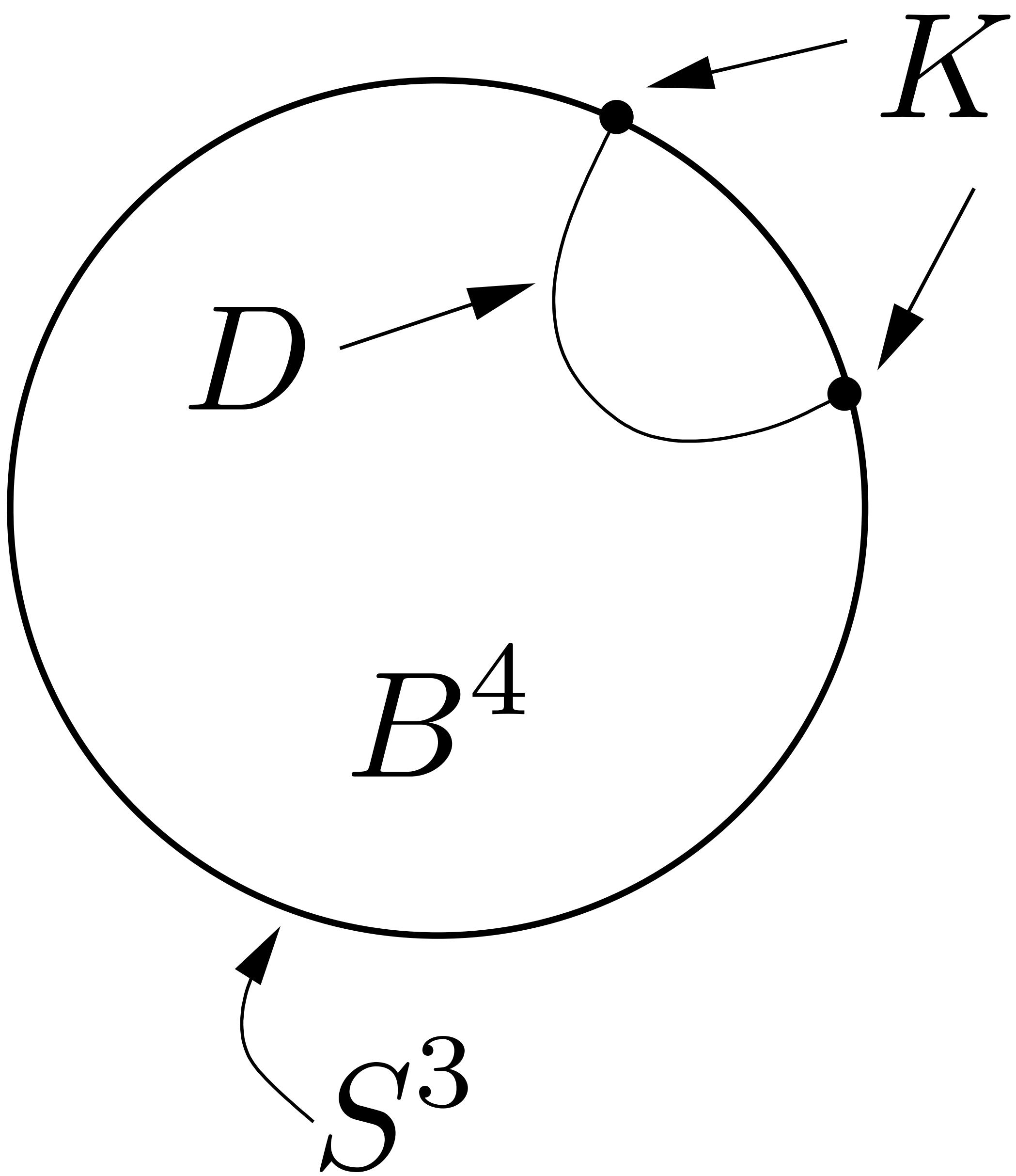
# Application: Knot concordance




$g_4(K) =$ minimal genus of such F

"Link cobordisms"

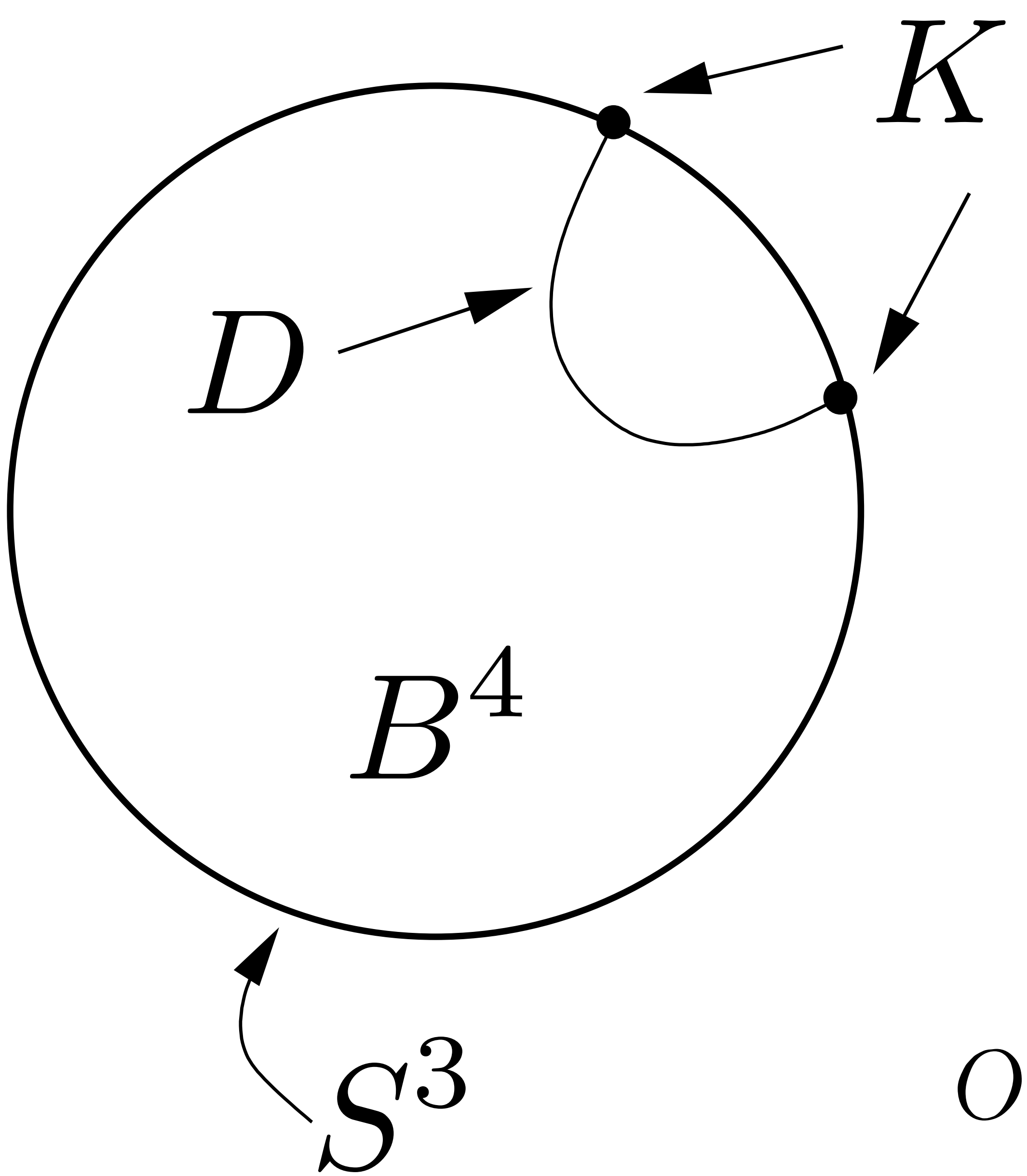
# Application: Knot concordance




$g_4(K) = 0$

K is "slice"

