FURTHER STUDY ON MACMAHON-TYPE SUMS OF DIVISORS

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ABSTRACT. This paper is devoted to the study of

$$U_t(a,q) := \sum_{1\leq n_1<n_2<\cdots<n_t} \frac{q^{n_1+n_2+\cdots+n_t}}{(1+aq^{n_1}+q^{2n_1})(1+aq^{n_2}+q^{2n_2})\cdots(1+aq^{n_t}+q^{2n_t})}$$

when a is one of $0, \pm1, \pm2$. The idea builds on our previous treatment of the case $a=-2$. It is shown that all these functions lie in the ring of quasimodular forms. Among the more surprising findings is

$$U_2(1,q) = \sum_{n\geq 1} \frac{q^{3n}}{(1-q^{3n})^2}.$$

1. Introduction

The object of this paper is the study of

$$U_t(a,q) := \sum_{1\leq n_1<n_2<\cdots<n_t} \frac{q^{n_1+n_2+\cdots+n_t}}{\prod_{k=1}^{t}(1+aq^{n_k}+q^{2n_k})} = \sum_{n\geq 0} MO(a,t;n)q^n, \qquad (1)$$

where a is among $0, \pm1, \pm2$. As will become clear, the $U_t(a;q)$ are most interesting when the roots of

$$1+ax+x^2=0$$

are also roots of unity, and this occurs precisely for $a=0,\pm1,\pm2$.

We shall prove that these functions are quasimodular forms. Thus we can expect interesting arithmetic consequences some of which we describe in detail. In Section 2, we prove a necessary theorem connecting $U_t(a,q)$ with modified Chebychev polynomials and prove that $MO(1,t;3n+2)=0$. In Section 3, we prove analytically the following result:

Theorem 1.1. *We have that*

$$U_2(1,q) = \sum_{n\geq 1} \frac{q^{3n}}{(1-q^{3n})^2}. \qquad (2)$$

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In Section 4, we give a combinatorial proof of the result in (2). Section 5 provides alternative forms of $U_t(a,q)$ necessary to discerning the quasimodular nature of these functions. Section 6 treats quasimodularity in the cases $a=\pm 2$. Section 7 explores the cases $t=1$, paving the way for Section 8 where we reveal quasimodularity of the remaining cases $a=0,\pm 1$. The paper concludes with two appendices, the first uses the Wilf-Zeilberger (WZ) method to treat some of the binomial coefficient summations needed in this paper, and the second utilizes Riordan arrays for the same objective.

2. The Chebychev connection

In this section, we begin by recalling the Andrews-Rose identity [4] which involves the *Chebychev polynomials of the first kind* defined by

$$T_n(\cos\theta) = \cos(n\theta).$$

In [4, Theorem 1], it is proved that

$$\begin{aligned} 2\sum_{n\geq 0} T_{2n+1}\left(\frac{x}{2}\right) q^{n^2+n} &= x(q^2;q^2)_\infty^3 \prod_{n\geq 1}\left(1+\frac{x^2q^{2n}}{(1-q^{2n})^2}\right) \\ &= (q^2;q^2)_\infty^3 \sum_{t\geq 0} U_t(-2,q)x^{2t+1}, \end{aligned} \tag{3}$$

where $U_t(a,q)$ is defined in (1), and

$$(A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}) \qquad \text{for } 0\leq n\leq\infty.$$

We note that $T_n(x)$ is alternately an even and odd function, depending on n. To make (3) more manageable for our purposes, we write

$$to_n(x) = \frac{T_{2n+1}(\sqrt{x})}{\sqrt{x}}. \tag{4}$$

Thus equation (3) now becomes

$$\sum_{n\geq 0} to_n\left(\frac{x}{4}\right) q^{\binom{n+1}{2}} = (q;q)_\infty^3 \prod_{n=1}^{\infty}\left(1+\frac{xq^n}{(1-q^n)^2}\right) = (q;q)_\infty^3 \sum_{t\geq 0} U_t(-2,q)x^t.$$

Consequently, we obtain the following.

Theorem 2.1. *We have*

$$\sum_{t\geq 0} U_t(a,q)x^t = \prod_{n=1}^{\infty}\frac{1}{(1+aq^n+q^{2n})(1-q^n)}\cdot\sum_{n\geq 0} to_n\left(\frac{x+a+2}{4}\right) q^{\binom{n+1}{2}}. \tag{5}$$

Proof. We proceed as follows,

$$\begin{aligned}\sum_{t\geq 0} U_t(a,q)x^t &= \prod_{n=1}^{\infty}\left(1+\frac{xq^n}{1+aq^n+q^{2n}}\right)\\ &= \prod_{n=1}^{\infty}\frac{1}{1+aq^n+q^{2n}}\cdot\prod_{n=1}^{\infty}\left(1+(x+a)q^n+q^{2n}\right)\\ &= \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{1+aq^n+q^{2n}}\cdot\prod_{n=1}^{\infty}\left(1+\frac{(x+a+2)q^n}{(1-q^n)^2}\right)\\ &= \prod_{n=1}^{\infty}\frac{1}{(1+aq^n+q^{2n})(1-q^n)}\cdot\sum_{n\geq 0} to_n\left(\frac{x+a+2}{4}\right)q^{\binom{n+1}{2}}. \qquad \square\end{aligned}$$

This core identity allows us an immediate congruence.

Theorem 2.2. *We have* $MO(1,t;3n+2)=0$.

Proof. By (5),

$$\begin{aligned}\sum_{t\geq 0} U_t(1,q)x^t &= \prod_{n\geq 1}\frac{1}{(1+q+q^{2n})(1-q^n)}\sum_{n\geq 0} to_n\left(\frac{x+3}{4}\right)q^{\binom{n+1}{2}}\\ &= \frac{1}{(q^3;q^3)_\infty}\sum_{n\geq 0} to_n\left(\frac{x+3}{4}\right)q^{\binom{n+1}{2}}.\end{aligned}$$

Now, $\binom{n+1}{2}$ is never congruent to 2 modulo 3. Hence there are no powers of q in $U_t(1,q)$ congruent to 2 modulo 3. This establishes our theorem. $\square$

3. Analytic proof of Theorem 1.1

We start with the following preliminary lemma.

