From contact structures to fluid flows ...

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the First National Forum of Young Topologists

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Morocco
Marrakech
Alexandria
Timimoun
Algeria
Laayoune
Raggane
In Salah
Libya
Western Sahara
Sahara
Nouadhibou
Atar
Mauritania
Mali
Niger
Largeau
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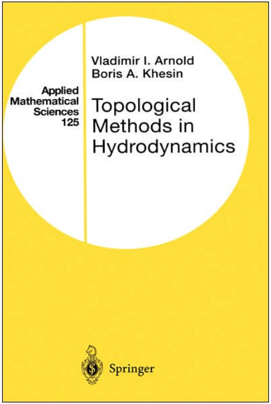
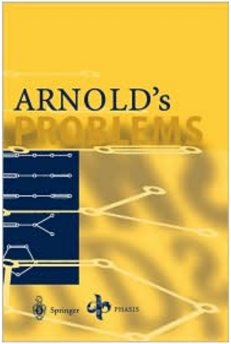
Topological methods in hydrodynamics



Advisor:

> *Read this book it has a lot of interesting open problems. . .*

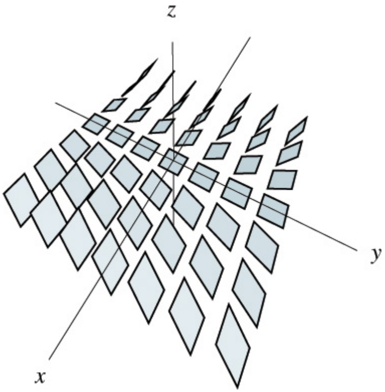
In fact there are so many open problems that there is another book



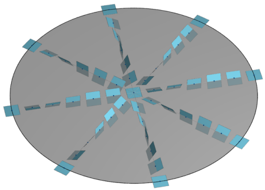
"Arnold's Problems" by Vladimir I. Arnold, available for free at
`http://www.phasis.ru/Arnold-Problems/index.html`

Contact structures

Are *plane distributions* ξ such that for any pair of vector fields X, Y spanning ξ, the field $[X, Y] = XY - YX$ is transverse to ξ.



$\xi_0 = \ker\{dz + x\,dy\}$



$\xi_1 = \ker\{\cos r dz + r \sin r\, d\theta\}$

Contact forms as the curl eigenfields

Let $\alpha = \langle B, \cdot \rangle$ be a 1-form on a 3-manifold M, then $\xi = \ker \alpha$, defines a contact structure if and only if

$$\alpha \wedge d\alpha \neq 0, \quad \alpha \in \Omega^1(M).$$

Fact: For every *contact form* α there exist an ample set of Riemannian metrics $\langle \cdot, \cdot \rangle$ for which the dual vector field B satisfies

$$\nabla \times B = \mu\, B, \qquad \mu \neq 0$$

i.e. B is an eigenfield of the curl operator $\nabla \times$. In terms of the contact form α this equation can be written as follows

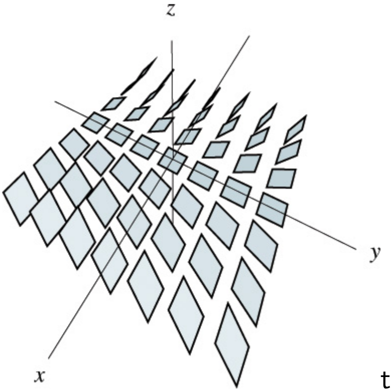
$$* \, d\, \alpha = \mu\, \alpha,$$

where $*$ is the Hodge star operator, d is the exterior differentiation.

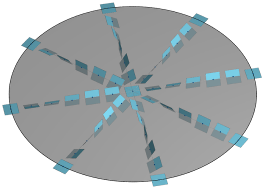
Isotopy of contact structures

- two contact structures ξ_0 and ξ_1 are *isotopic* iff. there exists a homotopy of plane fields ξ_t, $0 \leq t \leq 1$, such that ξ_t is a contact plane distribution for all t.
- a contact structure ξ is *overtwisted* if and only if there exists an embedded disk $D^2 \subset M$ such that D is transverse to ξ near ∂D but ∂D is tangent to ξ. Any contact structure which is not overtwisted is called *tight*. If all the covers of a structure are tight then we call it *universally tight*.
- full classification only on certain manifolds such us S^3, T^3 most of the 3d Seifert bundles. [Bennequin, Eliashberg, Etnyre, Giroux, Honda, . . .]. In general a big open problem in the field.