# Application: Knot concordance




(Ozsváth-Szabó, Rasmussen):

$$\tau(K) \in \mathbb{Z}$$

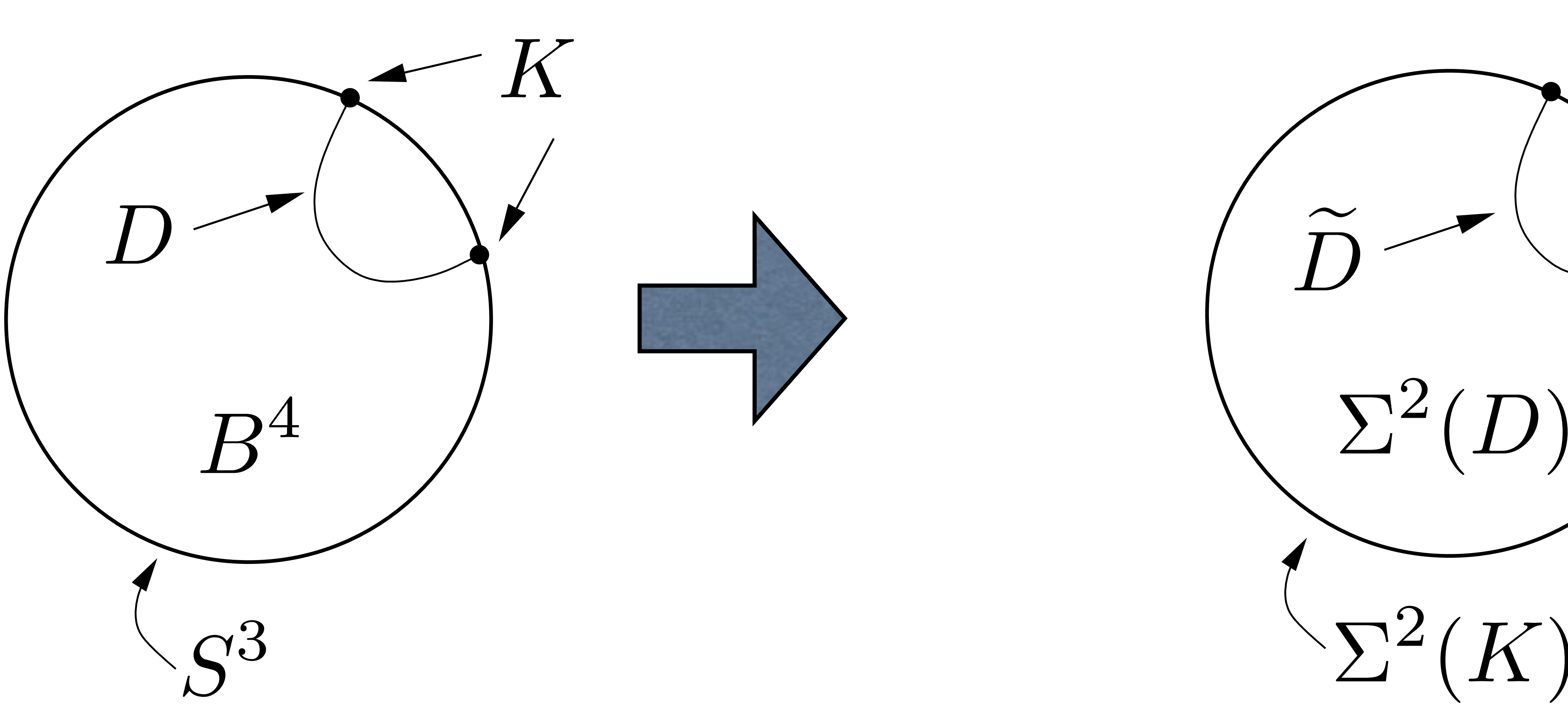
If K is slice, then

$$\tau(K) = 0$$

*Obstruction to K being slice.*

# Application: Knot concordance

(Casson-Gordon)

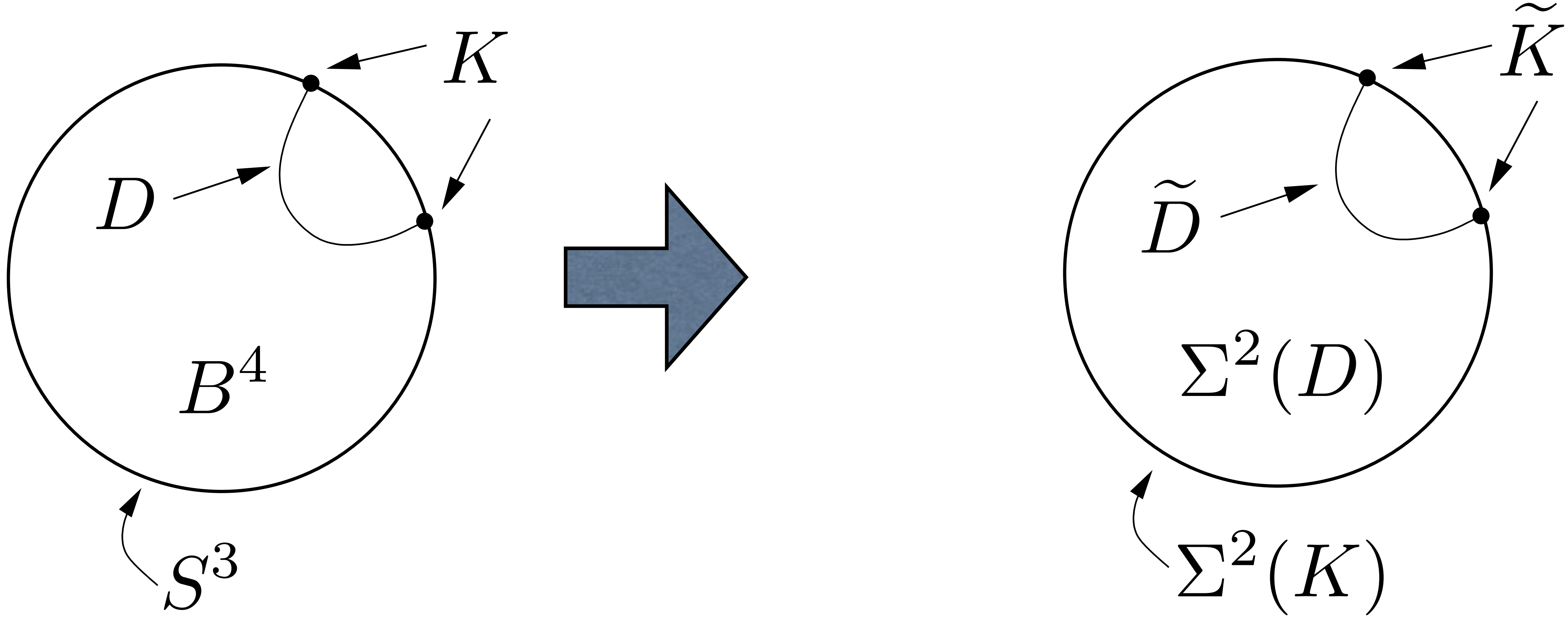



$\mathbb{Q}HB^4$

K slice $\Rightarrow$ $\widetilde{K}$ bounds $\widetilde{D} \subset \Sigma(D)$

# Application: Knot concordance

(Ruberman-Strle)