Lemma 3.1. *Let* $\omega(n) := \frac{n(3n+1)}{2}$. *Then,*

$$\sum_{n=1}^{\infty}\frac{q^{3n}}{(1-q^{3n})^2} = \frac{1}{(q^3;q^3)_\infty}\cdot\sum_{n\in\mathbb{Z}}(-1)^{n-1}\omega(n)\,q^{3\omega(n)}.$$

Proof. Consider

$$\begin{aligned}q\frac{d}{dq}(q^3;q^3)_\infty &= -q\cdot(q^3;q^3)_\infty\cdot\sum_{n=1}^{\infty}\frac{3nq^{3n-1}}{1-q^{3n}}\\ &= -3\cdot(q^3;q^3)_\infty\cdot\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{3nm}\\ &= -3\cdot(q^3;q^3)_\infty\cdot\sum_{m=1}^{\infty}\frac{q^{3m}}{(1-q^{3m})^2}.\end{aligned}$$

On the other hand, the identity $(q;q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\omega(n)}$ implies that

$$q\frac{d}{dq}(q^3;q^3)_\infty = q\frac{d}{dq}\sum_{n=-\infty}^{\infty}(-1)^n q^{3\omega(n)} = 3\sum_{n=-\infty}^{\infty}(-1)^n\omega(n)q^{3\omega(n)}.$$

The conclusion is immediate from here. □

3.1. **Proof of Theorem 1.1.** Let us now recall [1] one of the representations of $T_n(x)$ adjusted to what we defined (4) as $to_n(x)$. Namely,

$$to_n(x) = \frac{T_{2n+1}(\sqrt{x})}{\sqrt{x}} = (2n+1)\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k+1}{2k+1}\frac{(4x)^k}{n+k+1}. \tag{6}$$

Next we note that $U_2(1,q)$ is the coefficient of x^2 in the expansion (see Theorem 2.1)

$$\sum_{t\geq 0}U_t(1,q)x^t = \prod_{n\geq 1}\left(1+\frac{xq^n}{1+q^n+q^{2n}}\right) = \frac{1}{(q^3;q^3)_\infty}\cdot\sum_{n\geq 0}to_n\left(\frac{x+3}{4}\right)q^{\binom{n+1}{2}}.$$

Based on the expression

$$\begin{aligned} to_n\left(\frac{x+3}{4}\right) &= (2n+1)\sum_{k=0}^{n}(-1)^{n+k}\binom{n+k+1}{2k+1}\frac{(x+3)^k}{n+k+1} \qquad (7)\\ &= (2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\sum_{i=0}^{k}\binom{k}{i}3^{k-i}x^i, \end{aligned}$$

we want the coefficient of x^2 in (7), which is

$$c_n := (2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{2}3^{k-2}.$$

We claim that

$$c_n = \begin{cases}(-1)^{j-1}\frac{j(3j+1)}{2} & \text{if } n=3j,\\ 0 & \text{if } n=3j+1,\\ (-1)^{j-1}\frac{j(3j-1)}{2} & \text{if } n=3j-1.\end{cases}$$

This is a standard binomial coefficient identity that can easily be verified (see Example 9.2 in Appendix 1 or Example 10.1 in Appendix 2). That means,

$$U_2(1,q) = \frac{1}{(q^3;q^3)_\infty}\cdot\left(\sum_{j\geq 0}(-1)^{j-1}\frac{j(3j+1)}{2}q^{\binom{3j+1}{2}} + \sum_{j\geq 0}(-1)^{j-1}\frac{j(3j-1)}{2}q^{\binom{3j}{2}}\right).$$

The proof is completed after comparing this formula with that of Lemma 3.1.

As a companion for Theorem 2.2, we are now in a position to prove the next result.

Theorem 3.1. *The following congruence holds true:*

$$MO(1,3;3n+1) \equiv 0 \pmod 3.$$

Proof. We will borrow a result for $U_3(1,q)=\sum_{n\geq 0} MO(1;3,n)q^n$ from Section 5:

$$\begin{aligned} U_3(1,q) &= \frac{1}{(q^3;q^3)_\infty}\sum_{n\geq 0} q^{\binom{n+1}{2}}\sum_{k=0}^{n}\frac{(-1)^{n+k}(2n+1)\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{3}3^{k-3} \\ &= \frac{1}{(q^3;q^3)_\infty}\sum_{n\geq 0} q^{\binom{n+1}{2}}\sum_{k=3}^{n}(-1)^{n+k}\left(\binom{n+k+1}{2k+1}+\binom{n+k}{2k+1}\right)\binom{k}{3}3^{k-3}. \end{aligned}$$

Since we are interested in the coefficients of q^{3n+1} in $U_3(1,q)$, and all the powers of q in $\frac{1}{(q^3;q^3)_\infty}$ are multiple of 3, it suffices to check the case when $\binom{n+1}{2}\equiv 1 \pmod 3$, that is $n=3m+1$, and show that the inner sum

$$\sum_{k=3}^{3m+1}(-1)^{3m+k+1}\left(\binom{3m+k+2}{2k+1}+\binom{3m+k+1}{2k+1}\right)\binom{k}{3}3^{k-3}$$

is divisible by 3. Actually, each summands is immediately divisible by 3 due to the term 3^{k-3} unless (perhaps) when $k=3$. Replacing this value to compute the corresponding term, we obtain

$$\begin{aligned} (-1)^{3m}\left(\binom{3m+5}{7}+\binom{3m+4}{7}\right)\binom{3}{3}3^0 &= (-1)^m\frac{3(2m+1)}{7}\binom{3m+4}{6} \\ &\equiv 0 \pmod 3. \end{aligned}$$

This completes the proof. ∎

4. A combinatorial proof of Theorem 1.1

Consider the following alternative representations of $U_2(1,q)$,

$$\begin{aligned} U_2(1,q) &= \sum_{1\leq n_1<n_2}\frac{q^{n_1+n_2}}{(1+q^{n_1}+q^{2n_1})(1+q^{n_2}+q^{2n_2})} \\ &= \sum_{1\leq n_1<n_2}\frac{q^{n_1+n_2}(1-q^{n_1})(1-q^{n_2})}{(1-q^{3n_1})(1-q^{3n_2})} = \sum_{1\leq n_1<n_2}\frac{(q^{n_1}-q^{2n_1})(q^{n_2}-q^{2n_2})}{(1-q^{3n_1})(1-q^{3n_2})} \\ &= \sum_{1\leq n_1<n_2}\left(\sum_{f_1\geq 0}q^{(3f_1+1)n_1}-\sum_{f_1\geq 0}q^{(3f_1+2)n_1}\right)\left(\sum_{f_2\geq 0}q^{(3f_2+1)n_2}-\sum_{f_2\geq 0}q^{(3f_2+2)n_2}\right) \\ &= \sum_{1\leq n_1<n_2}\left(\sum_{f_1,f_2\geq 0}q^{(3f_1+1)n_1+(3f_2+1)n_2}+\sum_{f_1,f_2\geq 0}q^{(3f_1+2)n_1+(3f_2+2)n_2}\right) \\ &\quad -\sum_{1\leq n_1<n_2}\left(\sum_{f_1,f_2\geq 0}q^{(3f_1+1)n_1+(3f_2+2)n_2}+\sum_{f_1,f_2\geq 0}q^{(3f_1+2)n_1+(3f_2+1)n_2}\right) \\ &:= \sum_{n\geq 1}\left(P_0(n)-P_1(n)\right)q^n, \end{aligned}$$

where $P_0(n)$ and $P_1(n)$ are desribed below. We use the following notation

$$n_1^{f_1} n_2^{f_2}$$

to denote the partition of $f_1 n_1 + f_2 n_2$ into different parts n_1 and n_2 wherein n_1 appears f_1 times and n_2 appears f_2 times.

