RECURSIVE FORMULAS FOR MACMAHON AND RAMANUJAN q-SERIES

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Dedicated to George Andrews and Bruce Berndt on their 1010101*-th birthdays*

Abstract. In the present work, we extend current research in a nearly-forgotten but newly revived topic, initiated by P. A. MacMahon, on a generalized notion which relates the divisor sums to the theory of integer partitions and two infinite families of q-series by Ramanujan. Our main emphasis will be on explicit representations for a variety of q-series, studied primarily by MacMahon and Ramanujan, with an eye towards their modular properties and their proper place in the ring of quasimodular forms of level one and level two.

1. Introduction and Statement of Results

The classical sequence of Eisenstein series are defined as (for example, see Chapter 1 of [16])

$$E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{1.1}$$

where B_k is the k-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$ is the power-sum of divisors of n.

In the present work, we chose to follow up on Ramanujan's predictions regarding two functions [18, page 369] that he himself defined. Namely that

$$U_{2t}(q) = \frac{1^{2t+1} - 3^{2t+1}q + 5^{2t+1}q^3 - 7^{2t+1}q^6 + 9^{2n+1}q^{10} - \cdots}{1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots} \qquad \text{and}$$
$$V_{2t}(q) = \frac{1^{2t} - 5^{2t}q - 7^{2t}q^2 + 11^{2t}q^5 + 13^{2t}q^7 - \cdots}{1 - q - q^2 + q^5 + q^7 - \cdots}.$$

After listing the first few expansions $U_0 = V_0 = 1, U_2 = V_2 = E_2$,

$$\begin{aligned}
U_4 &= \frac{1}{3}(5E_2^2 - 2E_4), & V_4 &= 3E_2^2 - 2E_4, \\
U_6 &= \frac{1}{9}(35E_2^3 - 42E_2E_4 + 16E_6), & V_6 &= 15E_2^2 - 30E_2E_4 + 16E_6, \\
U_8 &= \frac{1}{3}(35E_2^4 - 84E_2^2E_4 + 64E_2E_6 - 12E_4^2), & V_8 &= 105E_2^4 - 420E_2^2E_4 + 448E_2E_6 - 132E_4^2.
\end{aligned}$$

Key words and phrases. MacMahon's q-series, recurrence formula, quasimodular forms.
2020 *Mathematics Subject Classification.* 11F03, 05A17, 11M36.

Ramanujan declared that:

"In general U_{2t} and V_{2t} are of the form $\sum K_{\ell,m,n}\, E_2^{\ell} E_4^m E_6^n$, where $\ell + 2m + 3n = t$."

In modern language Ramanujan's declaration amounts to saying that both U_{2t} and V_{2t} are quasimodular forms of weight $2t$. Berndt, Chan, Liu, Yee, and Yesilyurt [11, 12] proved this claim using Ramanujan's identities [19]

$$D(E_2) = \frac{E_2^2 - E_4}{12}, \quad D(E_4) = \frac{E_2E_4 - E_6}{3}, \quad \text{and} \quad D(E_6) = \frac{E_2E_6 - E_4^2}{2}, \tag{1.2}$$

where $D := q\frac{d}{dq}$. However, their results are not explicit. Indeed, Andrews and Berndt (see p. 364 of [8]) proclaim that *"...it seems extremely difficult to find a general formula for all $K_{\ell,m,n}$."*

Ramanujan's claim is that $U_{2t}(q)$ and $V_{2t}(q)$ are weight $2t$ quasimodular forms, as the ring of quasimodular forms is the polynomial ring (for example, see [14])

$$\mathbb{C}[E_2, E_4, E_6] = \mathbb{C}[E_2, E_4, E_6, E_8, E_{10}, \ldots].$$

The first and the third authors together with K. Ono [3] chose the latter $\mathbb{C}[E_2, E_4, E_6, E_8, E_{10}, \ldots]$, involving all Eisenstein series expressed as "traces of partition Eisenstein series," to produce the first explicit formulas for both $U_{2t}(q)$ and $V_{2t}(q)$.

Our goal in this paper is to obtain recursive formulas in the ring $\mathbb{C}[E_2, E_4, E_6]$ just as Ramanujan originally proposed. One can argue that the main merit of our effort here lies in inviting the audience to a variety to the techniques employed for the present goal, that the authors believe should help in similar circumstances. The first of such installments appeared in [2] for the q-series $U_{2t}(q)$:

Theorem 1.1. *If t is a non-negative integer, then we have that*

$$U_{2t}(q) = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+2\beta+3\gamma=t}} c_u(\alpha, \beta, \gamma) E_2(q)^{\alpha} E_4(q)^{\beta} E_6(q)^{\gamma}$$

where the coefficients $c_u(\alpha, \beta, \gamma)$ are rational numbers defined by [2, eq'n (1.7)].

Our first result at present concerns the other Ramanujan function $V_{2t}(q)$. We require in introductions of a triple-indexed sequence of *integers* defined by

$$\begin{aligned} c_v(\alpha, \beta, \gamma) := {} & (2\alpha + 8\beta + 12\gamma - 1) \cdot c_v(\alpha - 1, \beta, \gamma) - 2(\alpha + 1) \cdot c_v(\alpha + 1, \beta - 1, \gamma) \\ & - 8(\beta + 1) \cdot c_v(\alpha, \beta + 1, \gamma - 1) - 12(\gamma + 1) \cdot c_v(\alpha, \beta - 2, \gamma + 1), \end{aligned} \tag{1.3}$$

where $\alpha, \beta, \gamma \geq 0$. To seed the recursion, we let $c_v(0, 0, 0) := 1$, and we let $c_v(\alpha, \beta, \gamma) := 0$ if any of the arguments are negative. Here we list the "first few" values:

$$c_v(1, 0, 0) = 1,\ c_v(0, 1, 0) = -2,\ c_v(0, 0, 1) = 16,\ c_v(1, 1, 0) = -30,\ c_v(1, 0, 1) = 448, \ldots.$$

Theorem 1.2. *If t is a non-negative integer, then we have that*

$$V_{2t}(q) = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+2\beta+3\gamma=t}} c_v(\alpha, \beta, \gamma) E_2(q)^{\alpha} E_4(q)^{\beta} E_6(q)^{\gamma}$$

where the coefficients $c_v(\alpha, \beta, \gamma)$ are defined by (1.3).