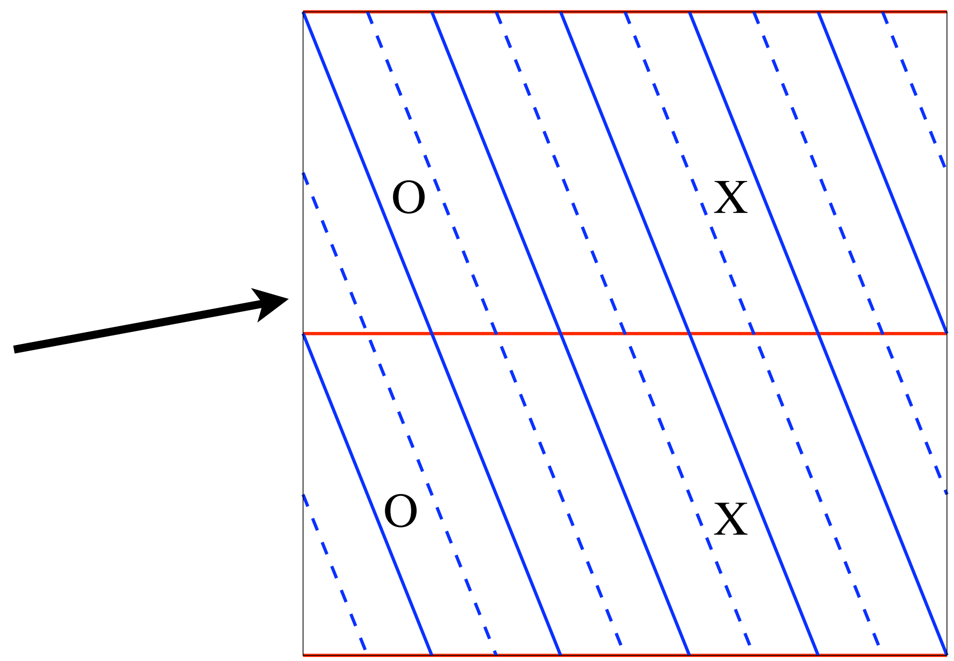



K slice $\Rightarrow$ $\tau(\widetilde{K}) = \vec{0}$ (for a certain subset)

# Application: Knot concordance

Calculations depended on new combinatorial techniques (special Heegaard diagrams generalizing Manolescu-Ozsvath-Sarkar)

special Heegaard diagram




...led to new concordance information about 2-bridge knots

# Lessons

- Sometimes a problem is more interesting than it seems at first.
- If you know something very well, it is probably useful to someone.
- Keep your eye out for new "technology"!

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# Application: Connections between Khovanov and Heegaard-Floer homology

(Ozsváth-Szabó, 2003)

There is an algebraic relationship between certain invariants (of links) defined by Khovanov and Heegaard-Floer invariants (of their double branched covers).

# Application: Connections between Khovanov and Heegaard-Floer homology

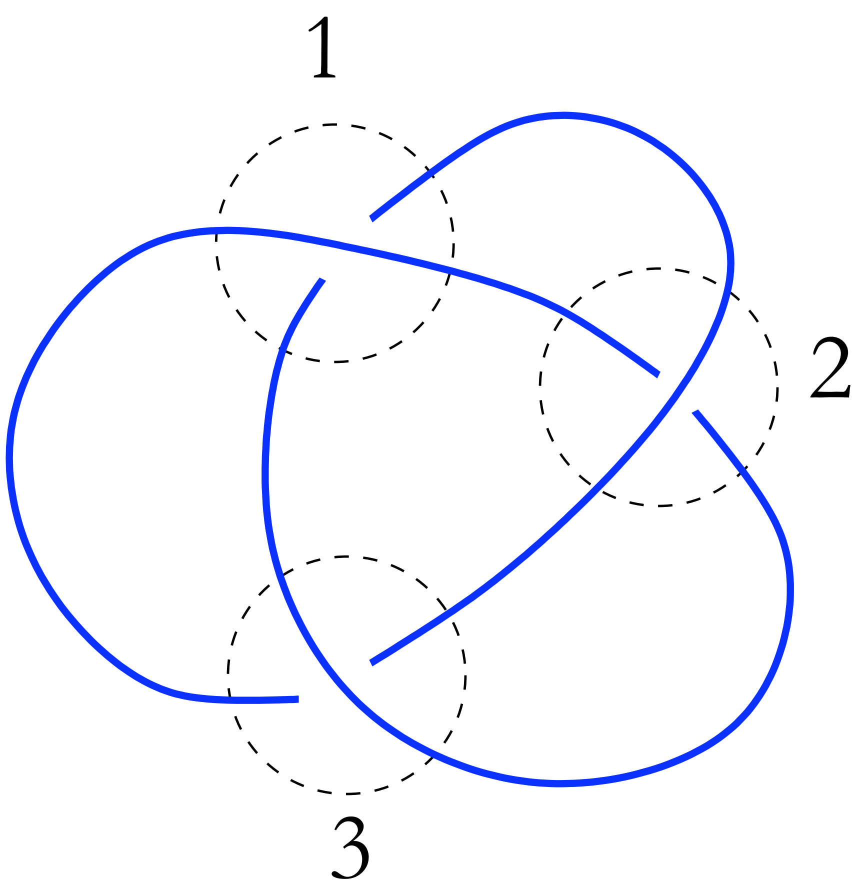
combinatorial

(Ozsváth-Szabó, 2003)

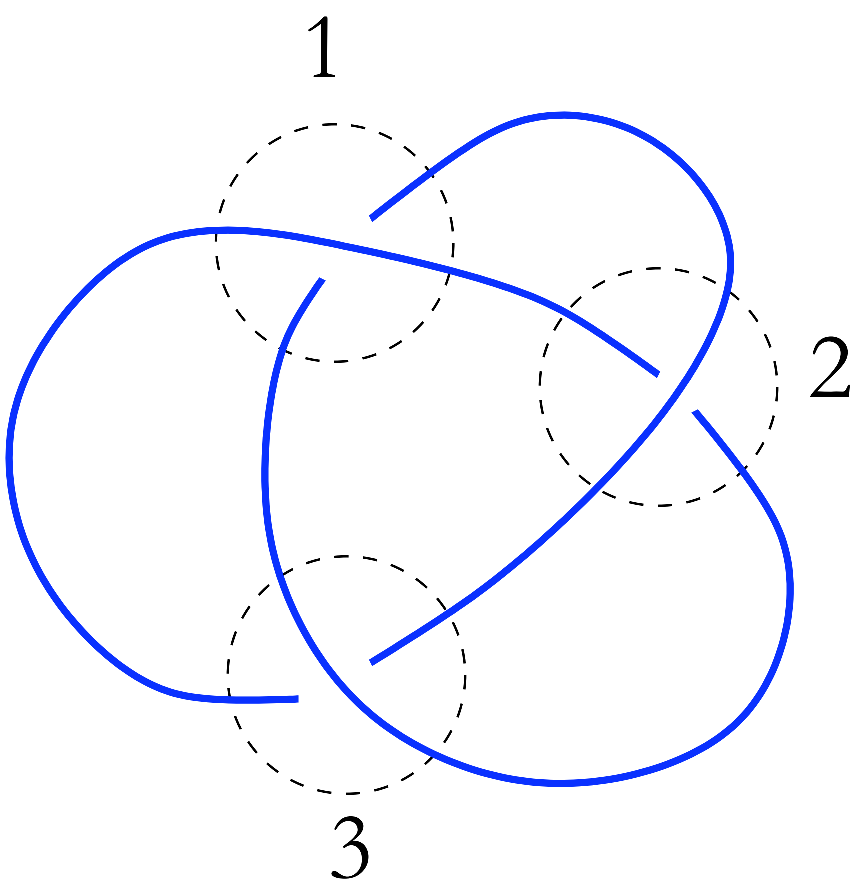
There is a *spectral sequence* connecting the (reduced) Khovanov homology of (the mirror of) a link and the Heegaard-Floer homology of the link's double branched cover.