Let $P_0(n)$ denotes the number of partitions of n involving two different parts (each may occur any number of times)

$$n_1^{f_1} n_2^{f_2}$$

where neither f_1 nor f_2 is divisible by 3, and $f_1 \equiv f_2 \pmod 3$.

Let $P_1(n)$ denotes the number of partitions of n involving two different parts

$$n_1^{f_1} n_2^{f_2}$$

where neither f_1 nor f_2 is divisible by 3, and $f_1 \not\equiv f_2 \pmod 3$.

Thus we may reformulate Theorem 1.1 as

$$P_0(n) - P_1(n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod 3, \\ \sigma\left(\frac{n}{3}\right) & \text{if } n \equiv 0 \pmod 3 \end{cases} \tag{8}$$

where $\sigma(m)$ denotes the sum of the divisors of m. The analytic form of (8) is clearly

$$\sum_{n \geq 1} (P_0(n) - P_1(n)) q^n = \sum_{n \geq 1} \frac{q^{3n}}{(1 - q^{3n})^2}.$$

To prove this, we begin with a proposed bijection between the partitions enumerated by $P_0(n)$ and those enumerated by $P_1(n)$.

First we map $P_1(n)$ partitions into $P_0(n)$ partitions. We begin with the partitions

$$n_1^{f_1} n_2^{f_2}$$

where $f_1 \not\equiv f_2 \pmod 3$ and 3 divides neither f_1 nor f_2. Without loss of generality we take $f_2 > f_1$ (equality is impossible because $f_1 \not\equiv f_2 \pmod 3$)

$$n_1^{f_1} n_2^{f_2} \mapsto (n_1 + n_2)^{f_1} n_2^{f_2 - f_1}.$$

Clearly the image is a P_0 partition because $f_2 - f_1 \not\equiv f_2 \pmod 3$ and thus must be congruent to f_1 (keep in mind there are only 2 non-zero residue classes modulo 3). This is evidently an injection of the P_1 partitions into P_0 partitions.

For the reverse mapping we assume (without loss of generality) $n_1 > n_2$ (neither f_1 nor f_2 being divisible by 3 and $f_1 \equiv f_2 \pmod 3$)

$$n_1^{f_1} n_2^{f_2} \mapsto (n_1 - n_2)^{f_1} n_2^{f_1 + f_2}.$$

The reverse image clearly has $f_1 \not\equiv f_1 + f_2 \pmod 3$ (and neither f_1 nor $f_1 + f_2$ is congruent to 0 modulo 3). Also both $n_1 - n_2$ and n_2 are positive integers. However, the image is not a P_0 partition precisely when $n_1 = 2n_2$ because then there are not two different parts.

So let us examine partitions of the form

$$(2d)^{f_1} d^{f_2}$$

with $f_1 \equiv f_2 \pmod 3$. The number being partitioned is

$$f_1(2d) + f_2 d = d(2f_1 + f_2) = n.$$

Now $2f_1 + f_2 \equiv -f_1 + f_2 \equiv 0 \pmod 3$; so $3 \mid n$ and d must be a divisor of $n/3$. Thus if $n \not\equiv 0 \pmod 3$, we have a bijection and $P_1(n) = P_0(n)$. Finally we may take $n = 3\nu$. In how many ways can we solve

$$f_1(2d) + f_2 d = 3\nu,$$

or equivalently

$$2f_1 + f_2 = 3\frac{\nu}{d}?$$

This is solvable for f_1 provided $3\frac{\nu}{d} - f_2$ is even. Thus f_2 may be chosen from

$$1 \text{ or } 2,\ 4 \text{ or } 5,\ 7 \text{ or } 8,\ \ldots,\ 3\frac{\nu}{d} - 2 \text{ or } 3\frac{\nu}{d} - 1.$$

Therefore, there are $\frac{\nu}{d}$ choices possible for f_2 (and f_1 is uniquely determined once f_2 is chosen). Hence for each divisor d of ν (recall $n = 3\nu$), there are $\frac{\nu}{d}$ partitions in $P_0(n)$ without an image in $P_1(n)$. Hence

$$P_0(3\nu) - P_1(3\nu) = \sum_{d|\nu} \frac{\nu}{d} = \sum_{d|\nu} d = \sigma(\nu) = \sigma\left(\frac{n}{3}\right)$$

and Theorem 1.1 is proved.

Example 4.1. *For $n = 9 = 3 \cdot 3$, we obtain*

$$\begin{aligned} &711 \mapsto 81 \\ &522 \mapsto 72 \\ &441 \mapsto 54 \\ &411111 \mapsto 51111, 63, 22221, 2211111, 21111111. \end{aligned}$$

5. A combined setup for all $U_t(a,q)$

As a result of (5), we may extract the coefficient, denoted $[\mathbf{x}^\mathbf{t}]$, of x^t from both sides:

$$U_t(a,q) = \prod_{m \geq 1} \frac{1}{(1 + aq^m + q^{2m})(1 - q^m)} \cdot \sum_{n \geq 0} q^{\binom{n+1}{2}} [\mathbf{x}^\mathbf{t}]\ to_n\left(\frac{x + a + 2}{4}\right).$$

From (6), we know that

$$[\mathbf{x}^\mathbf{t}]\ to_n\left(\frac{x + a + 2}{4}\right) = (2n+1) \sum_{k=0}^{n} (-1)^{n+k} \frac{\binom{n+k+1}{2k+1}}{n+k+1} \binom{k}{t} (a+2)^{k-t}. \tag{9}$$

The case $a=-2$ goes back to Andrews-Rose [4, Corollary 2]:

$$U_t(-2,q)=\frac{1}{(q;q)_\infty^3}\cdot\sum_{n\geq 0}(-1)^{n+t}\frac{2n+1}{2t+1}\binom{n+t}{2t}q^{\binom{n+1}{2}}. \tag{10}$$