Classification of contact structures up to isotopy



tight



overtwisted

How to detect tight/overtwisted?

- X is the *contact vector field* iff the flow of X preserves ξ.
- The set of tangencies $\Gamma_X = \{p \in M : X_p \in \xi_p\}$ is called the *characteristic hypersurface* of X in ξ.
- in terms of a contact form α, X is a contact vector field if $\mathcal{L}_X \alpha = v\,\alpha$, $f = \alpha(X)$ is called a *contact hamiltonian*.
- $\Gamma_X = f^{-1}(0)$.

How to detect tight/overtwisted?

[Giroux's criteria]: Given an embedded orientable surface Σ in the contact manifold (M, ξ) and a transverse to Σ contact vector field X define $\Gamma = \Gamma_X \cap \Sigma$. Then:

(i) if $\Sigma \neq S^2$, then ξ has a tight tubular neighborhood iff. none of the components in Σ/Γ bounds a disc.

(ii) if $\Sigma = S^2$, then ξ has a tight tubular neighborhood iff. Γ is connected.

(Σ is called the *convex surface* and Γ the *dividing set*).

Contact structures and magnetic relaxation.

The velocity field $v(\mathbf{x},t)$ of plasma and its magnetic field $B(\mathbf{x},t)$ are governed by the equations

$$\rho\Big(\frac{\partial v}{\partial t} + v\cdot\nabla v\Big) = -\nabla p + (\nabla\times B)\times B + \mu\,\Delta v,$$
$$\frac{\partial B}{\partial t} = \nabla\times(v\times B),$$
$$\mathsf{div}(v) = \mathsf{div}(B) = 0\ .$$

As a direct consequence of these equations

$$E_2(B(t)) + E_2(v(t)) = \int |B(t)|^2 + \int |v(t)|^2 \text{ decreases as } \quad t\to\infty$$

and in particular $E_2(B(t))$, which is known as *magnetic relaxation*.

Contact structures and magnetic relaxation.

The second equation $\frac{\partial B}{\partial t} = \nabla \times (v \times B)$ can also be written as follows

$$\frac{d}{dt}B + [v, B] = 0, \qquad \mathsf{div}(B) = \mathsf{div}(v) = 0.$$

As a consequence the evolution of $B_0 = B(0)$ occurs along a path $t \longrightarrow g(t) \in \mathsf{Diff}_0^{\mathsf{vol}}(M)$:

$$B(t) = g_*(t)B_0, \qquad \frac{d}{dt}g(t) = v$$

Therefore, candidates the stationary points (a.k.a. steady Euler flows) are minimizers to the problem

$$\text{Extremize} \quad E = \int_M |B|^2,$$
$$\text{on} \quad \Psi_{B_0} = \{B : B = g_*(B_0), g \in \mathsf{Diff}_0^{\mathsf{vol}}(M)\}.$$

What happens at infinity ...

...is not well understood.

Question

When do the minimizers exist?

Problem(s)

Existence results of generally nonsmooth minimizers are known only in dimension 2 [Burton & Alvion, Trombetti, Lions]. For dimension greater than 2, there is no proof that extremals exist except for certain partial results.

In even dimensions there are nonexistence examples for the smooth extremals [Ginzburg & Khesin] .

Open problem

Show nonexistence of smooth minimizers in dimension 3 for certain initial conditions.

Energy minimization

$$\text{Minimize} \quad E = \int_M |B|^2,$$

$$\text{on} \quad \Psi_{B_0} = \{B : B = g_*(B_0), g \in \mathsf{Diff}_0^{\mathsf{vol}}(M)\}.$$

Theorem [Arnold]

The "critical points" (i.e. extremals) of the above problem are divergence-free vector fields B which commute with their curls i.e. satisfy:

$$[B, \nabla \times B] = 0 \ .$$

Comment: Curl eigenfields naturally satisfy this condition.