In the same spirit but in an earlier paper [2], the first and the third authors together with K. Ono have found such an explicit description for MacMahon's quasimodular form

$$\mathcal{U}_{2t}(q) := \sum_{1\le k_1<\cdots<k_t} \frac{q^{k_1+\cdots+k_t}}{(1-q^{k_1})^2\cdots(1-q^{k_t})^2}. \tag{1.4}$$

An important observation is that Ramanujan's $U_{2t}(q)$ and MacMahon's $\mathcal{U}_{2t}(q)$ are directly linked to each other. In fact, this fact allows us to formulate Theorem 1.3 of [2] as follows:

$$\mathcal{U}_{2t}(q) = \sum_{a=0}^{t} w_a(t)\cdot U_{2a}(q) \qquad \text{where} \qquad w_a(t) := \frac{\binom{2t}{t}}{16^t(2t+1)} \sum_{0\le \ell_1<\cdots<\ell_a<t} \prod_{j=1}^{a} \frac{1}{(2\ell_j+1)^2}.$$

In light of this, the weight $2a$ part of $\mathcal{U}_{2t}(q)$ becomes precisely $U_{2a}(q)$ and we can evidently borrow the corresponding expansion from Theorem 1.1 above.

In a related rendition, the first author together with Andrews and Tauraso [1] introduced a q-series which is intimately connected to MacMahon's $\mathcal{U}_{2t}(q)$ and given by

$$\mathcal{U}^\star_{2t}(q) := \sum_{1\le k_1\le\cdots\le k_t} \frac{q^{k_1+\cdots+k_t}}{(1-q^{k_1})^2\cdots(1-q^{k_t})^2}. \tag{1.5}$$

In the same paper [1, Theorem 6.1], the authors have shown that each of these $\mathcal{U}^\star_{2t}(q)$ are quasimodular forms of weight at most $2t$. The first and the third authors together with K. Ono followed this through and furnished the expansion [2, Theorem 1.4]

$$\mathcal{U}^\star_{2t}(q) = \sum_{a=0}^{t} w^\star_a(t)\cdot \mathbb{E}^\star_{2a}(q) \qquad \text{where} \qquad \mathbb{E}^\star_{2a}(q) := \sum_{(1^{m_1},\ldots,a^{m_a})\vdash a} \prod_{j=1}^{a} \frac{1}{m_j!}\left(-\frac{B_{2j}\,E_{2j}(q)}{(2j)\cdot(2j!)}\right)^{m_j}$$

and for some coefficients $w^\star_a(t)$ akin to the above $w_a(t)$.

In the interest of exhibiting the tight link between the two q-series, $\mathcal{U}_{2t}(q)$ and $\mathcal{U}^\star_{2t}(q)$, we brought to bear the following relation which expresses one in terms of the other:

$$\mathcal{U}^\star_{2t}(q) = (-1)^t \sum_{(1^{m_1},\cdots,t^{m_t})\vdash t} (-1)^{m_1+\cdots+m_t}\binom{m_1+\cdots+m_t}{m_1,\ldots,m_t}\prod_{k=1}^{t}\left(\mathcal{U}_{2k}(q)\right)^{m_k}.$$

The proof relies on a well-known convolution between the elementary symmetric functions $\mathbf{e}_t$ and the complete homogeneous symmetric functions $\mathbf{h}_t$, effectively utilized in [1, Lemma 6.1]:

$$\sum_{i=0}^{t}(-1)^i\mathbf{e}_i\mathbf{h}_{t-i} = 0.$$

In addition to the $\mathcal{U}_{2t}(q)$, MacMahon also introduced [15] the q-series

$$\mathcal{C}_{2t}(q) = \sum_{n=1}^{\infty}\mathfrak{m}_{\text{odd}}(t;n)q^n := \sum_{0<s_1<s_2<\cdots<s_t} \frac{q^{2s_1+2s_2+\cdots+2s_t-t}}{(1-q^{2s_1-1})^2(1-q^{2s_2-1})^2\cdots(1-q^{2s_t-1})^2}. \tag{1.6}$$

The number $\mathfrak{m}_{\text{odd}}(t;n)$ is a sum of the products of the *part multiplicities* for partitions of n with t distinct odd part sizes. Furthermore, in analogy with the work of Andrews and Rose [9, 20], Bachmann [10] proved that each $\mathcal{C}_{2t}(q)$ is a finite linear combination of quasimodular forms on $\Gamma_0(2)$ of weight at most $2t$. In recent years, the literature saw a flurry of research activities related to the present context, see [4, 5, 6, 17].