For $a=2$, the coefficient of x^t in (5) is

$$c_n:=(2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{t}4^{k-t}=\binom{n+t}{2t}.$$

This is known as *Moriarty identity*, see Gould's collection [8, 3.161] or Appendix 2 or Example 9.1 in Appendix 1 below. Therefore, we have

$$U_t(2,q)=\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\cdot\sum_{n\geq 0}\binom{n+t}{2t}q^{\binom{n+1}{2}}. \tag{11}$$

The case $a=-1$ is:

$$U_t(-1,q)=\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(q;q)_\infty^2(q^6;q^6)_\infty}\cdot\sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^{n+k}(2n+1)\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{t}q^{\binom{n+1}{2}}.$$

The case $a=1$ is:

$$U_t(1,q)=\frac{1}{(q^3;q^3)_\infty}\cdot\sum_{n\geq 0}\sum_{k=0}^{n}\frac{(-1)^{k}(2n+1)\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{t}3^{k-t}q^{\binom{n+1}{2}}.$$

For $a=0$, by the computations made in the Appendix 2 below, we may write

$$U_t(0,q)=\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty}\cdot\sum_{n\geq 0}(-1)^{n+\lfloor (n+t)/2\rfloor}\binom{\lfloor (n+t)/2\rfloor}{t}q^{\binom{n+1}{2}}.$$

6. Quasimodular structure for $a=\pm 2$

We introduce the rational functions

$$Q_m(a,q):=\frac{q^m}{1+aq^m+q^{2m}}$$

and write a natural generating function (see the proof of Theorem 2.1) as

$$F(x;a,q):=\prod_{m\geq 1}(1+Q_m(a,q)\,x)=\sum_{t\geq 0}U_t(a,q)\,x^t.$$

One also has that

$$-\log\left(F(-x;a,q)\right)=\sum_{r\geq 1}H_r(a,q)\frac{x^r}{r}\quad\text{where}\quad H_r(a,q):=\sum_{m\geq 1}Q_m^r(a,q).$$

6.1. **The case** $a = -2$. Let's revive the umbral expansion [2, Example 7.1]

$$(2r-1)!\, H_r(-2,q) = \mathbf{S}(\mathbf{S}^2 - 1^2)(\mathbf{S}^2 - 2^2)\cdots(\mathbf{S}^2 - (r-1)^2) \tag{12}$$

where

$$H_r(-2,q) = \sum_{k\geq 1} \frac{q^{rk}}{(1-q^k)^{2r}} \qquad \text{and} \qquad \mathbf{S}_j(q) := \sum_{k\geq 1} \frac{k^j q^k}{1-q^k}.$$

From Andrews-Rose [4], we know that the $U_t(-2,q)$ lie in the ring of quasimodular forms and hence freely generated by the Eisenstein series $E_2 = 24D(\eta)/\eta, E_4$ and E_6 where $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ denotes the *Dedekind eta-function*. So, the derivative ∂_{E_2} is meaningful here.

Proposition 6.1. *It is true that*

$$\partial_{E_2} U_t(-2,q) = -\frac{1}{2}\sum_{j=1}^{t} \frac{U_{t-j}(-2,q)}{j^2\binom{2j}{j}}.$$

Proof. Operating with ∂_{E_2} on $-\log\left(F(-x;-2,q)\right) = \sum_{r\geq 1} H_r(-2,q)\frac{x^r}{r}$ leads to

$$-\sum_t (-x)^t \partial_{E_2} U_t(-2,q) = F(-x;-2,q)\sum_r \frac{x^r}{r}\partial_{E_2} H_r(-2,q)$$

from which we obtain (by comparing powers of x^t)

$$\begin{aligned}
\partial_{E_2} U_t(-2,q) &= -\sum_{j=1}^{t} (-1)^j \frac{U_{t-j}(-2,q)}{j} \cdot \partial_{E_2} H_j(-2,q) \\
&= -\sum_{j=1}^{t} (-1)^j \frac{U_{t-j}(-2,q)}{j(2j-1)!} \cdot \partial_{E_2}\left[\mathbf{S}\prod_{\ell=1}^{j-1}(\mathbf{S}^2 - \ell^2)\right] \\
&= -\sum_{j=1}^{t} (-1)^j \frac{U_{t-j}(-2,q)}{j(2j-1)!} \cdot (-1)^j \cdot \frac{1^2 2^2\cdots(j-1)^2}{24}
\end{aligned}$$

where the two facts $E_2 = 1 - 24\,\mathbf{S}_1$ and the identity in (12) have been utilized. So, we infer that

$$\partial_{E_2} U_t(-2,q) = -\frac{1}{12}\sum_{j=1}^{t} \frac{U_{t-j}(-2,q)}{j^2\binom{2j}{j}}.$$

□

6.2. **The case** $a = 2$.

Lemma 6.1. *Let* $D := q\frac{d}{dq}$. *If* $A_t(q) := \sum_{n\geq 0}(2n+1)^t q^{\binom{n+1}{2}}, B(q) := \sum_{n\geq 0} q^{\binom{n+1}{2}}$ *and* $C(q) := q^{\frac{1}{8}}\prod_{m\geq 1}(1+q^m)(1-q^{2m})$ *then we have the property that*

$$8^t q^{-\frac{1}{8}} \cdot D^t C(q) = (1+8D)^t B(q) = A_{2t}(q) \qquad \textit{when } t \geq 0 \textit{ is an integer.}$$

Proof. First recall *Jacobi's triple product* (see [5, p. 35, Entry 19]) which may be stated in the manner

$$(q;q)_\infty\,(-z^{-1};q)_\infty\,(-zq;q)_\infty=\sum_{n\in\mathbb{Z}} q^{\binom{n+1}{2}}\,z^n. \tag{13}$$

Choosing $z=1$ leads to a well-known identity

$$\prod_{m\geq 1}(1+q^m)(1-q^{2m})=\sum_{n\geq 0} q^{\binom{n+1}{2}} \tag{14}$$

which ensures validity of the case $t=0$ of this lemma. In the next step, apply logarithmic differentiation in (14) in tandem with the product rule for derivatives on $C(q)$. Then proceed with induction on t to complete the proof. □

Lemma 6.2. *Let* $T_t(q):=4^t\sum_{n\geq 0}\frac{(n+t)!}{(n-t)!}q^{\binom{n+1}{2}}$. *Preserving notations from Lemma 6.1, we have the umbral relation*

$$T_t=A^0(A^2-1^2)(A^2-3^2)\cdots(A^2-(2t-1)^2).$$

Proof. The statement boils down to the elementary fact that

$$4^t\cdot\frac{(n+t)!}{(n-t)!}=\prod_{\ell=1}^{t}((2n+1)^2-(2\ell-1)^2). \quad □$$

Theorem 6.1. *We have the differential-difference equation*

$$U_t(2,q)=\frac{1}{t(2t-1)}\left(D+U_1(2,q)-\binom{t}{2}\right)U_{t-1}(2,q)$$

together with $U_1(2,q)=\mathbf{S}_1(q)-4\,\mathbf{S}_1(q^2)=-\frac{1}{8}-\frac{1}{24}E_2(q)+\frac{1}{6}E_2(q^2)$.