...but there is a closer connection to curl eigenfields

There is a well known quantity (the only such known for general vector fields!) associated with $B(t)$ which stays invariant in time, known as helicity

$$\mathsf{H}(B) \;=\; \int_M \langle B, A \rangle, \quad \nabla \times A = B \;.$$

i.e. $\mathsf{H}(B) = \mathsf{H}(g_* B)$ for every $g \in \mathsf{Diff}_0^{\mathsf{vol}}(M)$.
Thus, in the context of magnetic relaxation it makes perfect sense to consider a constrained problem:

$$\text{Minimize } E := E_2(B) \text{ on } \Phi_{B_0} \text{ subject to } \mathsf{H} := \mathsf{H}(B) = \text{const.}$$

Critical points of the constrained problem.

The method of Lagrange multipliers tells us that extremals B satisfy

$$\delta E(B,h) - \lambda\, \delta \mathsf{H}(B,h) = 0, \qquad \text{for all} \quad h.$$

where $\delta(*)(B,h) := \frac{d}{dt}(*)(B + t\,h)|_{t=0}$ where h is traditionally denoted by δB. Thus $\delta(*)$ is just a directional derivative at B in the direction of δB. Calculate to obtain

$$\begin{aligned}\delta E(B,h) &= \frac{d}{dt}\int \langle B + t\,h, B + t\,h\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\int \langle B, B\rangle + 2\,t\langle h, B\rangle + t^2\langle h, h\rangle\Big|_{t=0} \\ &= 2\int \langle B, \delta B\rangle = 2\int \langle B, \nabla\times(\delta A)\rangle\end{aligned}$$

Analogously, for $\mathsf{H}(B) = \int \langle B, A\rangle$, $B = \nabla \times A$:

$$\delta \mathsf{H}(B,h) = \int \big(\langle A, \nabla\times(\delta A)\rangle + \langle B, \delta A\rangle\big).$$

Critical points of the constrained problem.

The Lagrange equations now read

$$\int \langle B, \nabla \times (\delta A)\rangle - \lambda \int \big(\langle A, \nabla \times (\delta A)\rangle + \langle B, \delta A\rangle\big) = 0 \ .$$

Applying the standard calculus identity:

$$\mathsf{div}(X \times Y) = \langle Y, \nabla \times X\rangle - \langle X, \nabla \times Y\rangle$$

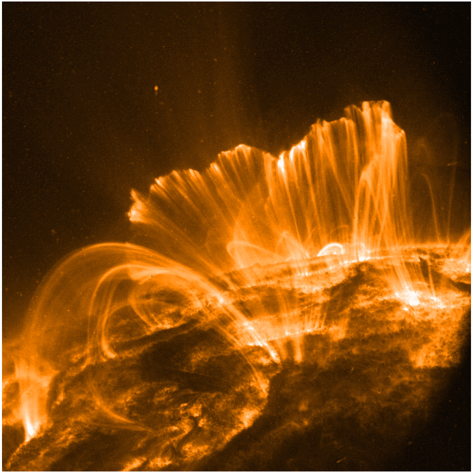
yields: $0 = \int \mathsf{div}(B \times \delta A) = \int \langle \delta A, \nabla \times B\rangle - \int \langle B, \nabla \times (\delta A)\rangle$,

$$\int \langle \delta A, \nabla \times B - \lambda\, B\rangle = 0, \qquad \text{for all} \quad \delta A \ .$$

thus any smooth critical point B satisfies

$$\nabla \times B = \lambda\, B \ .$$

Conclusion: λ becomes the eigenvalue μ and B is the curl eigenfield and defines a contact structure on the whole domain whenever B is nonvanishing everywhere. In particular B minimizes $E_2(B)$ whenever $\mu = \mu_1$ is the first eigenvalue of the curl operator $\nabla\times$.



Contact structures in fluid dynamics

Important physics question

Are the curl eigenfields stable critical points ? Physicists suggest that the only stable critical point is the principal curl eigenfield.

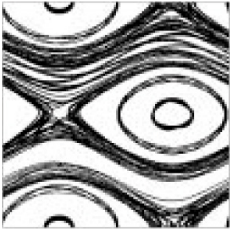
Theorem [Etnyre & Ghrist]

The curl eigenfield defined by an overtwisted contact structure is linearly unstable (i.e. an unstable critical point of the linearized Euler equations).

