A Q-SERIES IN DISTANCE GEOMETRY

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Abstract. We give a short and elementary proof of a result by K. Stolarsky.

K. Stolarsky [S] defined q-sequences having *root-mean-square distance*

$$P_n(q)=n^2q^{2(n-1)}-\sum_{k=1}^{n-1}kq^{2(k-1)},\qquad S_n(q)=\frac{\sqrt{P_n(q)}}{n}+\sum_{m=1}^{n-1}\frac{\sqrt{P_m(q)}}{m(m+1)}.$$

Theorem [S]:

$$\sum_{k=0}^{n}(S_n(q)-S_k(q))^2=nq^{2(n-1)}.$$

Proof: Let us denote $b_m:=\sqrt{P_m(q)}$, $S_n:=S_n(q)$, $g_n:=\sum_{k=0}^n(S_n-S_k)$, $f_n:=\sum_{k=0}^n(S_n-S_k)^2$ and $h_n:=nq^{2(n-1)}$. Now, observe that

$$(1)\qquad S_n=\sum_{m=1}^{n}\frac{b_m-b_{m-1}}{m},\qquad g_{n+1}-g_n=\sum_{k=0}^{n}(S_{n+1}-S_n)=b_{n+1}-b_n,\qquad \text{hence}\qquad g_n=b_n.$$

Consequently, we obtain the relations

$$(2)\qquad \begin{aligned} f_{n+1}-f_n&=(S_{n+1}-S_n)\sum_{k=0}^{n}((S_{n+1}-S_k)+(S_n-S_k))=\frac{b_{n+1}-b_n}{n+1}(g_{n+1}+g_n)=\frac{b_{n+1}^2-b_n^2}{n+1}\\ &=\frac{P_{n+1}-P_n}{n+1}=\frac{(n+1)^2q^{2n}-n(n+1)q^{2(n-1)}}{n+1}=h_{n+1}-h_n.\end{aligned}$$

Hence the theorem is proved since $f_0=h_0=0$. □

REMARKS: The argument used in the proof above is clearly valid for any sequence b_m. Hence the first line in equation (2) is basically the content of the Lemma in [S].

References

[S] K. Stolarsky, *q-Analogue Triangular Numbers and Distance Geometry*, Proc. Amer. Math. Soc. **125 #1** (1997), 35-39.

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