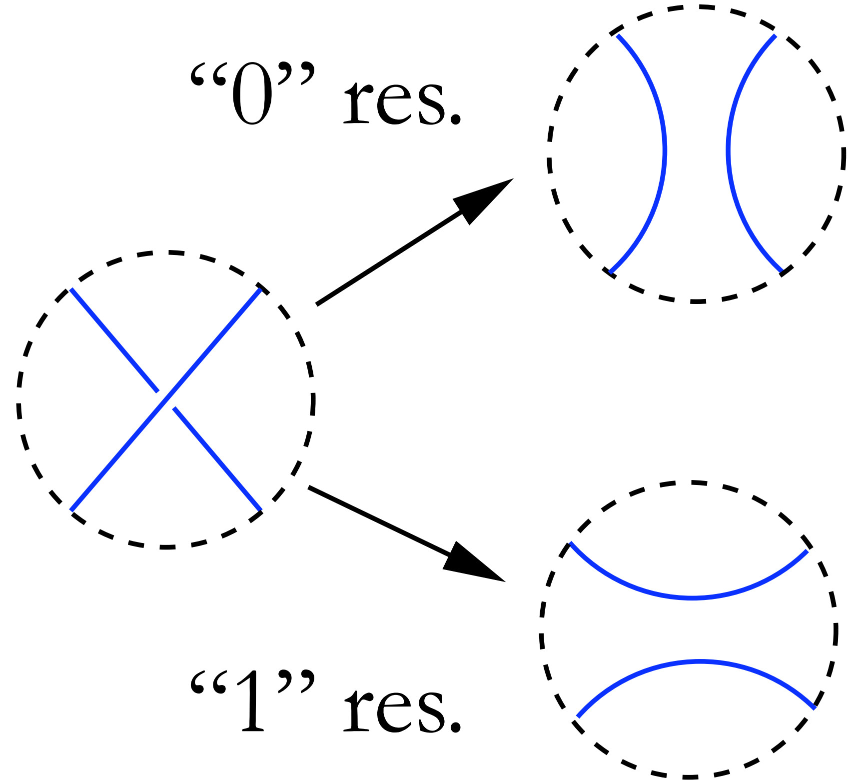
*"Deforming" Khovanov homology (of a link) yields Heegaard Floer homology (of the double branched cover).*

# Definition: Khovanov Homology



# Definition: Khovanov Homology



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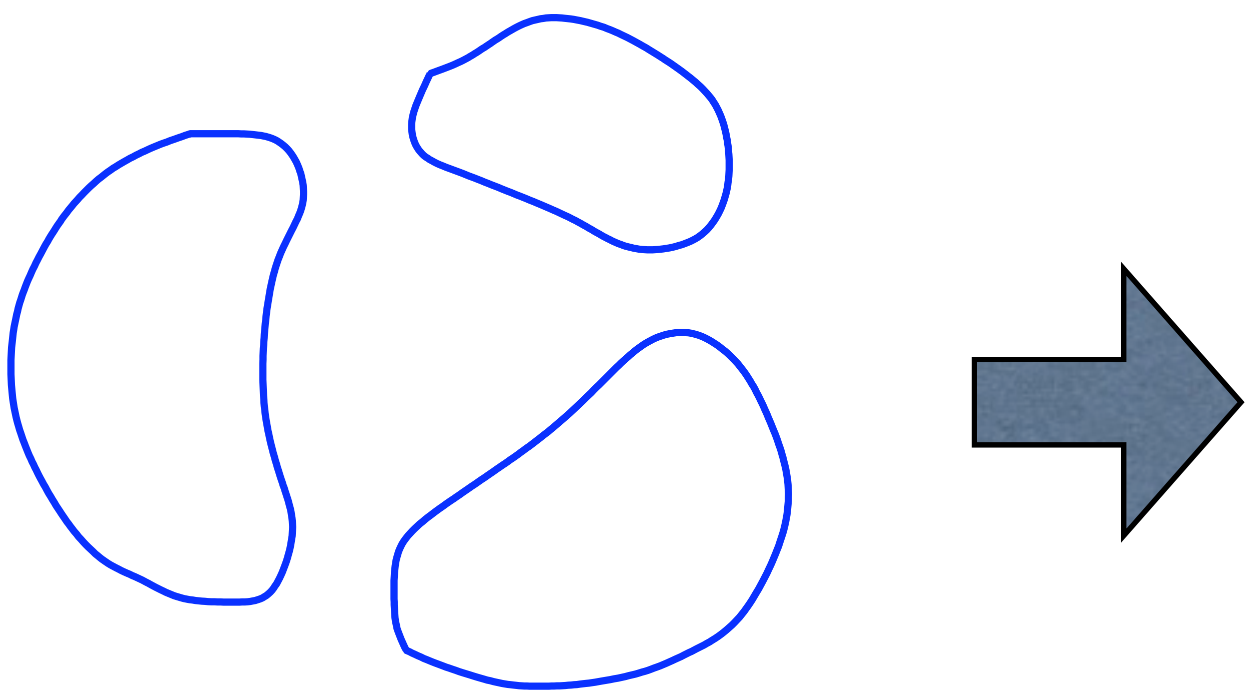
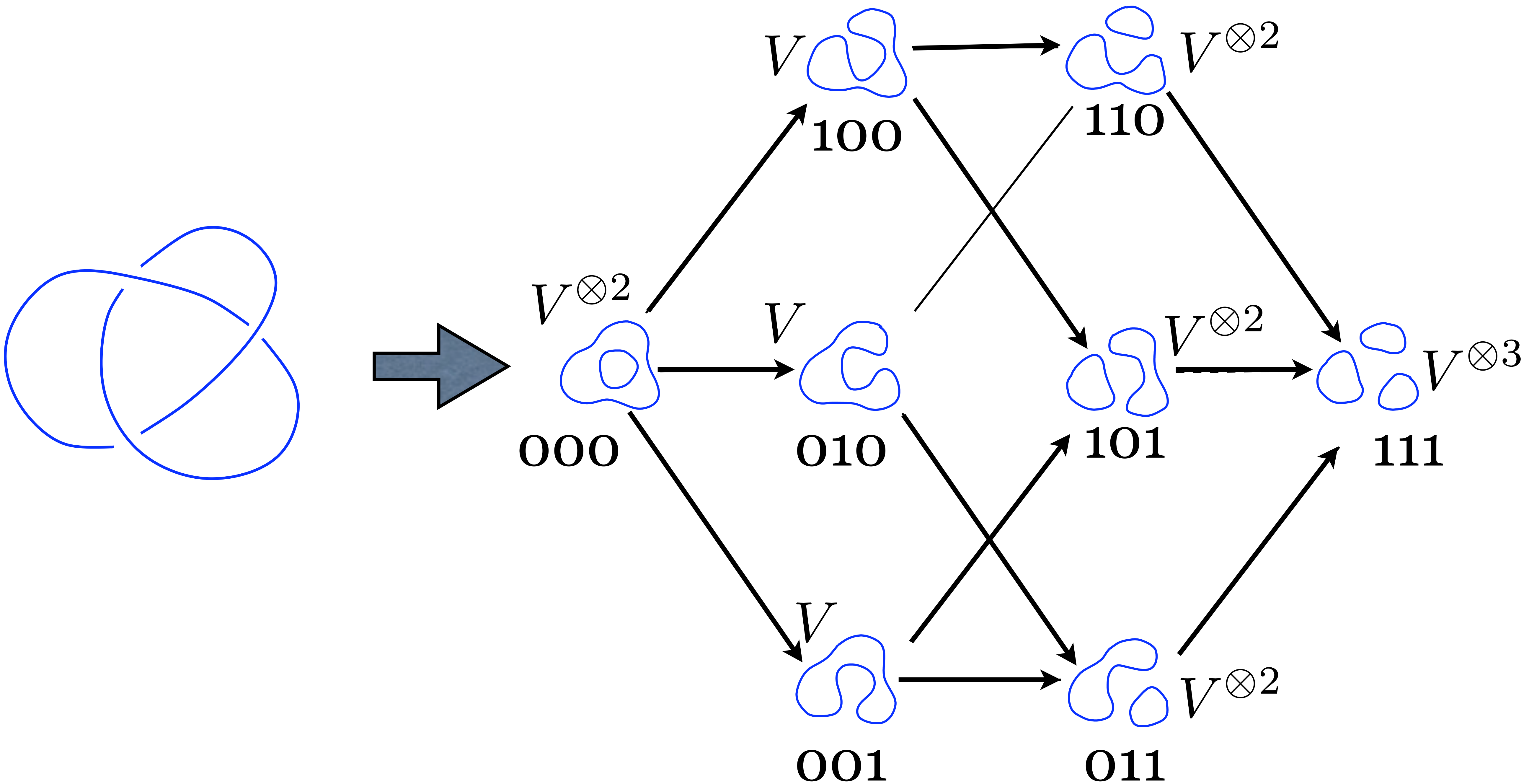