Principal curl eigenfields (ABC-fields)

- ABC-fields on $T^3 \cong S^1 \times S^1 \times S^1$,

$$\dot{x} = A\sin(z) + C\cos(x),$$
$$\dot{y} = B\sin(y) + A\cos(z),$$
$$\dot{z} = C\sin(x) + B\cos(y).$$

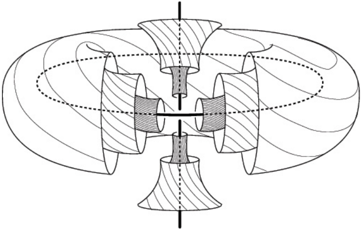


(author: Ghrist)

Principal curl eigenfields (Hopf fields)

- Hopf fields on S^3,

$$\dot{x} = -y;\ \dot{z} = -w$$
$$\dot{y} = x;\ \dot{w} = z.$$



(author: Ghrist)

Contact structures in fluid dynamics

Fact: Both the ABC-fields and the Hopf fields are tight as contact structures.

Question

Are the curl eigenfields B defined by the *tight* contact structures always energy minimizing?

Conjecture [Etnyre & Ghrist]

The μ_1-curl-eigenfields always define *tight* contact structures.

. . .

. . . but the truth is that this motivation came after some calculations . . .

How to get nonvanishing curl-eigenfields?

. . . or even just curl-eigenfields?

$$*\, d\,\alpha = \mu\,\alpha \quad \text{or} \quad \nabla \times B = \mu\, B,$$

(for $\alpha = \langle B, \cdot \rangle$).

- $\delta\alpha = 0$, i.e. α is "divergence free".
- $\Delta^1|_{\mathcal{H}} = d\delta + \delta d = (*d)^2$ on $\mathcal{H} = \{\beta \in \Omega^1(M); \delta\,\beta = 0\}$

$$\Delta^1\alpha = \delta\, d\,\alpha = *\, d * d\alpha = \mu^2\,\alpha,$$

- For the converse define:

$$\beta_\pm = \mu\,\alpha \pm *\, d\alpha,$$

  where α a co-closed eigenform of Δ^1.
-

$$*\, d\beta_\pm = \mu * d\alpha \pm \Delta^1\alpha = \mu * d\alpha \pm \mu^2\alpha = \pm\mu\,\beta_\pm.$$

...it is never bad to calculate some simple examples.

Starting from the simplest eigenform: $\alpha = f\,dt$, on $S^1 \times \Sigma$ where $\Delta^0 f = \mu^2 f$, where f is an eigenfunction on the surface we get $\delta\alpha = 0$ and an eigenfield of the curl:

$$\beta = \mu\, f \pm *d(f\,dt) = \mu f\,dt \pm (f_x\,dy - f_y\,dx)\ .$$

β defines a contact structure whenever $\beta \neq 0$ or equivalently

$$\{f = 0\} \cap \{\nabla f = 0\} = \emptyset$$

Curl eigenfields on $P = (S^1 \times \Sigma, 1 \oplus g_\Sigma)$

- On $S^1 \times \Sigma$ for any $\alpha \in \Omega^1(P)$:

$$\alpha = f(t,x)\,dt + \beta(t,x), \qquad (t,x) \in S^1 \times \Sigma$$

- the Hodge Laplacian on P is given as

$$\Delta^1_P \alpha = (-\mathcal{L}^2_{\partial_t} f + \Delta^0_\Sigma f)dt + (-\mathcal{L}^2_{\partial_t}\,\beta + \Delta^1_\Sigma\,\beta).$$

- Δ^1_P respects the decomposition of $\mathcal{H} = \{\alpha \in \Omega^1(P); \delta\,\alpha = 0\}$

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp, \quad \mathcal{H}_1 = \{\alpha : \alpha = f\,dt, \delta\alpha = 0\}.$$