Here we give yet another explicit formula for $\mathcal{C}_{2t}(q)$ analogous to the MacMahon's $\mathcal{U}_{2t}(q)$ series as described in [2, Theorem 1.3]. We require some preliminary concepts and terminologies. Let's recall the Jacobi theta series

$$\theta_2(q) = \sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{m^2}, \qquad \theta_3(q) = \sum_{m\in\mathbb{Z}} q^{m^2} \qquad \text{and} \qquad \theta_4(q) = \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2},$$

satisfying the identity $\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q)$ and having the derivatives

$$\frac{D(\theta_2)}{\theta_2} = \frac{E_2 - \theta_2^4 + 5\theta_3^4}{24}, \qquad \frac{D(\theta_3)}{\theta_3} = \frac{E_2 + 5\theta_2^4 - \theta_3^4}{24}, \qquad \frac{D(\theta_4)}{\theta_4} = \frac{E_2(q) - \theta_2^4 - \theta_3^4}{24}. \tag{1.7}$$

Observe that modular forms over the congruence subgroup $\Gamma_0(2)$ can be generated by $\Theta_{0,1}(q)$ and $\Theta_{1,1}(q)$ where these symmetric combinations are defined by

$$\Theta_{r,s}(q) := \theta_2^{4r}(q)\cdot\theta_3^{4s}(q) + \theta_2^{4s}(q)\cdot\theta_3^{4r}(q)$$

which carry weight $2r+2s$. Construct the sequences $c_c(\alpha,\beta,\gamma)$ defined recursively by

$$\begin{aligned} c_c(\alpha,\beta,\gamma) &= \frac{2\alpha+4\beta+4\gamma-1}{24}c_c(\alpha-1,\beta,\gamma) + \frac{20\gamma-4\beta+3}{24}c_c(\alpha,\beta-1,\gamma) \\ &\quad + \frac{20\beta-4\gamma+3}{24}c_c(\alpha,\beta,\gamma-1) - \frac{7(\alpha+1)}{6}c_c(\alpha+1,\beta-1,\gamma-1) \\ &\quad - \frac{\alpha+1}{12}c_c(\alpha+1,\beta-2,\gamma) - \frac{\alpha+1}{12}c_c(\alpha+1,\beta,\gamma-2). \end{aligned} \tag{1.8}$$

To seed the recursion, we let $c_c(\alpha,\beta,\gamma) := 0$ if any of the arguments is negative while we let $c_c(0,0,0) := 1$. We are now ready to state the content of the promised results.

Theorem 1.3. *If t is a positive integer, then we have*

$$\frac{D^t(\theta_4(q))}{\theta_4(q)} = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ 2\alpha+2\beta+2\gamma=2t}} c_c(\alpha,\beta,\gamma)\cdot E_2^{\alpha}(q)\,\Theta_{\beta,\gamma}(q)$$

where the coefficients $c_c(\alpha,\beta,\gamma)$ are defined by (1.8)*.*

Example. In view of Theorem 1.3, we calculate the following weight 4 modular form as:

$$\frac{D^2(\theta_4)}{\theta_4} = \frac{1}{192}E_2^2 - \frac{1}{96}E_2\Theta_{0,1} + \frac{1}{192}\Theta_{0,2} - \frac{11}{96}\Theta_{1,1}. \tag{1.9}$$

We require the role of the *elementary symmetric functions* given by $\mathbf{e}_0 = 1$ and

$$\mathbf{e}_k(x_1,\ldots,x_r) = \sum_{1\leq i_1<\cdots<i_k\leq r} x_{i_1}\cdots x_{i_k} \qquad \text{for } 1\leq k\leq r.$$

Theorem 1.4. *If t is a positive integer, then we have that*

$$\mathcal{C}_{2t}(q) = \sum_{k=1}^{t} (-1)^k \,\frac{\mathbf{e}_{t-k}(0^2, 1^2, \ldots, (t-1)^2)}{(2t)!} \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+\beta+\gamma=t}} c_c(\alpha,\beta,\gamma)\cdot E_2^{\alpha}(q)\,\Theta_{\beta,\gamma}(q).$$

This paper is organized as follows. In Section 2, we prove Theorem 1.2. In Section 3, we prove Theorem 1.3 and by using the theory of elementary symmetric function, we prove Theorem 1.4. In Section 4, we give the generalization of Ramanujan's derivatives (1.2) and in Section 5, we give sketch of another proof of Theorem 1.2.

Acknowledgements

The third author is grateful for the support of a Fulbright Nehru Postdoctoral Fellowship.

2. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\eta(q) := q^{\frac{1}{24}} \prod_{k\geq 1}(1-q^k)$ be the Dedekind's eta function. We calculate $V_{2t}(q) = \frac{D^t(\eta(q))}{\eta(q)}$ by inducting on t. First observe $D(\eta(q)) = \frac{1}{24}\eta(q)E_2(q)$. The functions $V_{2t}(q)$ are known (see, for example [3, 11, 12]) to be quasimodular forms of weight $2t$, over $\mathrm{SL}_2(\mathbb{Z})$, thus we are ensured that there exist numbers $\widetilde{c}_v(\alpha,\beta,\gamma)$ for which

$$\frac{D^t(\eta(q))}{\eta(q)} = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} \widetilde{c}_v(\alpha,\beta,\gamma)\cdot E_2^{\alpha}(q)E_4^{\beta}(q)E_6^{\gamma}(q).$$

Writing $\widetilde{c}_v$ instead of $\widetilde{c}_v(\alpha,\beta,\gamma)$, one more derivative $D = q\frac{d}{dq}$ turns the last equation into

$$D^{t+1}(\eta) = D(\eta)\cdot\left(\sum_{\alpha,\beta,\gamma} \widetilde{c}_v\cdot E_2^{\alpha}E_4^{\beta}E_6^{\gamma}\right) + \eta\cdot\sum_{\alpha,\beta,\gamma}\widetilde{c}_v\cdot D(E_2^{\alpha}E_4^{\beta}E_6^{\gamma}).$$