k circles

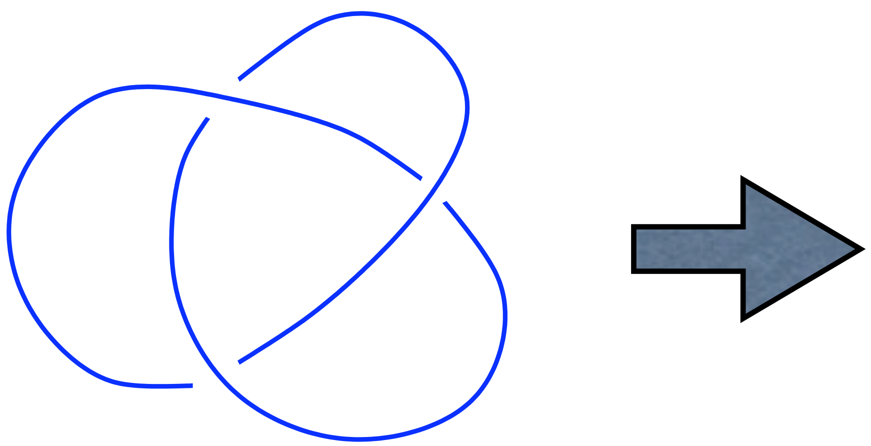
$$V^{\otimes k}$$

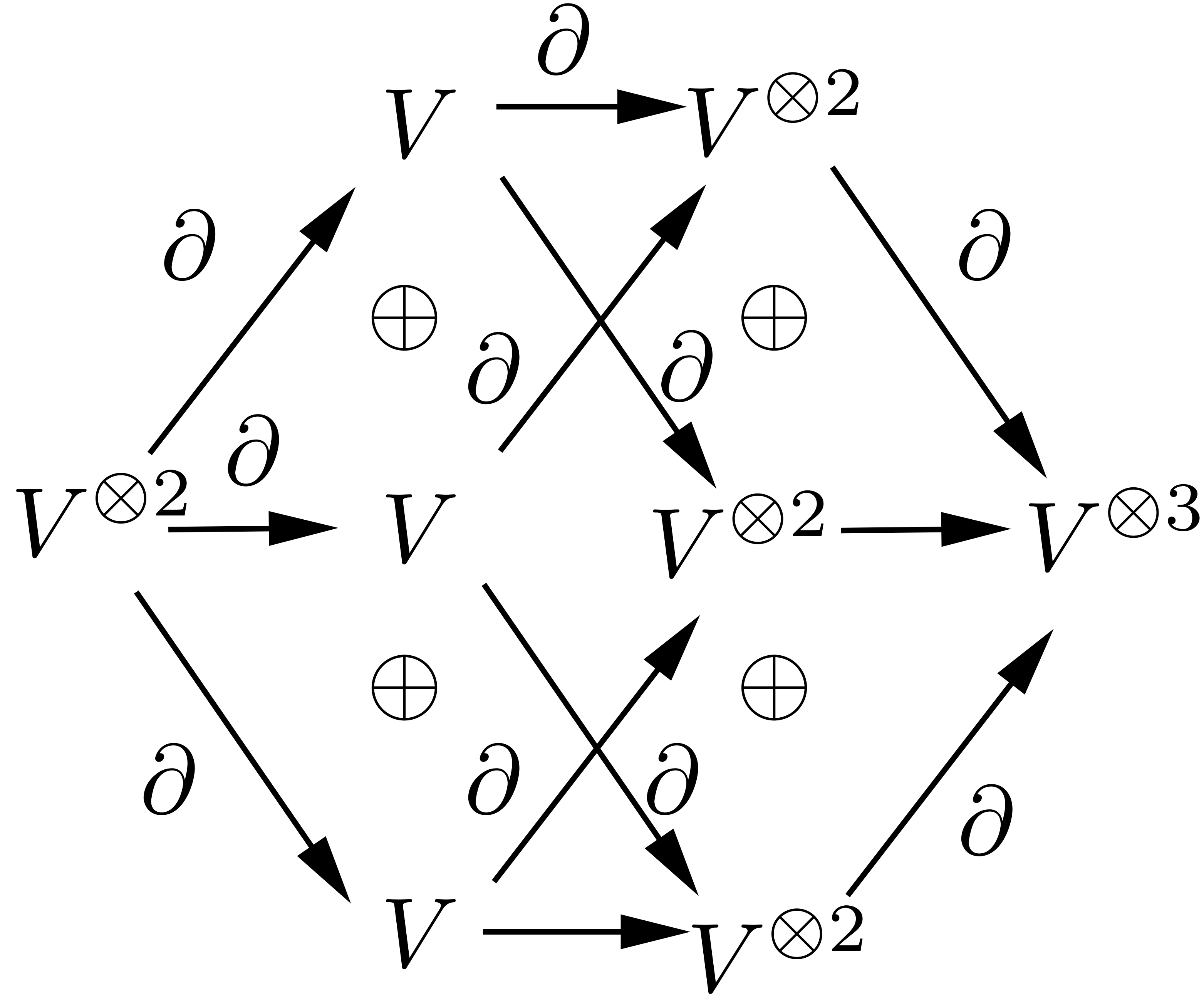
$$V = \mathrm{Span}_{\mathbb{F}}(v_+, v_-)$$