Curl eigenfields on $P = (S^1 \times \Sigma, 1 \oplus g_\Sigma)$

- Eigenforms:

$$\mathcal{H}_1 : \ \alpha = f\,dt \quad \Delta^0_\Sigma f = \lambda f,\ f \in C^\infty(\Sigma)$$

$$\mathcal{H}_1^\perp : \ \Big(a \sin(\frac{2\pi n t}{l}) + b \cos(\frac{2\pi n t}{l})\Big)\,\beta, \qquad \Delta^1_\Sigma \beta = \nu\beta,\ \beta \in \Omega^1(\Sigma)$$

  where $l = \text{length}(S^1)$ in $S^1 \times \Sigma$.
- Eigenvalues of Δ^1_P: $\{\lambda_j, \gamma_k\}$; $\gamma_k = \nu_m + (\frac{2\pi n}{l})^2$:

$$\begin{aligned} \lambda_j &: \quad \text{eigenvalues of } \Delta^0_\Sigma, \\ \nu_m &: \quad \text{eigenvalues of } \Delta^1_\Sigma, \\ (\frac{2\pi n}{l})^2 &: \quad \text{eigenvalues of } -\mathcal{L}^2_{\partial_t}. \end{aligned}$$

- with the Hodge star L^2-isometry one shows: $\lambda_j = \nu_j$

Curl eigenfields on $P = (S^1 \times \Sigma, 1 \oplus g_\Sigma)$

- The first eigenvalue μ_1 of the curl operator $* \, d$ on P satisfies,

$$\mu_1^2 = \min\left\{\lambda_1, \left(\frac{2\pi}{l}\right)^2\right\}.$$

- For small l: μ_1-curl eigenfield α is S^1-invariant:

$$\alpha = f(x)\, dt + \beta(x) \quad f \in C^\infty(\Sigma),\ \beta \in \Omega^1(\Sigma).$$

where: $\beta = *_\Sigma \, d\, f$, $\Delta^0_\Sigma f = \mu_1^2\, f$, $\mu_1^2 = \lambda_1$.

Curl eigenfields on $P = (S^1 \times \Sigma, 1 \oplus g_\Sigma)$

- $\alpha(p) = 0$ iff $f(p) = 0$ and $\nabla f(p) = 0$, i.e. $\Gamma = f^{-1}(0)$ is singular.
- if $\Gamma = f^{-1}(0)$ is nonsingular then $\xi = \ker \alpha$ defines a contact structure on P.

Observation: The vertical vector field $\frac{\partial}{\partial t}$ is a contact vector field !

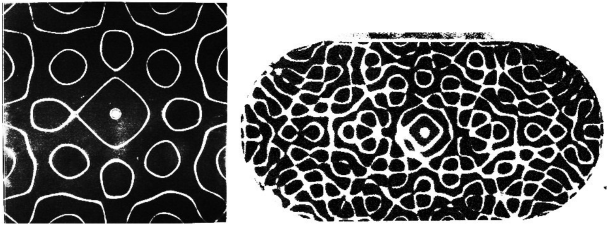
How to detect tight/overtwisted?

[Giroux's criteria for $S^1 \times \Sigma$]: The contact vector field $\frac{\partial}{\partial t}$ define $\Gamma = \Gamma_{\frac{\partial}{\partial t}} \cap \Sigma = f^{-1}(0)$. Then:

(i) if $\Sigma \neq S^2$, then ξ has a tight tubular neighborhood iff. none of the components in Σ/Γ bounds a disc.

(ii) if $\Sigma = S^2$, then ξ has a tight tubular neighborhood iff. Γ is connected.

Observation: The vertical vector field $\frac{\partial}{\partial t}$ is a contact vector field, and the dividing set Γ is the same as the *nodal set* i.e. the zero set of f!



(downloaded from: http://www.physics.utoronto.ca/$\sim$nonlin/chladni.html)

Therefore, if you construct an eigenfunction f with a contractible nodal curve the curl eigenfield, constructed above, will be *overtwisted* and *tight* it the nodal set does not have such a curve.

Questions...

Open problem $\#45$, stated by Schoen and Yau in *Lectures on Differential Geometry*:

- A. Melas proved that the nodal line of any second eigenfunction cannot enclose a compact subregion of a bounded convex domain in $\mathbb{R}^2$. *This is an open problem for general domains in* $\mathbb{R}^2$ *known as the Payne's conjecture (1967).*
- Is there a similar conclusion for higher dimensional Euclidean space?
- To what extend do these conclusions hold for compact manifolds with boundary?
- What is the topology of nodal sets of higher eigenvalues? For example, can one find an infinite sequence of eigenfunctions, which domains are disjoint union of cells?

