# ON SUM OF DISTANCES ON A SPHERE

Tewodros Amdeberhan

Department of Mathematics, Temple University, Philadelphia PA 19122, USA
tewodros@euclid.math.temple.edu

Abstract. An improved upper bound for the sum of distances between N points on a unit sphere is presented.

Let $P_1, \dots, P_N$ be points on the unit sphere $S^{n-1}$ of the Euclidean space $R^n$. Define the distance

$$S(N,n,\lambda) = max \sum_{i<j} |P_i - P_j|^\lambda$$

where the max is taken over all possible $P_1, \dots, P_N$. This function has been investigated by several authors [**1**] - [**5**]. Denoting the surface area of $S^{n-1}$ by $\sigma(S^{n-1})$ and

$$c(n,\lambda) = \frac{1}{2\sigma(S^{n-1})} \int_{S^{n-1}} |P_0 - P|^\lambda \sigma(P)$$

for any fixed point $P_0$, the following are the best known (to date) estimates for S.

1. The circle is nicely treated by Fejes Tóth [**4**] and

$$S(N,2,1) = \text{N } \cot(\frac{\pi}{2N}) = \frac{2}{\pi}N^2 - \frac{\pi}{6} + O(\frac{1}{N^2}). \tag{1}$$

2. The special case $n = 3, \lambda = 1$

$$S(N,3,1) < \frac{2}{3}N^2 - \frac{1}{2} \tag{2}$$

was proved by R. Alexander [**1**].

3. For the higher dimensions and any $0 \leq \lambda \leq 1$, this is due to K. Stolarsky [**5**]

$$S(N,n,\lambda) \leq c(n,\lambda)N^2 - \frac{1}{2}c(n,\lambda) \tag{3}$$

which holds true at least "half of the time", i.e. for any given N, inequality (3) is valid either for N or N+1.

In this note we present improved estimates for the special case $\lambda = 1$, but *any* dimension $n$. Our method relies heavily on the following two lemmata found in [**1**].

**Lemma 1:** Let $p_1, \ldots, p_N$ and $q_1, \ldots, q_N$ be any two set of real numbers. Then

$$\sum_{i,j} |p_i - q_j| \geq \sum_{i<j} |p_i - p_j| + \sum_{i<j} |q_i - q_j| + \sum_{j=1}^{N} \rho(p_i), \tag{4}$$

where $\rho(p_i) = \min_j |p_i - q_j|$.

**Lemma 2:** Let $q_1, \ldots, q_N$ be numbers in the interval [-1,1], and $\rho(x) = \min_j |x - q_j|$. Then

$$\int_{-1}^{1} \rho(x)dx \geq \frac{1}{N}. \tag{5}$$

Our main result is

**Theorem:**

$$S(N, n, 1) \leq c(n, 1)N^2 - \frac{1}{2}. \tag{6}$$

**Proof:** The variables $\tau, \tau_i$ will denote elements of the special orthogonal group $SO(n)$ acting on $S^{n-1}$ and integral on the $\tau$'s will be a Harr integral. Assume its measure is normalized. Take N points on the sphere $S^{n-1}$, say $P_1, \ldots, P_N$. Then observe that since

$$\int |P_i - \tau(P_j)| d\tau = \frac{1}{\sigma(S^{n-1})} \int |P_i - P| d\sigma(P),$$

it follows that

$$I := \sum_{i,j} \int |P_i - \tau(P_j)| d\tau = 2c(n, 1)N^2. \tag{7}$$

Next, using the fact that for any $P_0 \in S^{n-1}$

$$|P_i - P_j| = b_n \int |(P_i - P_j) \cdot \tau(P_0)| d\tau,$$

and Lemma 1, we obtain

$$\begin{aligned} I &= b_n \iint \sum_{i,j} |(P_i - \tau_1(P_j)) \cdot \tau_2(P_0)| d\tau_2 d\tau_1 \\ &\geq b_n \iint \left\{ \sum_{i<j} |(P_i - P_j) \cdot \tau_2(P_0)| + \sum_{i<j} |(\tau_1(P_i) - \tau_1(P_j)) \cdot \tau_2(P_0)| \right\} d\tau_2 d\tau_1 \\ &+ b_n \iint \sum_{j=1}^{N} II d\tau_2 d\tau_1. \end{aligned}$$

Upon taking the supremum over the set of all N-points $P_1, P_2, \ldots, P_n$, we have the inequality

$$I \geq 2S(N,n,1) + b_n \iint \sum_{i=1}^{N} II d\tau_2 d\tau_1. \tag{8}$$

Finally, we turn to estimate the missing expression in (8):

(9)

$$\begin{aligned}
b_n \sum_{j=1}^{N} \int \left( \int II d\tau_1 \right) d\tau_2 &= b_n \sum_{j=1}^{N} \int \left( \int \min_i |(P_i - \tau_1(P_j)) \cdot \tau_2(P_0)| d\tau_1 \right) d\tau_2 \\
&= b_n \sum_{j=1}^{N} \int \frac{1}{\sigma(S^{n-1})} \left( \int \min_i |(P_i - Q) \cdot \tau_2(P_0)| d\sigma(Q) \right) d\tau_2 \\
&= b_n \sum_{j=1}^{N} \int \frac{1}{\sigma(S^{n-1})} \left( d_n \int_{-1}^{1} \min_i |p_i' - x| dx \right) d\tau_2 \\
&\geq 1,
\end{aligned}$$

where the last inequality is justified using Lemma 2, and a simplification of the dimensional constants. The proof is complete after combining the results found in (7)-(9). □

## References


[1] R. Alexander, *On the sum of distances between n points on a sphere*, Acta Math. Acad. Sci. Hung. **23** (1972), 443-448.

[2] R. Alexander, K. Stolarsky, *Extremal problems of distance geometry related to energy integrals*, Tranc. Amer. Math. Soc. **193** (1974), 1-31.

[3] G. Björck, *Distributions of positive mass which maximize a certain generalized energy integral*, Ark. Mat. **3** (1955), 255-269.

[4] L. Fejes Tóth, *On the sum of distances determined by a pointset*, Acta Math. Acad. Sci. Hung. **7** (1956), 397-401.

[5] K. Stolarsky, *Sums of distances between points on a sphere*, Proc. Amer. Math. Soc. **35 #2** (1972), 547-549.