THE 'UNCENTERED' MAXIMAL FUNCTION AND L^p-BOUNDEDNESS

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Abstract. In this note, we prove that the 'uncentered' Maximal function is not a bounded operator from $L^p(d\mu)$ to $L^p(d\mu)$, when $d\mu$ is the Gaussian measure on R^n. In fact, it is not even weak-type (p,p) for any dimension n.

Let $\xi := (x,y)$, where $x \in R^n$ and y is real. Now, introduce the "uncentered" weighted-Maximal function

$$M_\mu f(\xi) := sup_B \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where the *supremum* runs over *all* euclidean balls B *containing* ξ, and the integral is taken with respect to the Gaussian measure

$$d\mu(\xi) := e^{-|\xi|^2/2} d\xi.$$

It has long been known (for an excellent exposition, see [S1] or [S2], and references therein) that if one replaces $d\mu$ by the Lebesgue measure dm, then M_μ is L^p-bounded, i.e.

$$||M_\mu f||_p \leq c_p ||f||_p, \qquad p > 1,$$

and weakly bounded for $p = 1$. Our aim here is to prove that such results fail miserably in the case of the Gaussian measure, and this will be the content of the

Theorem: M_μ is not weak-type (p,p), for any $p \geq 1$.

Proof: Take λ to be a unit mass at $(0, a+1)$, for $a > 0$ large. Let

$$E_\alpha := \{\xi : M_\mu \lambda(\xi) > \alpha\}.$$

Consider the finite cylinder

$$A := \{\xi : |x| < 1, a < y < a+2\},$$

and the unit ball $B_s := B_1((s, a+1))$, with $s \in R^n, |s| < 1$. Notice that $(0, a+1) \in B_s$.

Let $\wp$ be the paraboloid given by

$$\wp := \{\xi : y > a + \frac{|x-s|^2}{2}\},$$

and let $G := \wp \cap \{\xi : |x-s| < 1\}$.

Then, since

$$\int_x^\infty e^{-t^2/2} dt \leq \frac{1}{x} e^{-x^2/2}, \qquad x \gg 1$$

we have the following estimates

$$\begin{aligned}
\mu(B_s) = \int_{B_s} d\mu \leq \int_G d\mu &\leq \int_{|x|\leq 1} \int_{a+\frac{|x|^2}{2}}^\infty e^{-y^2/2} dy \\
&\leq \frac{c}{a} \int_{|x|\leq 1} e^{-\frac{1}{2}(a+\frac{|x|^2}{2})^2} dx \\
&\leq \frac{c}{a} e^{-a^2/2} \int_{|x|\leq 1} e^{-\frac{a|x|^2}{2}} dx \\
&\leq \frac{C_n}{a} \frac{1}{\sqrt{a^n}} e^{-a^2/2},
\end{aligned}$$

for some dimensional constant C_n.

This immediately implies that

$$M_\mu \lambda \geq \frac{C_n}{a} \frac{1}{\sqrt{a^n}} e^{-a^2/2}, \qquad in \qquad A = \{\xi : |x| < 1, a < y < a+2\}.$$

On the other hand, it is easy to see that

$$\mu(A) = \int_{|x|\leq 1} \int_a^{a+2} e^{-|x|^2/2} e^{-y^2/2} dy dx \geq \frac{C_n}{a} \frac{1}{\sqrt{a^n}} e^{-a^2/2}.$$

Now choosing $\alpha = C_n a \sqrt{a^n} e^{a^2/2}$, we get a lower bound on the distribution function (w.r.t. μ) of λ,

$$\mu(E_\alpha) \geq \mu(A) \geq \frac{C_n}{a} \frac{1}{\sqrt{a^n}} e^{-a^2/2}.$$

Suppose M_μ is weak-type (p,p) for some $p \geq 1$, then from the above estimates we gather that

$$\frac{C_n}{a} e^{-a^2/2} \leq C(p,n) \left(\frac{1}{a} \frac{1}{\sqrt{a^n}} e^{-a^2/2} \right)^p,$$

for some constant $C(p,n)$ depending only on p and n. The last estimate can be expressed as

$$\left(a e^{a^2/2} \right)^{p-1} \leq C_1(p,n) \frac{1}{\sqrt{a^{np}}}.$$

However, this inequality is absurd as soon as a is *large* enough! Whence the assertion of the theorem holds. $\square$

References

[S1] E. Stein, *Singular Integrals and Differentiablity Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[S2] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.