# Cube of resolutions



# Cube of resolutions






$$\partial \circ \partial = 0$$

# Application: Connections between Khovanov and Heegaard-Floer homology

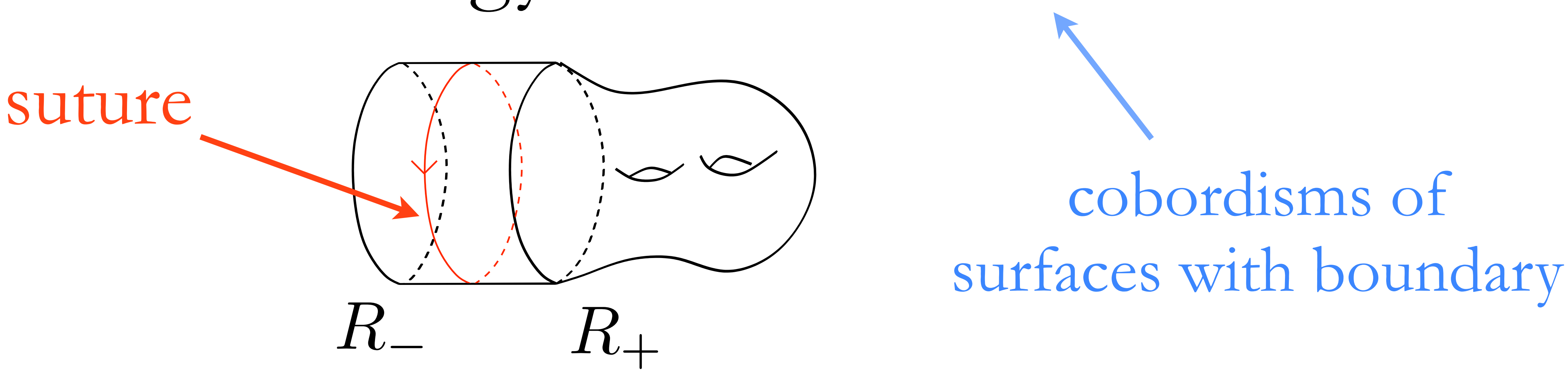
+ Stephan Wehrli

Question: Can the spectral sequence from Khovanov homology to Heegaard Floer homology be described combinatorially?

*about 6 months...*

# Application: Connections between Khovanov and Heegaard-Floer homology

(Juhasz): a version of Heegaard Floer homology for sutured manifolds




Heegaard Floer homology detects knot genus and knot fiberedness

# Application: Connections between Khovanov and Heegaard-Floer homology

Sutured structure ⇨ geometric structures

(Gabai): foliations

↕

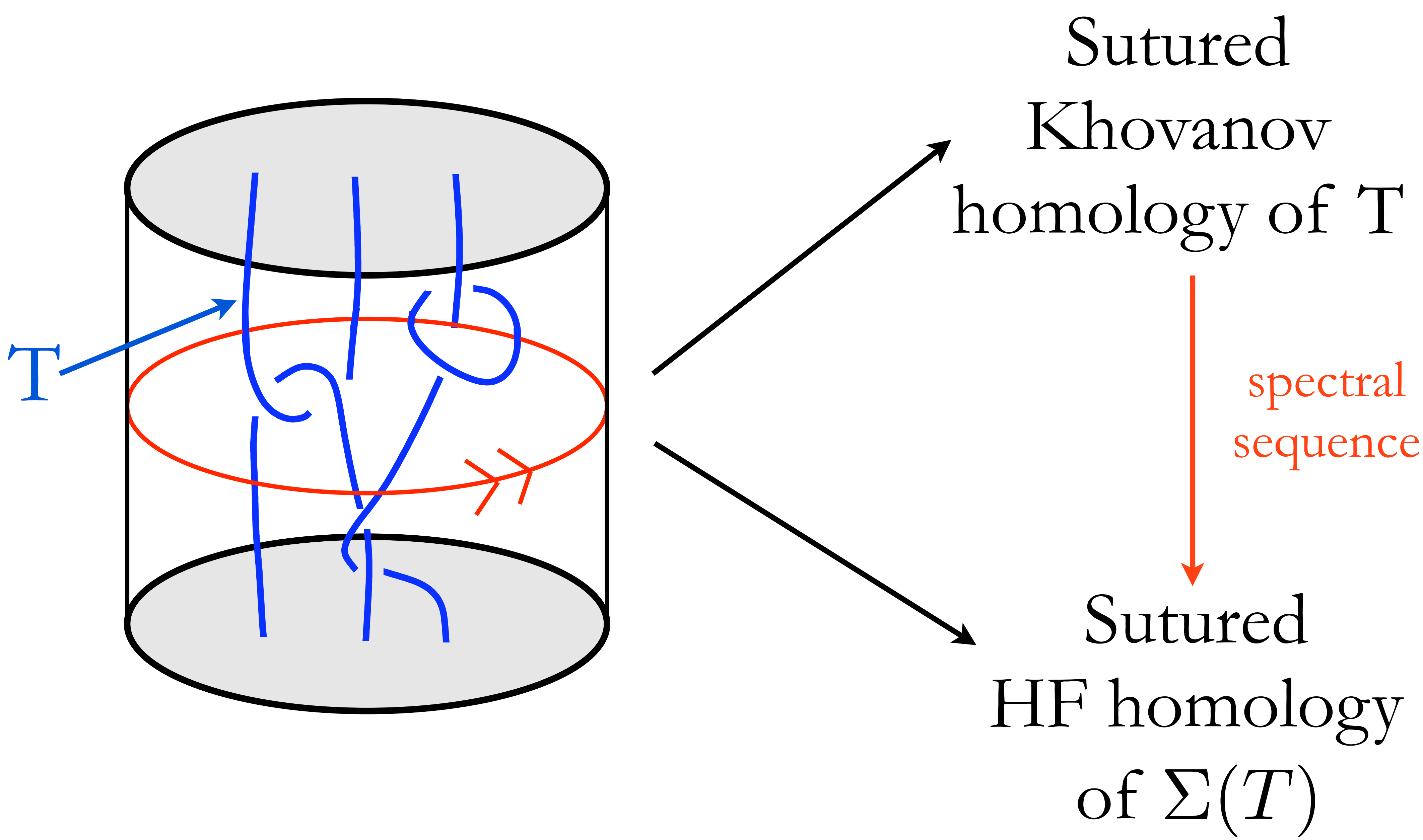
(Eliashberg-Thurston, Giroux, Honda-Kazez-Matic) contact structures

# Application: Connections between Khovanov and Heegaard-Floer homology

Question: What is the relationship between Khovanov homology and sutured Floer homology?

The “right” structure for understanding Khovanov homology (and its relationship to Heegaard Floer homology).

