ZEROS OF HERMITE-TYPE POLYNOMIALS

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Let $f(z)\frac{d^n}{dz^n}(1/f(z)) = P_n(z)$ be a monic polynomial having n distinct real roots, for $n = 0, 1, \ldots$. Assume also that $f(\pm\infty) = \infty$. Note that $f(z)$ is an *entire function of finite type.*

Claim: For q and m nonnegative integers,

$$P(z) = \sum_{\gamma} \binom{q}{\gamma} \frac{m!}{(m-\gamma)!} z^{m-\gamma} P_{q-\gamma}(z), \tag{1}$$

has $q + m$ distinct real roots if $q > m$, and $2q$ distinct nonzero real roots if $q \leq m$.

Remark: This generalizes Problem 5681 of the AMM [P], where this was proposed for $P_n(z) = H_n(z)$, the Hermite polynomials.

Proof: After rewriting the sum in (1),

$$P(z) = f(z)\frac{d^q}{dz^q}(z^m/f(z)),$$

using $z^m/f(z) \to 0$ when $z \to \pm\infty$, Rolle's theorem and induction on q, the claim follows. $\square$

Reference:

[**P**] P 5681, *American Mathematical Monthly,* **(72) # 6,** 1969.