Proof. The definition of $T_t(q)$ (see Lemma 6.2), notations from Lemma 6.1 and formula (11) imply the relation

$$U_t(2,q)=\alpha_t\,\frac{T_t(q)}{B(q)}\qquad\text{where}\qquad \alpha_t:=\frac{1}{4^t(2t)!}.$$

We prove this theorem by induction on t. The claim on $U_1(2,q)$ is obvious once we notice that $\mathbf{S}_1(q)=\sum_n\frac{q^n}{(1-q^n)^2}$. Therefore, we need to show that

$$\frac{\alpha_{t-1}}{8}\,\frac{T_t(q)}{B(q)}=\alpha_{t-1}D\left(\frac{T_{t-1}(q)}{B(q)}\right)+\alpha_{t-1}\,\frac{T_1(q)\,T_{t-1}(q)}{8B^2(q)}-\alpha_{t-1}\frac{t(t-1)}{2}\,\frac{T_{t-1}(q)}{B(q)}. \tag{15}$$

To this end, it suffices to observe that (using Lemma 6.1 and Lemma 6.2)

$$\begin{aligned}D\left(\frac{T_{t-1}}{B}\right)&=\frac{B(DT_{t-1})-T_{t-1}(DB)}{B^2}=\frac{DT_{t-1}}{B}-\frac{T_{t-1}}{B^2}\left(\frac{A_2-B}{8}\right)\\&=\frac{DT_{t-1}}{B}-\frac{T_{t-1}}{B^2}\left(\frac{A_2-A_0}{8}\right)=\frac{DT_{t-1}}{B}-\frac{T_{t-1}}{B^2}\frac{T_1}{8}.\end{aligned}$$

Thus, after some simplifications, equation (15) amounts to $T_t = (8D - 4t(t-1))T_{t-1}$. Using the definition of T_t and the derivation D, this is equivalent to

$$\sum_{n\geq 0} \frac{4^t(n+t)!}{(n-t)!} q^{\binom{n+1}{2}} = \sum_{n\geq 0} \left(\frac{4^t n(n+1)(n+t-1)!}{(n-t+1)!} - \frac{4^t t(t-1)(n+t-1)!}{(n-t+1)!} \right) q^{\binom{n+1}{2}}.$$

Finally, we rearrange the factorials on the right side to reduce to the left side. □

Corollary 6.1. *The function $U_t(2,q)$ belongs to $\widetilde{M_{\leq 2t}}(\Gamma_0(2))$, where $\widetilde{M_{\leq 2t}}(\Gamma_0(2))$ is the space of quasimodular forms of mixed weight of at most $2t$ on $\Gamma_0(2)$.*

Proof. From Theorem 6.1, we know that $U_1(2,q)$ is quasimodular of level 2, weight at most 2. Since the action of D on quasimodular forms increase the weight by 2, our assertion follows by Theorem 6.1. □

7. The special cases $t = 1$ and $a = 0$ or ± 1 for $U_t(a,q)$

We know that $U_t(-2,q)$ is quasimodular by work of Andrews-Rose [4], and $U_t(2,q)$ is quasimodular by Section 6 above. In the present section, we would note the following $t = 1$ cases for the other values of a.

We still use $\omega(n) = \frac{n(3n+1)}{2}$. Denote Jacobi's theta functions by

$$\theta_2(q) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2} \qquad \text{and} \qquad \theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

The Pentagonal Number Theorem gives $(q;q)_\infty = \sum_{n\in\mathbb{Z}} (-1)^n q^{\omega(n)}$. By a classical theorem of Jacobi on representations of a number as a sum of two squares, we have

$$U_1(0,q) = \sum_{n\geq 1} \frac{q^n}{1+q^{2n}} = \frac{\theta_3(q)^2 - 1}{4}.$$

On the other hand,

$$U_1(1,q) = \sum_{n\geq 1} \frac{q^n}{1+q^n+q^{2n}} = \frac{\sum_{n\in\mathbb{Z}} (-1)^n\, n\, q^{\omega(n)}}{\sum_{n\in\mathbb{Z}} (-1)^n\, q^{\omega(n)}}.$$

We prove this directly from Jacobi's Triple Product formula [5, p. 35, Entry 19]

$$\prod_{n\geq 1} (1-q^{3n})(1-\zeta q^{3n-1})(1-\zeta^{-1} q^{3n-2}) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\omega(n)} \zeta^n$$

and then computing the derivative $\frac{d}{d\zeta}$ at $\zeta = 1$ so that

$$\prod_{n\geq 1} (1-q^n) \cdot \left[\sum_{n\geq 1} \frac{q^{3n-2}}{1-q^{3n-2}} - \sum_{n\geq 1} \frac{q^{3n-1}}{1-q^{3n-1}} \right] = \sum_{n\in\mathbb{Z}} (-1)^n\, n\, q^{\omega(n)}.$$

The claim follows after dividing through by $\prod_n (1-q^n)$ and observing that

$$\sum_{n\geq 1} \frac{q^{3n-2}}{1-q^{3n-2}} - \sum_{n\geq 1} \frac{q^{3n-1}}{1-q^{3n-1}} = \sum_{n\geq 1} \frac{q^n}{1-q^{3n}} - \sum_{n\geq 1} \frac{q^{2n}}{1-q^{3n}} = \sum_{n\geq 1} \frac{q^n(1-q^n)}{1-q^{3n}}.$$

Next, we consider

$$1+2U_1(-1,q)=\sum_{n\in\mathbb{Z}}\frac{q^n}{1+q^{3n}}+\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{1+q^{3n}}.$$

Now by the Ramanujan ${}_1\psi_1$ formula [7, Eq. (5.2.1)], we may deduce

$$\sum_{n\in\mathbb{Z}}\frac{t^n}{1-cq^n}=\frac{(q)_\infty^2(ct;q)_\infty(q/ct;q)_\infty}{(c;q)_\infty(q/c;q)_\infty(t;q)_\infty(q/t;q)_\infty}.$$