On the other hand, Ramanujan's identities (1.2) imply that

$$\begin{aligned} D(E_2^{\alpha}E_4^{\beta}E_6^{\gamma}) = \left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}\right)E_2^{\alpha+1}E_4^{\beta}E_6^{\gamma} - \frac{\alpha}{12}E_2^{\alpha-1}E_4^{\beta+1}E_6^{\gamma}\\ -\frac{\beta}{3}E_2^{\alpha}E_4^{\beta-1}E_6^{\gamma+1} - \frac{\gamma}{2}E_2^{\alpha}E_4^{\beta+2}E_6^{\gamma-1}.\end{aligned}$$

We find that the homogeneous weight $2t+2$ form $D^{t+1}(\eta)$ satisfies

$$\begin{aligned}\frac{D^{t+1}(\eta)}{\eta} = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+2\beta+3\gamma=t}} \left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}+\frac{1}{24}\right)\widetilde{c}_v\cdot E_2^{\alpha+1}E_4^{\beta}E_6^{\gamma} - \sum_{\alpha,\beta,\gamma}\frac{\alpha}{12}\,\widetilde{c}_v\cdot E_2^{\alpha-1}E_4^{\beta+1}E_6^{\gamma}\\ -\sum_{\alpha,\beta,\gamma}\frac{\beta}{3}\,\widetilde{c}_v\cdot E_2^{\alpha}E_4^{\beta-1}E_6^{\gamma+1} - \sum_{\alpha,\beta,\gamma}\frac{\gamma}{2}\,\widetilde{c}_v\cdot E_2^{\alpha}E_4^{\beta+2}E_6^{\gamma-1}.\end{aligned}$$

By comparing the coefficients of $E_2^\alpha E_4^\beta E_6^\gamma$ on both sides of the equation above, we obtain the recursion (together with $\widetilde{c}_v(\alpha,\beta,\gamma)=\delta_{(0,0,0)}(\alpha,\beta,\gamma)$, a Kronecker delta)

$$\begin{aligned}\widetilde{c}_v(\alpha,\beta,\gamma) = \left(\frac{\alpha}{12}+\frac{\beta}{3}+\frac{\gamma}{2}-\frac{1}{24}\right)\widetilde{c}_v(\alpha-1,\beta,\gamma) - \frac{\alpha+1}{12}\cdot\widetilde{c}_v(\alpha+1,\beta-1,\gamma)\\ - \frac{\beta+1}{3}\cdot\widetilde{c}_v(\alpha,\beta+1,\gamma-1) - \frac{\gamma+1}{2}\cdot\widetilde{c}_v(\alpha,\beta-2,\gamma+1).\end{aligned}$$

To determine the exact weight $2t$ term, we take into account the prefactor $24^{\alpha+2\beta+3\gamma}$ so that

$$V_{2t}(q) = 24^{2t}\frac{D^t(\eta(q))}{\eta(q)} \qquad \text{and} \qquad c_v(\alpha,\beta,\gamma) := 24^{\alpha+2\beta+3\gamma}\cdot\widetilde{c}_v(\alpha,\beta,\gamma).$$

As a result, we obtain the desired

$$\begin{aligned}c_v(\alpha,\beta,\gamma) = (2\alpha+8\beta+12\gamma-1)\cdot c_v(\alpha-1,\beta,\gamma) - 2(\alpha+1)\cdot c_v(\alpha+1,\beta-1,\gamma)\\ - 8(\beta+1)\cdot c_v(\alpha,\beta+1,\gamma-1) - 12(\gamma+1)\cdot c_v(\alpha,\beta-2,\gamma+1).\end{aligned}$$

We thus conclude the constructions and the proof. □

3. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. We proceed by induction on $t\ge 1$. Since $\Theta_{0,1}=\theta_2^4+\theta_3^4$, the base case $t=1$ is recovered from (1.7) with $c_c(1,0,0)=\frac{1}{24}$ and $c_c(0,0,1)=-\frac{1}{24}$. Also, note the additional properties that

$$\Theta_{a,b}\Theta_{a',b'} = \Theta_{a+a',b+b'}+\Theta_{a+b',b+a'}, \qquad \Theta_{a,b}=\Theta_{b,a}, \qquad E_4=\Theta_{0,2}+7\Theta_{1,1}. \tag{3.1}$$

Moreover, the following relation follows from (1.7):

$$D(\Theta_{r,s}) = \frac{r+s}{6}E_2\Theta_{r,s} + \frac{5s-r}{6}\Theta_{r+1,s} + \frac{5r-s}{6}\Theta_{r,s+1}. \tag{3.2}$$

Assume the expansion of the weight $2t$ quasimodular form

$$\frac{D^t(\theta_4)}{\theta_4} = \sum_{\substack{\alpha,\beta,\gamma\ge 0\\ \alpha+\beta+\gamma=t}} c_c(\alpha,\beta,\gamma)\cdot E_2^\alpha\Theta_{\beta,\gamma}.$$

