A WZ PROOF OF A DETERMINANT EVALUATION CONJECTURED BY KUPERBERG AND PROPP

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The determinant evaluation:

$$det\left[\binom{i+j}{i}\binom{2n-i-j}{n-i}_{0\le i,j\le n}\right]=\prod_{i=0}^{n}\binom{2n+1}{i}=\frac{(2n+1)!^{n+1}}{(2n+1)!!}\tag{1}$$

was conjectured by G. Kuperberg and J. Propp [P]. Here we present a WZ-style proof of (1) (for WZ methods the reader is highly encouraged to refer to [WZ], [Z]).

Using row operations we transform the above matrix to a lower-right-triangular matrix T. To this end we need only justify the identities:

for $i=0,\dots,n-s-1$ and $s=0,\dots,n$:

$$\sum_{j=0}^{n-s}(-1)^j\binom{i+j+s}{j+s}\binom{2n-i-j-s}{n-j-s}\binom{n+1}{j}=0,\tag{1a}$$

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+j+s}{j+s}\binom{n-j}{s}\binom{n+1}{j}=0.\tag{1b}$$

Introduce the functions

$$F:=(-1)^{j+1}\binom{n+j+s-k}{j+s}\binom{n+k-j-s}{n-j-s}\binom{n+1}{j},\qquad F_1:=\frac{(n+s+1)!n!(n-s)!}{(n-k+s)!(2n+1)!}F.$$

Then, the wonderful package **EKHAD** accompanying [PWZ] applied to F_1 produces:

$G:=-j(j+s)(n+k-j-s+1)F_1Q$ where

$$Q=\frac{n[21n+29+5n^2-3kn-9k]+13-7k-j[3n^2+8n-2kn+5-3k]+s[kn-jn-sn-s+k-j]}{2(n+2-j)(n+1-j-s)(2n+3)(n-k+1)(n-k+s+1)(n+1)}$$

such that

$$F_1(n+1,j,k,s)+F_1(n,j,k,s)=G(n,j+1,k,s)-G(n,j,k,s).\tag{2}$$

Now, summing over the index j in (2) combined with the initial conditions for n=0 verify (1a) and (1b). But the terms in the product (1) are precisely the elements in the principal diagonal of T. This completes the proof. □

Let $M_n := [\binom{i+j}{i}\binom{2n-i-j}{n-i}_{0\le i,j\le n}]$ and define the matrices:

$$M_{n,m}(a,b) := \left[\binom{i+j+a+b}{i+a}\binom{2n-i-j-a-b}{n-i-a}_{0\le i,j\le m}\right].$$

Remark 1: Noting that $\mathrm{M}_n = M_{n,n}(0,0)$, the above reduction process is easily summed up in

$$det M_n = det M_{n,n}(0,0) = det M_{n,n}(1,0) = \cdots = det M_{n,n}(n+1,0).$$

Remark 2: It may also be interesting to see that

$$det M_{n,n}(a,b) = (-1)^{ab}\frac{(2n+1)!^{n+1}}{(2n+1)!!}\frac{(a+b)!^2}{a!^2b!^2}\frac{n!!^2a!!^2b!!^2(n+a+b)!!(n-a-b)!!}{(a+b)!!^2(n-a)!!(n-b)!!(n+a)!!(n+b)!!}.$$

Remark 3: Similarly, one can show that

$$det\left[\binom{i+j}{i}\binom{n+m+1-i-j}{n-i}_{0\le i,j\le n}\right] = \frac{(n+m+1)!^{n+1}m!!}{(n+m+1)!!n!!}.$$

References

[P] J. Propp *E-mail message to D. Zeilberger*, dated 1 July 1996, 15:47:56 (EDT).

[PWZ] M. Petkovšek, H.S. Wilf, D.Zeilberger, *"A=B"*, A.K. Peters Ltd., 1996.
`The package EKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

[WZ] H.S. Wilf, D. Zeilberger, *Rational functions certify combinatorial identities*, Jour. Amer. Math. Soc. **3** (1990), 147-158.

[Z] D. Zeilberger, *Closed Form (pun intended!)*, Contemporary Mathematics **143** (1993), 579-607.