Hence with $q\to q^3, c=-1$ and $t=q$, we obtain

$$\sum_{n\in\mathbb{Z}}\frac{q^n}{1+q^{3n}}=\frac{(q^3;q^3)_\infty^2(-q;q^3)_\infty(-q^2;q^3)_\infty}{2\,(-q^3;q^3)_\infty^2(q;q^3)_\infty(q^2;q^3)_\infty}=\frac{1}{2}\,\frac{\theta_3(-q^3)^3}{\theta_3(-q)}.$$

And sending $n\to -n$,

$$\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{1+q^{3n}}=\sum_{\mathbb{Z}}\frac{q^n}{1+q^{3n}}.$$

Hence

$$U_1(-1,q)=\frac{1}{2}\left(\frac{\theta_3(-q^3)^3}{\theta_3(-q)}-1\right).$$

Alternate formula for $U_1(1,q^4)$.

$$\begin{aligned}U_1(-1,q)-\sum_{n\geq 1}\frac{q^n}{1+(-q)^n+q^{2n}}&=\sum_{n\geq 1}\left(\frac{q^n}{1-q^n+q^{2n}}-\frac{q^n}{1+(-q)^n+q^{2n}}\right)\\&=\sum_{n\geq 1}\left(\frac{q^{2n}}{1-q^{2n}+q^{4n}}-\frac{q^{2n}}{1+q^{2n}+q^{4n}}\right)\\&=\sum_{n\geq 1}\frac{2q^{4n}}{(1+q^{4n})^2-q^{4n}}=\sum_{n\geq 1}\frac{2q^{4n}}{1+q^{4n}+q^{8n}}\\&=2U_1(1,q^4).\end{aligned}$$

Now by $A113661$ in OEIS [10],

$$\sum_{n\in\mathbb{Z}}\frac{q^n}{1+(-q)^n+q^{2n}}=\frac{1}{6}\left(\frac{\theta_3(q)^3}{\theta_3(q^3)}-1\right).$$

Hence

$$\begin{aligned}U_1(1,q^4)&=\frac{1}{2}\,U_1(-1,q)-\frac{1}{12}\left(\frac{\theta_3(q)^3}{\theta_3(q^3)}-1\right)\\&=\frac{1}{4}\left(\frac{\theta_3(-q^3)^3}{\theta_3(-q)}-1\right)-\frac{1}{12}\left(\frac{\theta_3(q)^3}{\theta_3(q^3)}-1\right).\end{aligned}$$

So

$$U_1(1,q^4)=\frac{1}{4}\,\frac{\theta_3(-q^3)^3}{\theta_3(-q)}-\frac{1}{12}\,\frac{\theta_3(q)^3}{\theta_3(q^3)}-\frac{1}{6}.$$

Lemma 7.1. *We have the following three identities:*

$$\begin{aligned}
U_1(0,q) &= \frac{\theta_3(q)^2-1}{4},\\
U_1(1,q) &= \frac{\theta_2(q)\theta_2(q^3)+\theta_3(q)\theta_3(q^3)-1}{6},\\
U_1(-1,q) &= \frac{2\theta_2(q^2)\theta_2(q^6)+2\theta_3(q^2)\theta_3(q^6)+\theta_2(q)\theta_2(q^3)+\theta_3(q)\theta_3(q^3)-3}{6}.
\end{aligned}$$

Proof. The second formula holds due to the classical result [9] that

$$\sum_{a,b\in\mathbb{Z}} q^{a^2+ab+b^2} = 1+6\left(\sum_{n\ge1}\frac{q^{3n-2}}{1-q^{3n-2}}-\sum_{n\ge1}\frac{q^{3n-1}}{1-q^{3n-2}}\right)$$

and the equality of the two multisets defined as $\mathcal{A} := \{a^2+ab+b^2 : a,b\in\mathbb{Z}\}$ and $\mathcal{B} := \{n^2+3m^2 : n,m\in\mathbb{Z}\}\cup\{n^2+3m^2+n+3m+1 : n,m\in\mathbb{Z}\}$ (equivalence of quadratic forms) which lead to

$$\sum_{a,b\in\mathbb{Z}} q^{a^2+ab+b^2} = \sum_{n\in\mathbb{Z}} q^{(n+\frac12)^2}\sum_{m\in\mathbb{Z}} q^{3(m+\frac12)^2}+\sum_{n\in\mathbb{Z}} q^{n^2}\sum_{m\in\mathbb{Z}} q^{3m^2}.$$

The last identity follows from the elementary observation that

$$U_1(-1,q)-U_1(1,q) = \sum_{n\ge1}\left(\frac{q^n(1+q^n)}{1+q^{3n}}-\frac{q^n(1-q^n)}{1-q^{3n}}\right) = 2\sum_{n\ge1}\frac{q^{2n}(1-q^{2n})}{1-q^{6n}}. \qquad \square$$

Example 7.1. *We have that*

$$\begin{aligned}
\theta_3(q)\theta_3(q^3) &= 2U_1(1,q)+4U_1(1,q^4)+1,\\
U_2(1,q) &= -\,D\left(\log(q)_\infty\right)\Big|_{q\to q^3} = -\frac{\sum_{n\in\mathbb{Z}}(-1)^n\,\omega_n\,q^{3\omega_n}}{\sum_{n\in\mathbb{Z}}(-1)^n\,q^{3\omega_n}} = \frac{1-E_2(q^3)}{24},\\
U_1(0,q) &= \frac{(q^2)_\infty\,\sum_{n\in\mathbb{Z}}(-1)^n\,n\,q^{\binom{2n+1}{2}}}{(q)_\infty(q^4)_\infty} \qquad \text{(see Section 5).}
\end{aligned}$$

Proposition 7.1. *Suppose that*

$$f_t(n) := \sum_{k=0}^{n}\frac{(-1)^k(2n+1)\binom{2n+1-k}{k}}{2n+1-k}\binom{n-k}{t}3^{n-k-t}.$$

Then, the recursive formula $f_t(n+2)-f_t(n+1)+f_t(n)=f_{t-1}(n)$ *holds and hence*

$$U_{t-1}(1,q) = U_t(1,q)+\frac{1}{(q^3;q^3)_\infty}\left(\sum_{n\ge0} f_t(n+2)\,q^{\binom{n+1}{2}}-\sum_{n\ge0} f_t(n+1)\,q^{\binom{n+1}{2}}\right).$$

8. Quasimodular structure when $a \in \{-1, 0, 1\}$

We require some definitions first (see for example [6]).