Overtwisted principal curl eigenfields (a counterexample to the conjecture)...

Theorem

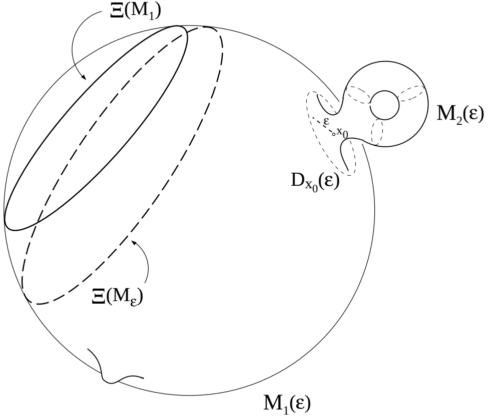
Overtwisted principal curl-eigenfields exist on products $(S^1 \times \Sigma, 1 \oplus g_\Sigma)$ for a carefully chosen Riemannian metric on g_Σ.

Ideas behind the proof:
Recall that for small l: μ_1-curl eigenfield α is S^1-invariant:

$$\alpha = f(x)\, dt + \beta(x) \quad f \in C^\infty(\Sigma),\ \beta \in \Omega^1(\Sigma).$$

and the dividing set is $\Gamma = f^{-1}(0)$. Thus is suffices to "produce" an eigenfunction which has a contractible circle in its nodal set.

The idea behind the proof...



Contact structures in fluid dynamics

Theorem (Ghrist & ______)

The $\mathcal{E}$-curl eigenfield $\eta = \langle X, \cdot \rangle$ by the vertical unit Killing vector field X defined on a principal S^1-bundle: $\pi : (P, g_P) \mapsto (\Sigma, g_\Sigma)$ with constant length fibers is a tight energy minimizer if

$$\mathcal{E}^2 < \min\left(\frac{\nu_1}{3}, \frac{4\pi^2}{l^2}\right)$$

where ν_1 is the first nonzero eigenvalue of the scalar Laplacian on Σ, and l the length of the fiber.

Yet another interesting question ...

Problem

Provide a geometric characterization of *tight* (*overtwisted*) contact structures. Namely, indicate sufficient conditions (such as *injectivity radius, curvature, eigenvalues,...*) for an adapted Riemannian metric g to the contact form α which imply that $\xi = \ker \alpha$ is tight (overtwisted).

Geometric characterization of Γ_X

- (M, g, α) a Riemannian 3-manifold, α a contact form, g is adapted to α, i.e. $*\, d\alpha = \mu\, \alpha$,

- X a global nonsingular vector field on M such that $\mathcal{L}_X \alpha = 0$,

Geometric characterization of Γ_X

Theorem (_____)

- *Then the contact hamiltonian* $f = \alpha(X) \in C^\infty(M)$ *satisfies the following sub-elliptic equation;*

$$\Delta_E f - \langle \nabla \ln h, \nabla f \rangle + \mu(\mathcal{E} - \mu) f = 0$$

where $\mathcal{E} = (*d\,\eta_1)(e_1)$, $\eta_1 = g(e_1, \cdot\,)$, $h = 1/(\mu\|X\|)$, *and* Δ_E *is the sub-Laplacian on* $E = \ker \eta_1$, $e_1 = X/\|X\|$.

Geometric tightness theorem.

Theorem (______)

- X *is a unit Killing field* $\|X\| = 1$, $\mathcal{L}_X g = 0$, *such that* $\mathcal{L}_X \alpha = 0$,
- X *has circular orbits and* $l_{\min}$ *is a lower bound for length of the orbits.*

(i) $\mathcal{E} = *\, d\eta(X) =$ *const,* $\mu =$ *const,* $\mathcal{E} \leq \mu$, *where* $\eta = \langle X, \cdot \rangle$;

(ii) *the sectional curvature* κ_E *of planes* $E = \ker\, \eta$, *satisfies:* $\kappa_E \leq -\frac{3}{4}\, \mathcal{E}^2$;

(iii) *for a constant* C_M *which depends only on* M *we have*

$$\frac{4\pi l_{min}}{\mu(\mu - \mathcal{E})} > C_M \, \mathrm{Vol}(M).$$

Then α *defines a universally tight contact structure on* M.