Then, we take one more derivative and apply (1.7), (3.1), (3.2) to obtain

$$\begin{aligned}
\frac{D^{t+1}(\theta_4)}{\theta_4} &= \frac{D(\theta_4)}{\theta_4}\left(\sum c_c\cdot E_2^{\alpha}\Theta_{\beta,\gamma}\right)+\sum c_c\cdot\left[\Theta_{\beta,\gamma}D(E_2^{\alpha})+E_2^{\alpha}D(\Theta_{\beta,\gamma})\right]\\
&= \left(\frac{E_2(q)-\Theta_{0,1}}{24}\right)\left(\sum c_c\cdot E_2^{\alpha}\Theta_{\beta,\gamma}\right)+\sum c_c\cdot\left[\frac{\alpha\Theta_{\beta,\gamma}E_2^{\alpha-1}(E_2^2-\Theta_{0,2}-7\Theta_{1,1})}{12}\right]\\
&+\sum c_c\cdot\left[\frac{E_2^{\alpha}\cdot[(\beta+\gamma)E_2\Theta_{\beta,\gamma}+(5\gamma-\beta)\Theta_{\beta+1,\gamma}+(5\beta-\gamma)\Theta_{\beta,\gamma+1}]}{6}\right]\\
&= \frac{1}{24}\sum c_c\cdot\left[E_2^{\alpha+1}\Theta_{\beta,\gamma}-E_2^{\alpha}\Theta_{\beta+1,\gamma}-E_2^{\alpha}\Theta_{\beta,\gamma+1}\right]\\
&+\frac{1}{12}\sum c_c\cdot\left[\alpha E_2^{\alpha+1}\Theta_{\beta,\gamma}-\alpha E_2^{\alpha-1}(\Theta_{\beta+2,\gamma}+\Theta_{\beta,\gamma+2})-14\alpha E_2^{\alpha-1}\Theta_{\beta+1,\gamma+1}\right]\\
&+\frac{1}{6}\sum c_c\cdot\left[(\beta+\gamma)E_2^{\alpha+1}\Theta_{\beta,\gamma}+(5\gamma-\beta)E_2^{\alpha}\Theta_{\beta+1,\gamma}+(5\beta-\gamma)E_2^{\alpha}\Theta_{\beta,\gamma+1}\right]\\
&= \frac{1}{24}\sum(2\alpha+4\beta+4\gamma+1)\cdot c_c\cdot E_2^{\alpha+1}\Theta_{\beta,\gamma}\\
&+\frac{1}{24}\sum(20\gamma-4\beta-1)\cdot c_c\cdot E_2^{\alpha}\Theta_{\beta+1,\gamma}\\
&+\frac{1}{24}\sum(20\beta-4\gamma-1)\cdot c_c\cdot E_2^{\alpha}\Theta_{\beta,\gamma+1}\\
&-\frac{1}{12}\sum\alpha\cdot c_c\cdot\left[E_2^{\alpha-1}\Theta_{\beta+2,\gamma}+E_2^{\alpha-1}\Theta_{\beta,\gamma+2}+14E_2^{\alpha-1}\Theta_{\beta+1,\gamma+1}\right];
\end{aligned}$$

where we wrote c_c for $c_c(\alpha,\beta,\gamma)$ assuming no confusion arises. By comparing coefficients on both sides of the last equation, one is lead to the recurrence

$$\begin{aligned}
c_c(\alpha,\beta,\gamma) &= \frac{2\alpha+4\beta+4\gamma-1}{24}c_c(\alpha-1,\beta,\gamma)+\frac{20\gamma-4\beta+3}{24}c_c(\alpha,\beta-1,\gamma)\\
&+\frac{20\beta-4\gamma+3}{24}c_c(\alpha,\beta,\gamma-1)-\frac{7(\alpha+1)}{6}c_c(\alpha+1,\beta-1,\gamma-1)\\
&-\frac{\alpha+1}{12}c_c(\alpha+1,\beta-2,\gamma)-\frac{\alpha+1}{12}c_c(\alpha+1,\beta,\gamma-2).
\end{aligned}$$

The proof is hence complete. □

Proof of Theoerem 1.4. Begin by expressing the quasimodular form $\mathcal{C}_{2t}(q)$ in the manner

$$\mathcal{C}_{2t}(q)=\sum_{k=0}^{t}(-1)^k\frac{v_t(k)}{(2t)!}\cdot\frac{D^k(\theta_4(q)}{\theta_4(q)}.$$

A direct utility of the relation (see [9, Corollary 3])

$$\mathcal{C}_{2t}(q)=\frac{1}{2t(2t-1)}\left[(2\mathcal{C}_1(q)+(t-1)^2)\mathcal{C}_{2t-2}(q)-D\mathcal{C}_{2t-2}(q)\right],$$

implies the recurrence $v_t(k) = (t-1)^2 v_{t-1}(k) + v_{t-1}(k-1)$. However, one easily checks that

$$v_t(k) = \mathbf{e}_{t-k}(0^2, 1^2, \dots, (t-1)^2)$$

is indeed the solution. The proof follows from Theorem 1.3. $\square$

4. Generalizing Ramanujan's derivatives for E_2, E_4 and E_6

Ramanujan famously obtained the following formulas [19, p. 181] for the action of $D = q\frac{d}{dq}$:

$$D(E_2) = \frac{E_2^2 - E_4}{12}, \qquad D(E_4) = \frac{E_2 E_4 - E_6}{3}, \qquad D(E_6) = \frac{E_2 E_6 - E_4^2}{2}.$$

Let B_s denote the Bernoulli numbers and recall the partition Eisenstein series [3, eq'n 1.5]

$$\mathbb{E}_{2t}(q) := \sum_{(1^{m_1},\dots,t^{m_t})\vdash t} \prod_{s=1}^{t} \frac{1}{m_s!}\left(\frac{B_{2s}\, E_{2s}(q)}{(2s)\cdot(2s)!}\right)^{m_s}.$$

These series play a valuable role (see, for example [2, Theorem 1.4 and eq'n (3.8)]) in the identification of weight $2t$ components of the quasimodular forms $\mathcal{U}_{2a}$ and $\mathcal{U}_{2a}^{\star}$, and they are *universal* (i.e. independent of the "a" up to a constant factor).