Definition 8.1. *A holomorphic function $\phi(\tau, z)$ on $\mathbb{H} \times \mathbb{C}$ is a Jacobi form for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of weight k and index m if it satisfies the following conditions:*
(1) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have the modular transformation

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \exp\left(2\pi i \cdot \frac{mcz^2}{c\tau + d}\right) \phi(\tau, z).$$

(2) For all integers a, b, we have the elliptic transformation

$$\phi(\tau, z + a\tau + b) = \exp\big(-2\pi i m(a^2\tau + 2az)\big)\phi(\tau, z).$$

(3) For each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have the Fourier expansion

$$(c\tau + d)^{-k} \exp\left(-2\pi i \cdot \frac{mcz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \sum_{n \geq 0} \sum_{r^2 \leq 4mn} b(n, r) q^n u^r;$$

where $b(n, r)$ are complex numbers and $u := e^{2\pi i z}$.

We also recall a result on Jacobi forms [6, Theorem 1.3].

Theorem 8.2. *Let ϕ be a Jacobi form on Γ of weight k and index m and λ, μ rational numbers. Then the function $f(\tau) = e^{m\lambda^2\tau}\phi(\tau, \lambda\tau + \mu)$ is a modular form (of weight k and on some subgroup of Γ' of finite index depending only on Γ and λ, μ).*

Next, we state and prove the main result of this section.

Theorem 8.3. *Fix $a \in \{-1, 0, 1\}$. Then, for each non-negative integer t, the functions $U_t(a, q)$ are quasimodular forms of mixed weight for a congruent subgroup Γ.*

Proof. It's our convention that $U_0(a, q) := 1$. Let's expand the generating function

$$\begin{aligned} \sum_{t \geq 0} U_t(a, q)\, x^{2t} &= \prod_{n \geq 1} \left(1 + \frac{x^2\, q^n}{1 + aq^n + q^{2n}}\right) \\ &= \prod_{n \geq 1} \frac{1 + (x^2 + a)q^n + q^{2n}}{1 + aq^n + q^{2n}}. \end{aligned}$$

Choose ζ such that $\zeta + \zeta^{-1} = x^2 + a$ in the Jacobi's Triple Product [3, Theorem 2.8] to obtain

$$\begin{aligned} \sum_{t \geq 0} U_t(a, q)\, x^{2t} &= \prod_{n \geq 1} \frac{(1 - q^n)\,(1 + \zeta q^n)(1 + \zeta^{-1} q^n)}{(1 - q^n)(1 + aq^n + q^{2n})} \\ &= \frac{1}{(\sqrt{\zeta} + \frac{1}{\sqrt{\zeta}})\, q^{\frac{1}{8}}} \cdot \frac{\sum_{m \in \mathbb{Z} + \frac{1}{2}} q^{\frac{1}{2}m^2} \zeta^m}{\prod_{n \geq 1}(1 - q^n)(1 + aq^n + q^{2n})}; \end{aligned}$$

where $\vartheta_{\frac{1}{2}}(q,\zeta) := \sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m$ is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$. Since $\sqrt{\zeta}+\frac{1}{\sqrt{\zeta}} = \sqrt{x^2+a+2}$, we may also write the above in the form

$$\sqrt{a+2}\cdot\sqrt{1+\frac{x^2}{a+2}}\cdot\sum_{t\geq 0} U_t(a,q)\,x^{2t} = \frac{\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{q^{\frac{1}{8}}\prod_{n\geq 1}(1-q^n)(1+aq^n+q^{2n})}.$$

Now, the right-hand side is a Jacobi form of weight 0 and index $\frac{1}{2}$ for $a=0,\pm 1$. While the left-hand side amounts to the convolution

$$\sqrt{a+2}\cdot\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{1}{(a+2)^k}\binom{\frac{1}{2}}{k}U_{n-k}(a,q)\right)x^{2n} = \frac{\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{q^{\frac{1}{8}}\prod_{n\geq 1}(1-q^n)(1+aq^n+q^{2n})}.$$

Recall the Dedekind eta-function $\eta(q) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)$ and $\binom{\frac{1}{2}}{k} = \frac{(-1)^k}{4^k(1-2k)}\binom{2k}{k}$.

The case $a=1$: Expand the RHS at $z=\frac{1}{6}$ or $\zeta=\frac{1+i\sqrt{3}}{2}=e^{2\pi iz}$ (so $\zeta^6=1$).

$$\sqrt{1+\frac{x^2}{3}}\cdot\sum_{t\geq 0} U_t(1,q)\,x^{2t} = \frac{1}{\sqrt{3}}\frac{\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q^3)}.$$

This is equal to the convolution

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{1}{3^k}\binom{\frac{1}{2}}{k}U_{n-k}(1,q)\right)x^{2n} = \frac{1}{\sqrt{3}}\frac{\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q^3)}.$$

The case $a=-1$: Expand the RHS at $z=\frac{1}{3}$ or $\zeta=\frac{-1+i\sqrt{3}}{2}=e^{2\pi iz}$ (so $\zeta^3=1$).

$$\sqrt{1+x^2}\cdot\sum_{t\geq 0} U_t(-1,q)\,x^{2t} = \frac{\eta(q^2)\,\eta(q^3)\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q)^2\,\eta(q^6)}.$$

This amounts to the convolution

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{\frac{1}{2}}{k}U_{n-k}(-1,q)\right)x^{2n} = \frac{\eta(q^2)\,\eta(q^3)\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q)^2\,\eta(q^6)}.$$

The case $a=0$: Expand the RHS at $z=\frac{1}{4}$ or $\zeta=i=e^{2\pi iz}$ (so $\zeta^4=1$).

$$\sqrt{1+\frac{x^2}{2}}\cdot\sum_{t\geq 0} U_t(0,q)\,x^{2t} = \frac{1}{\sqrt{2}}\frac{\eta(q^2)\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q)\,\eta(q^4)}.$$

This is equal to the convolution

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{1}{2^k}\binom{\frac{1}{2}}{k}U_{n-k}(0,q)\right)x^{2n} = \frac{1}{\sqrt{2}}\frac{\eta(q^2)\sum_{m\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}m^2}\zeta^m}{\eta(q)\,\eta(q^4)}.$$

By employing Theorem 8.2, we know that the Taylor series coefficients at $z = \frac{1}{6}, \frac{1}{3}, \frac{1}{4}$ of the respective right-hand sides are quasimodular forms of pure weight. On the other hand, the corresponding left-hand side coefficients can recursively determine that the functions $U_n(a,q)$ are themselves quasimodular forms (of mixed weight) once we realize $U_0(a,q) = 1$ are modular of weight 0 and by Lemma 7.1 each $U_1(a,q)$ is quasimodular of weight at most 1. $\square$

Remark 8.1. Although we did not explicitly pursue this point in Theorem 8.3, we believe that each function $U_t(a,q)$ belongs to $\widetilde{M}_{\leq t}(\Gamma_0(24))$ for the congruent subgroup $\Gamma_0(24)$ of $SL_2(\mathbb{Z})$.