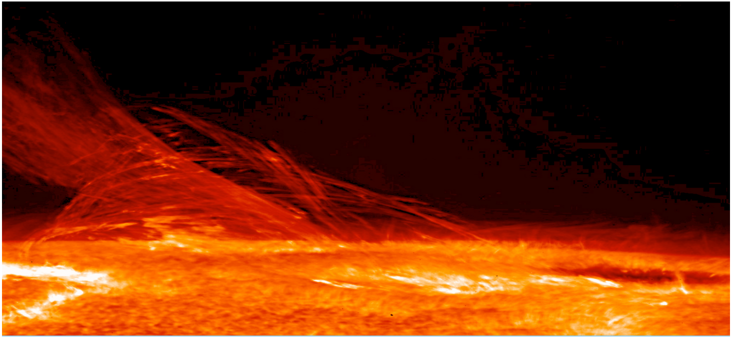
Special thanks to Margaret Symington here for a lot of help.

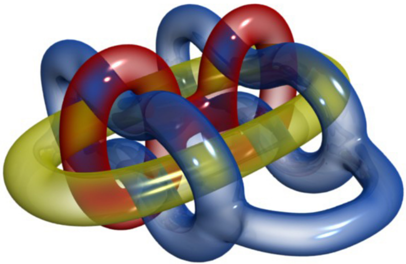
... When you go for a "postdoc" people expect you to detach from your advisor and your thesis, do something new and develop your own research program.

.. *so?*

Arnold and Khesin in "Topological Methods in Hydrodynamics" have proposed:

- *The higher helicity problem: ... The dream is to define such hierarchy of invariants for generic vector fields* B *such that, whereas all the invariants of order* $\leq k$ *have zero value for a given field and there exists a nonzero invariant of order* $k+1$*, this nonzero invariant provides a lower bound for the field energy*.



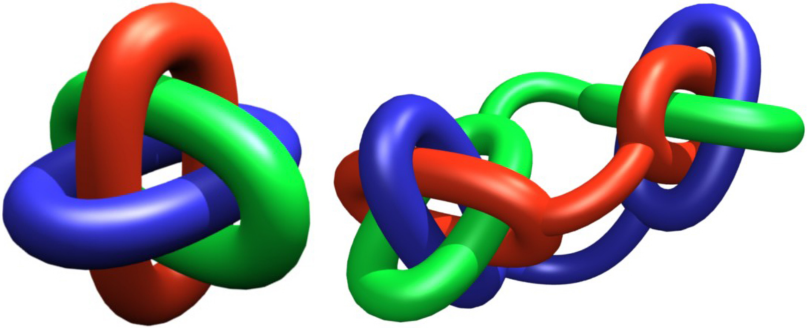


Theorem (___________)

Let $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \times \mathcal{T}_3$ *be a product of three unlinked handlebodies in* S^3*, the following limit (asymptotic* $\bar{\mu}_{123}$*-invariant of orbits) exists for almost all* $(x, y, z) \in \mathcal{T}$*:*

$$\bar{m}_B(x, y, z) = \lim_{T\to\infty} \frac{1}{T^3} \bar{\mu}_{123}\big(\bar{\mathcal{O}}_T^{X_1}(x), \bar{\mathcal{O}}_T^{X_2}(y), \bar{\mathcal{O}}_T^{X_3}(y)\big).$$

Moreover, the new helicity invariant can be defined as
$\mathsf{H}_{123}(B; \mathcal{T}) = \int_{\mathcal{T}} \bar{m}_B(x, y, z)\, \mu(x) \wedge \mu(y) \wedge \mu(z).$



... We mathematicians are measured in the sup norm not the L^2 *norm.*

... The impression of a hiring committee with amount of publications only lasts for about 1min, because the next question they usually ask is "what is the best result there..." .

... The way human brain works (when doing mathematics) is that you need to articulate your thoughts, even if something sounds like a complete nonsense articulating is the only way for me to really understand it..." .

... You must give good talks. People will never remember what this is you are doing, but they will always remember that you have given a good talk .

The End.