Remember the Dedekind's η-function can be given by $\eta(q) = q^{\frac{1}{24}}\prod_{k\geq 1}(1-q^k)$. Now, denote $\psi(q) = \eta^3(q)$ and $\mathcal{E}_t(q) = 8^t \frac{D^t(\psi(q))}{\psi(q)}$. In particular, $\mathcal{E}_0 = 1$ and $\mathcal{E}_1 = E_2$.

Lemma 4.1. *The functions $\mathcal{E}_t(q)$ satisfy the differential equation $\mathcal{E}_t(q) = (E_2(q) + 8D)\mathcal{E}_{t-1}(q)$.*

Proof. Employing $8\frac{D(\psi)}{\psi} = E_2 = \mathcal{E}_1$, it is easy to check that

$$\begin{aligned}\psi\mathcal{E}_t &= 8\cdot D\left[8^{t-1}D^{t-1}(\psi)\right] = 8\cdot D[\psi\cdot\mathcal{E}_{t-1}]\\ &= [\mathcal{E}_{t-1}\cdot 8D(\psi) + \psi\cdot 8D(\mathcal{E}_{t-1})]\\ &= [\mathcal{E}_{t-1}\cdot \psi E_2 + \psi\cdot 8D(\mathcal{E}_{t-1})].\end{aligned}$$

Dividing through by ψ, we arrive at the desired conclusion. $\square$

The next result can be regarded as a generalization of the derivative rules (1.2).

Theorem 4.2. *We have the differential equation*

$$D(\mathbb{E}_{2t-2}(q)) = t(2t+1)\cdot\mathbb{E}_{2t}(q) - 3\mathbb{E}_2(q)\cdot\mathbb{E}_{2t-2}(q).$$

Proof. For brevity, write $g(t) := \frac{1}{4^t(2t+1)!}$. Once more, we utilize $8\frac{D(\psi)}{\psi} = E_2 = \mathcal{E}_1 = 24\,\mathbb{E}_2$. From [2, equation (3.3)] we discern $\mathbb{E}_{2a} = g(a)D(\mathcal{E}_a)$. Together with Lemma 4.1, these allow

$$\begin{aligned}
D(\mathbb{E}_{2t-2}) &= g(t-1)\cdot D(\mathcal{E}_{t-1}) = \frac{1}{8}\,g(t-1)[\mathcal{E}_t - E_2\cdot\mathcal{E}_{t-1}] \\
&= \frac{1}{8}\,g(t-1)\left[\frac{1}{g(t)}\,\mathbb{E}_{2t} - E_2\cdot\frac{1}{g(t-1)}\mathbb{E}_{2t-2}\right] \\
&= \frac{1}{8}[8t(2t+1)\cdot\mathbb{E}_{2t} - E_2\cdot\mathbb{E}_{2t-2}] \\
&= \frac{1}{8}[8t(2t+1)\,\mathbb{E}_{2t} - (24\,\mathbb{E}_2)\cdot\mathbb{E}_{2t-2}].
\end{aligned}$$

Therefore, we gather $D(\mathbb{E}_{2t-2}) = t(2t+1)\cdot\mathbb{E}_{2t} - 3\mathbb{E}_2\cdot\mathbb{E}_{2t-2}$. □

We chose to record the following result $\partial_{E_2}\mathcal{U}_{2t} = \sum_{j=1}^{t}\epsilon_j\mathcal{U}_{2t-2j}$ (similarly for $\mathcal{U}_{2t}^{\star}$) which determines $\mathcal{U}_{2t}$ up to an E_2 independent term (the pure modular part) and expressing it recursively in terms of those of lower weight members.

Proposition 4.3. *Preserving the notation and definitions from* (1.4) *and* (1.5), *we have that*

$$\partial_{E_2}\mathcal{U}_{2t}(q) = -\frac{1}{12}\sum_{j=1}^{t}\frac{\mathcal{U}_{2t-2j}(q)}{j^2\binom{2j}{j}} \qquad \textit{and} \qquad \partial_{E_2}\mathcal{U}_{2t}^{\star}(q) = \frac{1}{12}\sum_{j=1}^{t}\frac{\mathcal{U}_{2t-2j}^{\star}(q)}{j^2\binom{2j}{j}}.$$

Proof. Denote $\mathfrak{U}^{\star}(x) := \sum_{t\geq 0}\mathcal{U}_{2t}^{\star}(q)\,x^t$. Direct calculation implies that

$$\log\mathfrak{U}^{\star}(x) = \sum_{r\geq 1}H_r(q)\frac{x^r}{r} \qquad \text{and} \qquad H_r(q) = \sum_{k\geq 1}\frac{q^{rk}}{(1-q^k)^{2r}}.$$

Let $\mathbf{S}_j(q) = \sum_{m\geq 1}\frac{m^j q^m}{1-q^m}$. It is proven in [1, Example 7.1] that

$$(2r-1)!H_r = \mathbf{S}(\mathbf{S}^2-1^2)(\mathbf{S}^2-2^2)\cdots(\mathbf{S}^2-(r-1)^2) \tag{4.1}$$

with the *umbral* convention (i.e. $\mathbf{S}^m$ understood as $\mathbf{S}_m$ after expansion).