9. Appendix 1 - WZ's approach

In this section, we opt to verify at least two of the binomial coefficient identities which appeared in the earlier sections of this paper. Our proof is the so-called Wilf-Zeilberger (WZ) method of automated procedure [12], effective for identities of hypergeometic type.

Example 9.1. *We have*

$$(2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{t}4^{k-t} = \binom{n+t}{2t}.$$

Proof. Define two functions

$$f_1(n,k) := \frac{(-1)^{n+k}(2n+1)}{n+k+1}\frac{\binom{n+k+1}{2k+1}\binom{k}{t}}{\binom{n+t}{2t}}4^{k-t}, \qquad \text{and}$$

$$g_1(n,k) := f_1(n,k)\cdot\frac{2(n+1)(k-t)(2k+1)}{(2n+1)(n+t+1)(n-k+1)}$$

where the second function is generated by *Zeilberger's algorithm*. Then, one checks (using symbolic software!) that $f_1(n+1,k) - f_1(n,k) = g_1(n,k+1) - g_1(n,k)$. Next, sum both sides of the last equation over all integers k. The outcome is the right-hand side vanishes and hence the sum $\sum_k f_1(n,k)$ is a constant (independent of n). Keep in mind that actually these sums have "finite support", namely the summands are zero outside of a finite interval. To complete the argument, evaluate say at $n = t$ to obtain the value 1. That means $\sum_k f_1(n,k) = 1$ hence the desired claim follows. $\square$

Example 9.2. *We have*

$$(2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{2}3^{k-2} = \begin{cases}(-1)^{j-1}\frac{j(3j+1)}{2} & \text{if } n = 3j,\\ 0 & \text{if } n = 3j+1,\\ (-1)^{j-1}\frac{j(3j-1)}{2} & \text{if } n = 3j-1.\end{cases}$$

Proof. We limit our justification to just one of the cases, say $n \to 3n$, since the remaining two are worked out analogously. To this end, introduce the discrete functions

$$f_2(n,k) := \frac{(-1)^{k-1}(6n+1)}{n(3n+1)(3n+k+1)}\binom{3n+k+1}{2k+1}\binom{k}{2}3^{k-2}, \qquad \text{and}$$
$$g_2(n,k) := f_2(n,k)\cdot R(n,k)$$

where $R(n,k)$ is some rational function of n and k which is *too long* to exhibit here but it can be furnished upon request. The next few steps are entirely similar to the above example, hence are omitted. □

10. Appendix 2 - Riordan's approach

Here, we give a detailed account of some computations made in Section 5. From equation (9), we have that

$$[x^t]\, to_n\left(\frac{x+a+2}{4}\right) = (2n+1)\sum_{k=0}^{n}(-1)^{n+k}\frac{\binom{n+k+1}{2k+1}}{n+k+1}\binom{k}{t}(a+2)^{k-t}$$
$$= (-1)^{n-t}\sum_{k=0}^{n}\frac{2n+1}{2k+1}\binom{n+k}{2k}\binom{k}{t}(-a-2)^{k-t}.$$

By applying Riordan arrays (see [11] for more details) to the last binomial sum, we find that

$$[x^t]\, to_n\left(\frac{x+a+2}{4}\right) = (-1)^{n-t}[z^n]\frac{z^t(1+z)}{(1+az+z^2)^{t+1}}.$$

If $a=2$ then we immediately obtain

$$[x^t]\, to_n\left(\frac{x+4}{4}\right) = (-1)^{n-t}[z^n]\frac{z^t}{(1+z)^{2t+1}} = \binom{n+t}{2t}.$$

In a similar way, for $a=-2$,

$$[x^t]\, to_n\left(\frac{x}{4}\right) = (-1)^{n-t}[z^n]\frac{z^t(1+z)}{(1-z)^{2t+2}} = (-1)^{n-t}\left(\binom{n+t+1}{2t+1}+\binom{n+t}{2t+1}\right).$$

Moreover, for $a=0$,

$$[x^t]\, to_n\left(\frac{x+2}{4}\right) = (-1)^{n-t}[z^n]\frac{z^t(1+z)}{(1+z^2)^{t+1}} = \begin{cases}(-1)^{n+j}\binom{j}{t} & \text{if } n+t=2j,\\ (-1)^{n+j}\binom{j}{t} & \text{if } n+t=2j+1.\end{cases}$$

Finally, if $a=1$ then

$$[x^t]\, to_n\left(\frac{x+3}{4}\right) = (-1)^{n-t}[z^n]\frac{z^t(1+z)(1-z)^{t+1}}{(1-z^3)^{t+1}}$$
$$= (-1)^{n-t}\left([z^{n-t}]h(z)+[z^{n-t-1}]h(z)\right)$$

where $h(z) = (\frac{1-z}{1-z^3})^{t+1}$.

Example 10.1. *If $a = 1$ and $t = 2$ then*

$$[x^2]\, to_n \left(\frac{x+3}{4}\right) = (-1)^n \left([z^{n-2}] \left(\frac{1-z}{1-z^3}\right)^3 + [z^{n-3}] \left(\frac{1-z}{1-z^3}\right)^3 \right)$$
$$= \begin{cases} (-1)^{j-1} \frac{j(3j+1)}{2} & \text{if } n = 3j, \\ 0 & \text{if } n = 3j+1, \\ (-1)^{j-1} \frac{j(3j-1)}{2} & \text{if } n = 3j-1. \end{cases}$$

where $h(z) = \left(\frac{1-z}{1-z^3}\right)^3 = \frac{1}{(1+z+z^2)^3} = \sum_{n=0}^{\infty} a_n z^n$ *with*

$$a_n = \begin{cases} j+1 & \text{if } n = 3j, \\ -\frac{3(j+1)(j+2)}{2} & \text{if } n = 3j+1, \\ \frac{3j(j+1)}{2} & \text{if } n = 3j-1. \end{cases}$$

(*see* $A128504$ *in OEIS* [10]).

Data Availability Statement. No datasets were generated or analysed during the current study.

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