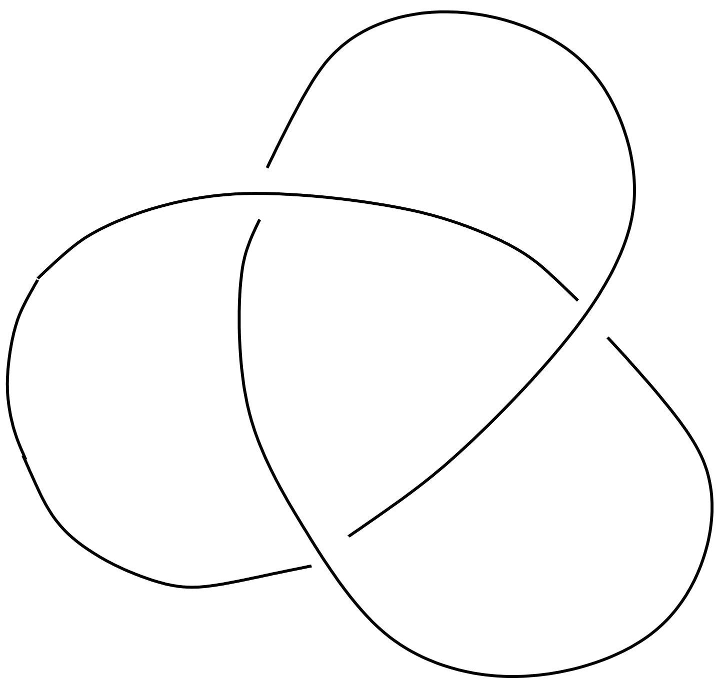
# Sutured Khovanov homology and Sutured Heegaard-Floer homology



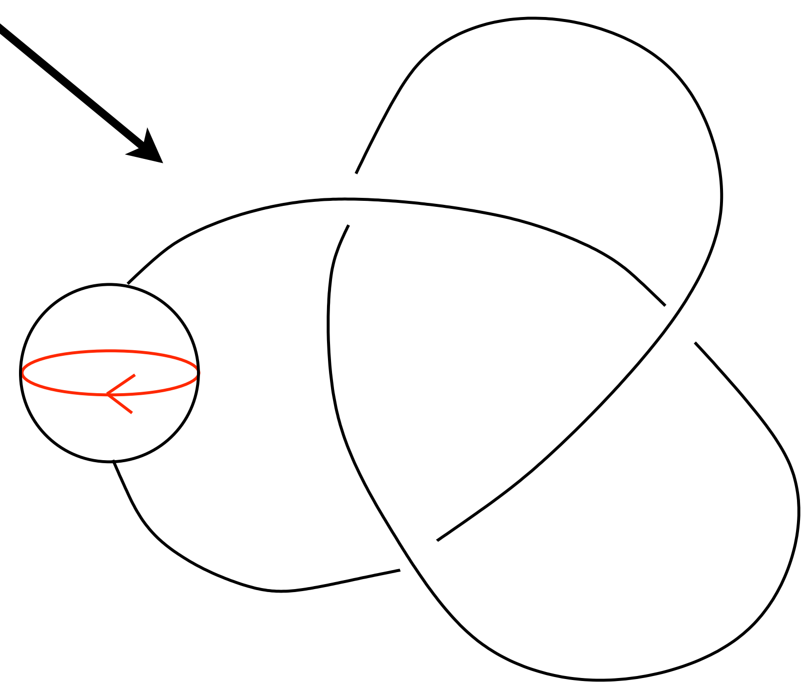
# Sutured Khovanov homology and Sutured Heegaard-Floer homology

Application:
Sutured Khovanov homology detects the unknot.