Operating with the derivative ∂_{E_2} on the equation $\log\mathfrak{U}^{\star}(x) = \sum_{r\geq 1}H_r(q)\frac{x^r}{r}$ leads to $\sum_t x^t\partial_{E_2}\mathcal{U}_{2t}^{\star} = \mathfrak{U}^{\star}\sum_r\frac{x^r}{r}\partial_{E_2}H_r$ from which we obtain

$$\begin{aligned}
\partial_{E_2}\mathcal{U}_{2t}^{\star} &= \sum_{j=1}^{t}\frac{\mathcal{U}_{2t-2j}^{\star}}{j}\cdot\partial_{E_2}H_j \\
&= \sum_{j=1}^{t}\frac{\mathcal{U}_{2t-2j}^{\star}}{j(2j-1)!}\cdot\partial_{E_2}\left[\mathbf{S}\prod_{s=1}^{j-1}(\mathbf{S}^2-s^2)\right] \\
&= \sum_{j=1}^{t}\frac{\mathcal{U}_{2t-2j}^{\star}}{j(2j-1)!}\cdot(-1)^j\cdot\frac{1^2 2^2\cdots(j-1)^2}{24}
\end{aligned}$$

where the two facts $E_2 = 1 - 24\,\mathbf{S}_1$ and the identity in (4.1) have been utilized. We gather

$$\partial_{E_2}\mathcal{U}^{\star}_{2t} = \frac{1}{12}\sum_{j=1}^{t}\frac{(-1)^j\,\mathcal{U}^{\star}_{2t-2j}}{j^2\binom{2j}{j}}.$$

Starting with $\mathfrak{U}(x) := \sum_{t\geq 0}\mathcal{U}_{2t}(q)\,x^a$, it is evident that $-\log\mathfrak{U}(-x) = \sum_r H_r(q)\frac{x^r}{r}$. Then, applying an analogous argument as in above, one can show the remaining claim. □

Remark. The relation depicted in (4.1) has a natural generalization

$$(\beta t-1)!\sum_{k\geq 1}\frac{q^{tk}}{(1-q^k)^{\beta t}} = \prod_{s=1}^{\beta t-1}(\mathbf{S}-t+s).$$

5. Concluding remarks

Define the modular form $G_2(q) := 2E_2(q^2) - E_2(2)$. We observe that $\mathbb{C}[E_2,\Theta_{0,1},\Theta_{1,1}] = \mathbb{C}[E_2,G_2,E_4]$, which is due to the conversion formulas for the modular forms G_2 and E_4:

$$G_2(q) = \Theta_{0,1}(q) \qquad \text{and} \qquad E_4(q) = \Theta_{0,1}^2(q) + 6\Theta_{1,1}(q).$$

Thus all quasimodular forms over the congruence subgroup $\Gamma_0(2)$ form the ring $\mathbb{C}[E_2,G_2,E_4]$. Consequently, there exist some constants $\widetilde{c}_c(\alpha,\beta,\gamma)$ such that $\widetilde{c}_c(0,0,0) = 1$ and

$$\begin{aligned}\widetilde{c}_c(\alpha,\beta,\gamma) = {}& \frac{2\alpha+4\beta+8\gamma-1}{24}\widetilde{c}_c(\alpha-1,\beta,\gamma) + \frac{8\gamma-8\beta+7}{24}\widetilde{c}_c(\alpha,\beta-1,\gamma)\\ &- \frac{\alpha+1}{12}\widetilde{c}_c(\alpha+1,\beta,\gamma-1) + \frac{\beta+1}{6}\widetilde{c}_c(\alpha,\beta+1,\gamma-1)\\ &- \frac{4(\gamma+1)}{3}\widetilde{c}_c(\alpha,\beta-3,\gamma+1).\end{aligned}$$

This allows one to write an alternative formulation of Theorem 1.3.

Theorem 5.1. *If t is a positive integer, then we have that*

$$\frac{D^t(\theta_4(q))}{\theta_4(q)} = \sum_{\substack{\alpha,\beta,\gamma\geq 0\\ \alpha+\beta+2\gamma=t}}\widetilde{c}_c(\alpha,\beta,\gamma)\cdot E_2^{\alpha}(q)\,G_2^{\beta}(q)\,E_4^{\gamma}(q). \tag{5.1}$$

Proof. The proof is again by induction on t. We require a few basic calculations:

$$\frac{D(\theta_4)}{\theta_4} = \frac{E_2-G_2}{24}, \qquad D(E_2) = \frac{E_2^2-E_4}{12}, \tag{5.2}$$

$$D(G_2) = \frac{E_2G_2-2G_2^2+E_4}{6}, \qquad D(E_4) = \frac{E_2E_4-4G_2^3+3G_2E_4}{3}.$$

The base case $t=1$ is covered in (5.2) with $\widetilde{c}_c(1,0,0) = \frac{1}{24}$ and $\widetilde{c}_c(0,1,0) = -\frac{1}{24}$. Assume (5.1) holds true. The next steps are very similar to the other proofs in this paper, so these are omitted to avoid unduly replications. □

We record the below results (with a great deal of overlaps with [7] and [13]) which reveal that if we add infinite families of MacMahon's quasimodular forms $\mathcal{U}_{2t}(q)$, $\mathcal{C}_{2t}(q)$ and $\mathcal{U}^{\star}_{2t}(q)$ then the outcomes are weight 0 modular forms although apparently they are of weights $\leq 2t$.

Proposition 5.2. *Adopt the notation $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$ and the short-hand $(q)_\infty = (q;q)_\infty$. We have the identities*

$$\sum_{t\geq 0}\mathcal{U}_{2t}(q) = \frac{(q^6)_\infty}{(q)_\infty(q^2)_\infty(q^3)_\infty}, \qquad \sum_{t\geq 0}\mathcal{C}_{2t}(q) = \frac{(q^4)_\infty(q^6)_\infty^2}{(q)_\infty(q^3)_\infty(q^{12})_\infty},$$

$$\sum_{t\geq 0}\mathcal{U}^{\star}_{2t}(q) = \sum_{n\geq 1}\frac{(-1)^{n-1}(1-q^n)(1-q^{2n})q^{\binom{n}{2}}}{1-3q^n+q^{2n}}.$$

Proof. From [9, Corollary 2], we obtain

$$\sum_{t\geq 0}\mathcal{U}_{2t}(q) = \frac{1}{(q)_\infty^3}\sum_{n\geq 0}q^{\binom{n+1}{2}}\left((-1)^n(2n+1)\sum_{t\geq 0}\frac{(-1)^t\binom{n+t}{2t}}{2t+1}\right) = \frac{\sum_{n\geq 0}q^{\binom{n+1}{2}} - 3\sum_{n\geq 0}q^{\binom{3n+2}{2}}}{(q)_\infty^3}.$$

On the other hand, we can simplify

$$\frac{(q^6)_\infty}{(q)_\infty(q^2)_\infty(q^3)_\infty} = \frac{1}{(q)_\infty^3}\prod_{k\geq 1}\frac{1-q^{6k}}{(1+q^k)(1+q^k+q^{2k})} = \frac{1}{(q)_\infty^3}\prod_{k\geq 1}(1-q^k+q^{2k})(1-q^k).$$

It suffices to recognize the identity $\prod_k(1-q^k+q^{2k})(1-q^k) = \sum_n q^{\binom{n+1}{2}} - 3\sum_n q^{\binom{3n+2}{2}}$.
Again, from [9, Corollary 2], we obtain

$$\theta_4(q)\sum_{t\geq 0}\mathcal{C}_{2t}(q) = \theta_4(q) + \sum_{n\geq 0}q^{n^2}\left((-1)^n(2n)\sum_{t\geq 1}\frac{(-1)^t\binom{n+t}{2t}}{n+t}\right) = \sum_{\mathbb{Z}}q^{(3n)^2} - \sum_{\mathbb{Z}}q^{(3n+1)^2}$$

to compare against ($\theta_4(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$ is used here)

$$\frac{\theta_4(q)\,(q^4)_\infty(q^6)_\infty^2}{(q)_\infty(q^3)_\infty(q^{12})_\infty} = \frac{(q)_\infty(-q^3,q^3)_\infty}{1-q^{2k}+q^{4k}} = \sum_{\mathbb{Z}}q^{(3n)^2} - \sum_{\mathbb{Z}}q^{(3n+1)^2}.$$

The conclusion is clear. The last identity follows directly from [1, Proposition 4.1] where

$$\mathcal{U}^{\star}_{2t}(q) = \sum_{n\geq 1}\frac{(-1)^{n-1}(1+q^n)q^{\binom{n}{2}+tn}}{(1-q^n)^{2t}}. \qquad \square$$

The following is an immediate consequence of Proposition 5.2.

Corollary 5.3. *If we let $\sum_{t\geq 0}\mathcal{U}_{2t}(q) = \sum_{n\geq 0}u(n)q^n$ and $\sum_{t\geq 0}\mathcal{C}_{2t}(q) = \sum_{n\geq 0}\kappa(n)q^n$, then*

$$u(3n+2)\equiv 0 \pmod 3 \qquad \textit{and} \qquad \kappa(9n+6)\equiv 0 \pmod 3.$$

In line with the above, consider the infinite series that is defined in [4, eq'n (1)] for which the authors have found a single-sum representation as

$$\mathcal{U}_{2t}(2;q) = \frac{(q)_\infty}{(q^2)_\infty^2} \sum_{n\geq 0} \binom{n+t}{2t} q^{\binom{n+1}{2}}.$$

We now state and prove our claim which may be regarded as a result of independent interest.

Theorem 5.4. *If we let* $\sum_{n\geq 0} y(n)\, q^n := \sum_{t\geq 0} \mathcal{U}_{2t}(2;q)$*, then*

$$y(n) \equiv (-1)^n \cdot \#\{(r,s) \in \mathbb{Z}^2 : n = r^2 + s^2\} \pmod 5.$$

Proof. Using the basic facts $\sum_{t\geq 0} \binom{n+t}{2t} = F_{2n+1}$ and $F_{2n+1} \equiv (-1)^n(2n+1) \pmod 5$, where F_k stands for the Fibonacci number, we proceed to compute modulo 5:

$$\sum_{t\geq 0} \mathcal{U}_{2t}(2;q) = \frac{(q)_\infty}{(q^2)_\infty^2} \sum_{n\geq 0} F_{2n+1} q^{\binom{n+1}{2}} \equiv \frac{(q)_\infty}{(q^2)_\infty^2} \sum_{n\geq 0} (-1)^n (2n+1) q^{\binom{n+1}{2}} = \frac{(q)_\infty (q)_\infty^3}{(q^2)_\infty^2} = \theta_4^2(q).$$

However, $\theta_4^2(q) = \sum_{r,s\in\mathbb{Z}} (-1)^{r+s} q^{r^2+s^2} = \sum_{n\geq 0} [(-1)^n \cdot \#\{(r,s) \in \mathbb{Z} : n = r^2+s^2\}] q^n.$ $\square$

We close this section and our paper with a remark (the simple proof is left for the reader) regarding the sequence $c_v(\alpha,\beta,\gamma)$ defined in equation (1.3) and attributed to Ramanujan's q-series $V_{2t}(q)$. Namely, these coefficients carry a closed form when varying α while β and γ remain fixed. The merit or value of our formula can be viewed as a way of generating coefficients, in the expansion portrayed in Theorem 1.2, of quasimodular components on the basis of the corresponding modular forms.

Proposition 5.5. *Letting* $(x)_n = x(x+1)\cdots(x+n-1)$ *be the Pochhammer symbol, it holds true that*

$$\frac{c_v(\alpha,\beta,\gamma)}{c_v(0,\beta,\gamma)} = \frac{(1+4\beta+6\gamma)_{2\alpha}}{2^\alpha\, \alpha!}.$$

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