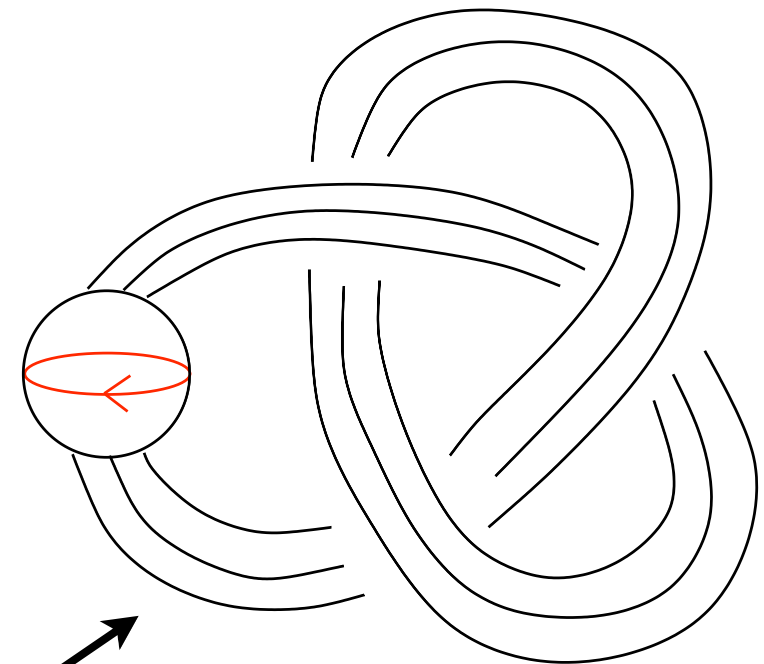
# Detecting the unknot



$K \subseteq S^3$

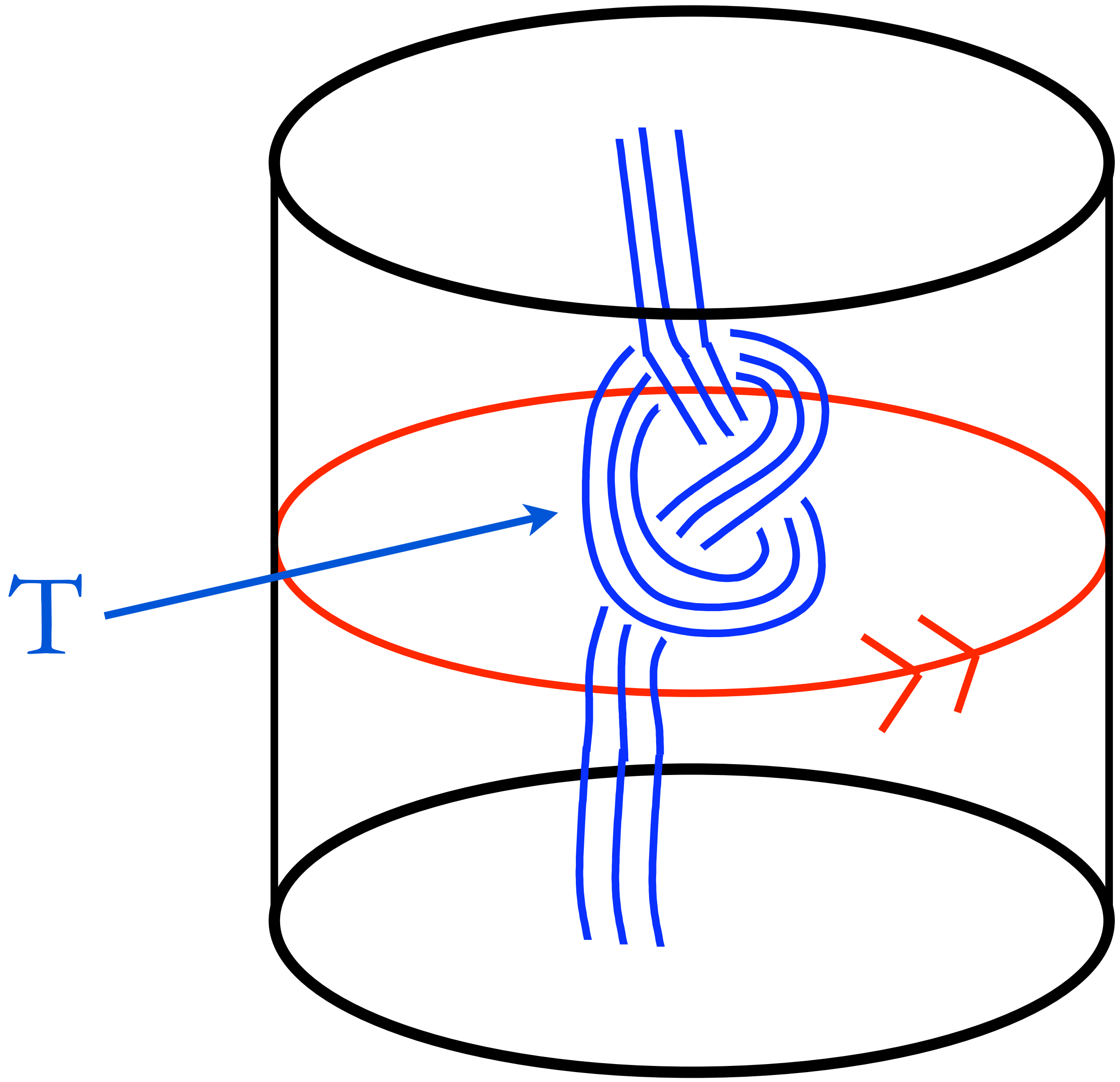


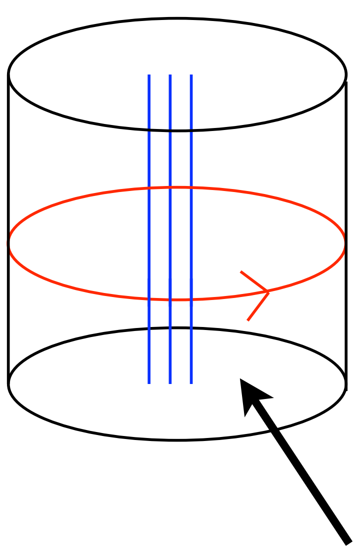
remove N(pt)



reduced
n-cable

# Detecting the unknot

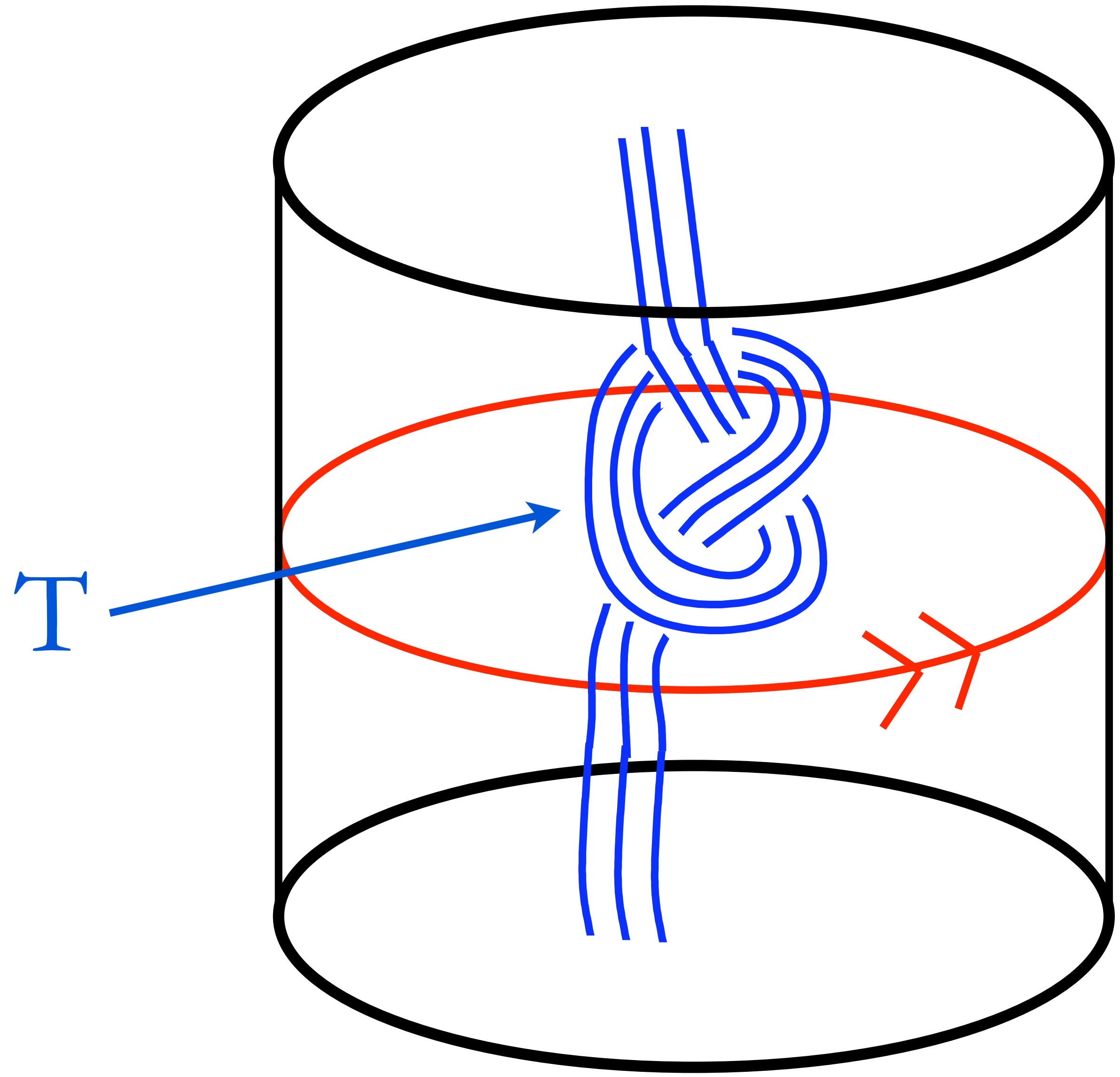


Not a product,
$n \geq 2$

sutured Khovanov homology
has rank $> 1$

# Detecting the unknot




uses connection to
sutured HF homology

# Sutured Khovanov homology and Sutured Heegaard-Floer homology

- Khovanov homology for tangles in other (sutured) manifolds?
- Khovanov homology for (sutured) 3-manifolds?
- Dependence of Khovanov homology on geometric sructures (like foliations, contact structures)?
- Relationship to 4-dimensional geometric structures (like broken fibrations)?

# Lessons

- Talking to people (especially those whose knowledge is disjoint from yours) is always useful.
- Sometimes you won't solve the problem you started working on. And that's OK. You will find another problem.

# A few more words...

(Rob Kirby): “Make sure you’re enjoying the math you’re doing. Otherwise, why are you doing it?”

# A few more words...

- Keep learning
- Enjoy the pain!